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INTERPOLATION SEQUENCES IN AREA PRIVALOV CLASSES IN DISK

E.G. RODIKOVA

Abstract. In the work we obtain a necessary and sufficient condition for the zeros of analytic functions in area Privalov classes $\tilde{\Pi}_q$ $(0<q<1)$ in the unit circle $D=\{z\in\mathbb{C}: |z|<1\}$ located in the Stolz angles. We solve the free interpolation problem in these classes under the condition that the interpolation nodes are located in the Stolz angles. We also solve the interpolation problem in the area Privalov classes in circle on Carleson sets.

Keywords: analytic function, zeros, interpolation, area Privalov class, Stolz angle, unit circle.

Mathematics Subject Classification: 30E05, 30H15, 30H50

1. Introduction

Let $\mathbb{C}$ be the complex plane, $D=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions analytic on D, Z_f be the set of all zeros on a non–trivial function $f\in H(D)$, $n(r)=\operatorname{card}\{z_k:|z_k|<r\}$ be the number of points of sequence $\{z_k\}_1^\infty$ in the circle $|z|<r<1$ counting their multiplicities,

$$M(r,f)=\max_{|z|=r}|f(z)|,\qquad 0<r<1.$$

For each $0<q<+\infty$ we introduce the class

$$\tilde{\Pi}_q=\left\{f\in H(D):\int\limits_0^1\int\limits_{-\pi}^{\pi}\left(\ln^+|f(re^{i\theta})|\right)^q d\theta dr<+\infty\right\}.$$

We call it area Privalov class or Privalov class by area. For $q=1$ the area Privalov class coincides with the well–known Nevanlinna class involved in the scale of Nevanlinna — Dzhrbashyan classes, see [2]. We note that the area Privalov classes naturally appear in studying the differentiation issues in Privalov spaces Π_q, $q>0$, introduced in the monograph [3], see [17],

$$\Pi_q=\left\{f\in H(D):\sup_{0<r<1}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(\ln^+|f(re^{i\theta})|\right)^q d\theta<+\infty\right\}.$$

The area Privalov class has the same position with respect to the Privalov classes as the planar Nevanlinna classes $\tilde{\Pi}_1$ do to the classes of functions of bounded type $N=\Pi_1$.

In order to show the position of the classes $\tilde{\Pi}_q$ in the structure of known classes, for $\alpha>-1$, $0<q<+\infty$ we consider the spaces S_α^q

$$S_\alpha^q=\left\{f\in H(D):\int\limits_0^1(1-r)^\alpha T^q(r,f)dr<+\infty\right\},$$

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where $T(r,f)$ is the Nevanlinna characteristics of a function $f \in H(D)$,

$$T(r,f) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \ln^+ |f(re^{i\varphi})| d\varphi,$$

$\ln^+ |a| = \max(0, \ln |a|)$, $a \in \mathbb{C}$.

The classes S_α^q were first studied in [8] by Shamoyan, they generalize well–known Nevannlinna — Dzhrbashyan classes S_α^1, see [2].

Using the Hölder inequality, it is easy to prove that

$$\tilde{\Pi}_q \subset S_0^q \quad \text{for} \quad q > 1,$$

and

$$\tilde{\Pi}_q \supset S_0^q \quad \text{for} \quad 0 < q < 1.$$

The present work is devoted to studying the zero sets of functions in the classes $\tilde{\Pi}_q$, $0 < q < 1$, and to the interpolation in these classes.

The problem of characterizing the zero sets of analytic functions in a circle from different classes was repeatedly raised by specialists in the complex analysis. A fundamental result in this area is the coincidence of the root sets of classes of bounded analytic functions and the classes of functions of bounded type $N = \Pi_1$ that was established in the works by Blaschke [12] and Nevanlinna [2]: the sequences of zeros $\{z_k\}_1^\infty$ of these classes are characterized by the Blaschke condition

$$\sum_{k=1}^{+\infty} (1 - |z_k|) < +\infty.$$

As it follows from the results in works by Nevanlinna [2] and Shamoyan [7], the root sets of area Privalov classes, that is, $\tilde{\Pi}_1$, are characterized by the condition

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^2 < +\infty.$$

In the general case (for all $q > 0$) the problem on characterization of root sets for the Privalov classes and their planar analogues is not solved, however, for the classes Π_q there are results close to sharp ones, see [4], [9], [10]. The first part of the present work is devoted to studying the root sets of area Privalov class $\tilde{\Pi}_q$ in the disk for all $0 < q < 1$: we describe the roots of the functions in this classes, which are located in Stolz angles.

Definition 1.1. *The Stolz angle $\Gamma_\delta(\theta)$ with the vertex at a point $e^{i\theta}$ is the angle of opening $\pi\delta$, $0 < \delta < 1$, the bisectrix of which coincides with the segment $re^{i\theta}$, $0 \leqslant r < 1$, that is, it is the set of points $z \in D$, for which the inequalities hold*

$$\left|\arg\left(e^{i\theta} - z\right) - \theta\right| \leqslant \frac{\pi\delta}{2},$$

$$\left|e^{i\theta} - z\right| < \cos\frac{\pi\delta}{2}, \qquad 0 < \delta < 1.$$

The second and third part of this work is devoted to solving the interpolation problems in the area Privalov class in circle for all $0 < q < 1$. We formulate the interpolation problem on the set of simple nodes in the class $\tilde{\Pi}_q$. Let $\{z_k\}_1^\infty$ be a sequence of distinct points in D, $\{w_k\}_1^\infty$ be an arbitrary sequence complex numbers. What are the conditions for the nodes $\{z_k\}_1^\infty$ and the sequence of points $\{w_k\}_1^\infty$, under which we can construct a function in the class $\tilde{\Pi}_q$ such that

$$f(z_k) = w_k, \qquad k = 1, 2, \ldots? \tag{1.1}$$

In this case the sequence $\{z_k\}_1^\infty$ is called the *interpolation sequence*.

In the second part of this work we solve the interpolation problem on so–called Carleson sets in the class $\tilde{\Pi}_q$ $(0 < q < 1)$, while in the third part of the work we solve the free interpolation problem in the mentioned classes under the condition that the interpolation nodes are located in the Stolz angles.

We note that the fundamental result in the interpolation theory belongs to L. Carleson. In the work [13] he completely characterized interpolation sequences in the class of bounded analytic functions. A constructive solution for the free interpolation problem in the class H^∞ in the form of a series was proposed by Jones in [15]. The free interpolation problem in classes of functions of bounded form was addressed by Naftalevich [1], Hartmann et al. [14]; Shapiro and Shields [20] and Seip [19] studied this problem in the Hardy classes, while the case of Smirnov classes was treated by Yanagihara [21].

The interpolation problem under the uniform separation of uniform nodes (on the Carleson sets) in the Privalov classes Π_q for $q > 1$ was resolved in the work [16], while for $0 < q < 1$ this was done in the work by the author and Bednazh [6].

2. Characterization of zeros located in Stolz angles

In this part of the work we study the zero sets of the functions in the classes $\tilde{\Pi}_q$ $(0 < q < 1)$.

To formulate the main result, we introduce additional notation and definitions. For each $\beta > -1$ by the symbol $\pi_\beta(z, z_k)$ we denote the infinite Dzhrbashyan product with the zeros at the points of sequence $\{z_k\}_1^{+\infty} \subset D$, $0 < |z_k| \leqslant |z_{k+1}| < 1$, $k = 1, 2, \ldots$, see [11].

If $\beta = m \in \mathbb{Z}_+$, the Dzhrbashyan product reads

$$\pi_m(z, z_k) = \prod_{k=1}^{+\infty} \frac{\overline{z}_k(z_k - z)}{1 - \overline{z}_k z} \exp \sum_{j=0}^{m} \frac{1}{j+1} \left(\frac{1 - |z_k|^2}{1 - \overline{z}_k z} \right)^{j+1}.$$

The product $\pi_\beta(z, z_k)$ converges absolutely and uniformly in D if and only if

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\beta+2} < +\infty.$$

The main result of this part of work is the following statement.

Theorem 2.1. *Let*

$$0 < q < 1, \quad \{z_k\}_1^\infty \subset D, \quad 0 < |z_k| \leqslant |z_{k+1}| < 1, \quad k = 1, 2, \ldots.$$

If $\{z_k\}_1^\infty = Z_f$ *for some* $f \in \tilde{\Pi}_q$, *then*

$$\sum_{k=0}^{+\infty} (1 - |z_k|)^{\frac{2}{q}} < +\infty. \tag{2.1}$$

And vice versa, if the points of sequence $\{z_k\}_1^\infty$ *are located in finitely many Stolz angles and satisfy the condition*

$$\int_0^1 (1 - r) n^q(r) dr < +\infty, \tag{2.2}$$

then there exists a function $g \in \tilde{\Pi}_q$ *such that* $Z_g = \{z_k\}_1^\infty$.

In the proof of this results we employ the following statements.

Theorem 2.2 ([5])**.** *If* $f \in \tilde{\Pi}_q$, *then*

$$\ln^+ M(r, f) = o((1 - r)^{-\frac{2}{q}}), \quad r \to 1 - 0. \tag{2.3}$$

The estimate (2.3) *is sharp, that is, for each positive function* $\omega(r) = o(1)$, $r \to 1-0$, *there exists a function* $f \in \tilde{\Pi}_q$ *such that*

$$\ln^+ M(r,f) \neq O(\omega(r)(1-r)^{-\frac{2}{q}}), \quad r \to 1-0.$$

For the set $E \subset \mathbb{C}$ of functions $f, g : E \to \mathbb{R}$ we write $f(\zeta) \lesssim g(\zeta)$, $\zeta \in E$, if there exists a constant $C > 0$ such that $f(\zeta) \leqslant Cg(\zeta)$ for all $\zeta \in E$.

Lemma 2.1 ([11, Lm. 4.7]). *For all* $0 < q \leqslant 1$, $\gamma \geqslant 0$ *the inequality*

$$\left(\int_0^1 (1-r)^\gamma n(r)dr\right)^q \lesssim \int_0^1 (1-t)^{q(\gamma+1)-1} n^q(r)dr.$$

holds.

If else is not said, by C_α, $c(\beta, \ldots)$ we denote various positive inessential constants, depending on $\alpha, \beta, \ldots$

Proof of Theorem 2.1. We use the Jensen inequality

$$\int_0^r \frac{n(t)}{t} dt \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q \left(\ln^+ |f(re^{i\theta})|\right)^{1-q} d\theta.$$

We then apply the estimate (2.3)

$$\int_0^r \frac{n(t)}{t} dt \lesssim \frac{1}{(1-r)^{\frac{2(1-q)}{q}}} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta,$$

and this yields

$$\int_0^1 (1-r)^{\frac{2(1-q)}{q}} \int_0^r \frac{n(t)}{t} dt dr \leqslant C_q. \tag{2.4}$$

Integrating twice by parts, we obtain

$$\int_0^1 (1-r)^{\frac{2}{q}} dn(r) \leqslant C_q,$$

which is equivalent to the convergence of series (2.1).

Now we prove the inverse statement. We note that the case, when the points of sequence $\{z_k\}_1^\infty$ are inside certain Stolz angle does not differ essentially from the case, when they are located on some radius. This is why without loss of generality we suppose that the points of sequence $\{z_k\}_1^\infty$ are located on some radius $[0,1)$, that is, $z_k = r_k$, $k = 1, 2, \ldots$. We are going to show that the Dzhrbashyan product $\pi_\beta(z, r_k)$ with zeros at the points $\{r_k\}_1^\infty$ obeying the condition (2.2) belongs to the class $\tilde{\Pi}_q$.

Since the integral (2.2) converges, the product $\pi_\beta(z, r_k)$ converges as well and it satisfies the estimate

$$\ln^+ |\pi_\beta(z, r_k)| \lesssim \sum_{k=1}^{+\infty} \left(\frac{1-r_k}{|1-r_k z|}\right)^{\beta+2} \tag{2.5}$$

for all $\beta > \frac{2}{q} - 2$, see [11, Lm. 3.7].

This is why

$$I_\pi = \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ |\pi_\beta(re^{i\theta}, r_k)|\right)^q d\theta dr$$
$$\lesssim \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{k=1}^{+\infty} \left(\frac{1-r_k}{|1-r_k z|} \right)^{\beta+2} \right)^q d\theta dr$$
$$= \int_0^1 \int_{-\pi}^{\pi} \left(\int_0^1 \left(\frac{1-t}{|1-tre^{i\theta}|} \right)^{\beta+2} dn(t) \right)^q d\theta dr.$$

Integrating by parts in the internal integral, we obtain

$$I_\pi \lesssim \int_0^1 \int_{-\pi}^{\pi} \left(\int_0^1 n(t) \frac{(1-t)^{\beta+1}}{((1-rt)^2+\theta^2)^{\frac{(\beta+2)}{2}}} dt \right)^q d\theta dr.$$

For further estimate we split the internal integral into parts

$$I_\pi \lesssim \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{k=0}^{+\infty} \int_{r_k}^{r_{k+1}} n(t) \frac{(1-t)^{\beta+1}}{((1-rt)^2+\theta^2)^{\frac{(\beta+2)}{2}}} dt \right)^q d\theta dr, \quad r_k = 1 - \frac{1}{2^k}, \quad k = 0, 1, 2, \dots.$$

We continue estimating

$$I_\pi \lesssim \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{k=0}^{+\infty} \frac{n(r_{k+1})}{((1-rr_{k+1})^2+\theta^2)^{\frac{(\beta+2)}{2}}} \int_{r_k}^{r_{k+1}} (1-t)^{\beta+1} dt \right)^q d\theta dr$$
$$\leqslant \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{k=0}^{+\infty} \frac{n(r_{k+1})}{2^{k(\beta+2)}\left((1-rr_{k+1})^2+\theta^2\right)^{\frac{(\beta+2)}{2}}} \right)^q d\theta dr$$
$$\leqslant \int_0^1 \int_{-\pi}^{\pi} \sum_{k=0}^{+\infty} \frac{n^q(r_{k+1})}{2^{kq(\beta+2)}\left((1-rr_{k+1})^2+\theta^2\right)^{\frac{(\beta+2)q}{2}}} d\theta dr$$
$$= \int_0^1 \sum_{k=0}^{+\infty} \int_{-\pi}^{\pi} \frac{n^q(r_{k+1})}{2^{kq(\beta+2)}\left((1-rr_{k+1})^2+\theta^2\right)^{\frac{(\beta+2)q}{2}}} d\theta dr,$$

and in view of the estimate from the proof of Lemma 2.1 in [11] we have

$$I_\pi \lesssim \int_0^1 \left(\sum_{k=0}^{+\infty} \frac{n^q(r_{k+1})}{2^{kq(\beta+2)}(1-rr_{k+1})^{(\beta+2)q-1}} \right) dr$$
$$\leqslant \sum_{k=0}^{+\infty} \frac{n^q(r_{k+1})(1-r_{k+1})^{(\beta+2)q}}{(1-r_{k+1})^{(\beta+2)q-2}} = \sum_{k=0}^{+\infty} n^q(r_{k+1})(1-r_{k+1})^2,$$

which is equivalent to

$$I_\pi \lesssim \int_0^1 n^q(r)(1-r)dr.$$

By the assumption, the integral in the right hand side of latter inequality converges and this is why $\pi_\beta(z, r_k) \in \tilde{\Pi}_q$ for all $\beta > \frac{2}{q} - 2$.

In conclusion we note that (2.2) implies (2.1). Indeed, it is sufficient to integrate by parts once in (2.4) and apply then Lemma 2.1. The proof is complete. □

Remark 2.1. *We note that a similar statement for the Privalov classes in circle was obtained in work* [9].

3. Interpolation on Carleson sets

Following L. Carleson, we introduce the following definition.

Definition 3.1. *A sequence of complex numbers* $\{z_n\}_1^\infty \subset D$ *obeying the Blaschke condition*

$$\sum_{n=1}^{+\infty}(1-|z_n|) < +\infty, \tag{3.1}$$

is called uniformly separated if there exists a number $0 < \delta < 1$ *such that*

$$\prod_{k\neq n}\left|\frac{z_k - z_n}{1-\overline{z}_k z_n}\right| \geqslant \delta \qquad \forall k \in \mathbb{N}. \tag{3.2}$$

The condition (3.2) *is also called the Carleson condition.*

For a given sequence of distinct points $\{z_n\}_1^\infty \subset D$ and s fixed $0 < q < +\infty$ we denote by $l^q(z_n)$ the space of sequences $\{w_n\}_1^\infty$, for which

$$\sum_{n=1}^{+\infty}(1-|z_n|)^2(\ln^+|w_n|)^q < +\infty.$$

For all $0 < p < +\infty$ by H^p we denote the well–known Hardy class

$$H^p := \left\{ f \in H(D) : \sup_{0<r<1}\int_{-\pi}^{\pi}|f(re^{i\varphi})|^p d\varphi < +\infty \right\},$$

H^∞ is the class of analytic in D functions.

The next theorem is true.

Theorem 3.1. *Let* $0 < q < 1$. *If a sequence* $\{z_n\}_1^\infty \subset D$ *is uniformly separated, that is, it satisfies the condition* (3.2), *then for each sequence* $\{w_n\}_1^\infty \in l^q(z_n)$ *there exists a function* $f \in \tilde{\Pi}_q$, *which solves the interpolation problem* $f(z_n) = w_n$, $n = 1, 2, \ldots$

Proof. We split the sequence $\{w_n\}_1^\infty$ into two subsequences $\{w_{n_{k'}}\}$, $\{w_{n_{k''}}\}$ such that $|w_{n_{k'}}| \leqslant 1$, $|w_{n_{k''}}| > 1$. Since the sequence $\{z_n\}$ satisfies the condition (3.2), then by the Carleson theorem, see [13], we can construct a function $G \in H^\infty$ such that $G(z_{n_{k'}}) = w_{n_{k'}}$, $G(z_{n_{k''}}) = 1$. Let us prove that there exists a function $F \in H^q$ such that $F(z_{n_{k'}}) = 0$, $F(z_{n_{k''}}) = \ln w_{n_{k''}}$, where the principal branch of logarithm is chosen. We have

$$|\ln w_{n_{k''}}| \leqslant \ln|w_{n_{k''}}| + |\arg w_{n_{k''}}| \leqslant \ln^+|w_{n_{k''}}| + 2\pi.$$

In view of the condition $\{w_n\}_1^\infty \subset l^q(z_n)$ of theorem we obtain

$$\sum_{n=1}^{+\infty}(1-|z_n|)^2|\ln w_n|^q < +\infty.$$

The well–known Shapiro — Shields theorem on interpolation in Hardy classes H^q, $0<q<1$, see [20], implies the existence of a function $F\in H^q$ with the mentioned properties.

We consider an analytic in D function $f=G\cdot\exp(F)$. We are going to show that it solves the interpolation problem in the area Privalov class.

Let us estimate the integral

$$I(q)=\int_0^1\int_{-\pi}^{\pi}\left(\ln^+|f(re^{i\theta})|\right)^q\,d\theta dr.$$

We have

$$I(q)\leqslant\int_0^1\int_{-\pi}^{\pi}(c_G+|F|)^q\,d\theta dr\leqslant c_q.$$

This is why $f\in\tilde{\Pi}_q$. Then

$$\begin{aligned}f(z_{n_{k'}})&=G(z_{n_{k'}})\cdot\exp\left(F(z_{n_{k'}})\right)=w_{n_{k'}}\cdot\exp 0,\\ f(z_{n_{k''}})&=G(z_{n_{k''}})\cdot\exp\left(F(z_{n_{k''}})\right)=1\cdot\exp\left(\ln w_{n_{k''}}\right)=w_{n_{k''}}.\end{aligned}$$

This is why $f(z_n)=w_n$ for all $n=1,2,\ldots$ The proof is complete. □

Theorem 3.2. *Let $0<q<1$. There exists a uniformly separated sequence of interpolation nodes $\{z_n\}_1^\infty\subset D$ and sequence $\{w_n\}_1^\infty$, which for each $\varepsilon\in(0,q)$ obey the condition*

$$\sum_{n=1}^{+\infty}(1-|z_n|)^2(\ln^+|w_n|)^{q-\varepsilon}<+\infty, \tag{3.3}$$

and for which in the class $\tilde{\Pi}_q$ there is no function solving the interpolation problem $f(z_n)=w_n$, $n=1,2,\ldots$

Proof. As $\{z_n\}_1^\infty$ we take the sequence of real numbers $z_n=1-\beta^n$, $0<\beta<1$, $n=1,2,\ldots$. This sequence obviously satisfies the Blaschke condition and is uniformly separated, that is, it satisfies the condition (3.2).

We also consider the sequence

$$w_n=\exp\left(\frac{n}{\beta^{\frac{2n}{q}}}\right),\qquad n=1,2,\ldots,$$

obeying the condition (3.3). It is obvious that

$$(1-|z_n|)^2(\ln^+|w_n|)^q\to+\infty,\qquad n\to+\infty. \tag{3.4}$$

But it follows from the estimate (2.3) that

$$(1-|z|)^2(\ln^+|f(z)|)^q=o(1),\qquad\forall f\in\tilde{\Pi}_q. \tag{3.5}$$

On the base of (3.4), (2.3) we conclude that in the class $\tilde{\Pi}_q$ there is no function solving the interpolation problem $f(z_n)=w_n$, $n=1,2,\ldots$, under the above choice of sequences $\{z_n\}_1^\infty$, $\{w_n\}_1^\infty$. The proof is complete. □

Remark 3.1. *In the proof of this theorem we have employed the idea by Yanagihara, see* [21, Thm. 4].

4. Free interpolation in area Privalov classes

For a fixed sequence of distinct points $\{z_n\}_1^\infty \subset D$ and fixed $0 < q < +\infty$ we denote by $\tilde{l}^q(z_n)$ the space of sequences $\{w_n\}_1^\infty$, for which

$$\ln^+|w_n| = o\left((1-|z_n|)^{-\frac{2}{q}}\right), \qquad n \to +\infty,$$

By Theorem 2.2, the operator $R(f) = (f(z_1), \ldots, f(z_n), \ldots)$ naturally maps the space $\tilde{\Pi}_q$ into the space $\tilde{l}^q(z_n)$. We are interesting in conditions, under which the mentioned mapping is surjective.

The next statement is true.

Theorem 4.1. *Let $0 < q < 1$, $\{z_k\}_1^\infty$ be an arbitrary sequence of distinct complex numbers in D located in finitely many Stolz angles, that is,*

$$\{z_k\}_1^\infty \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s)$$

for some $0 < \delta < q$.

If $\{z_k\}_1^\infty$ is an interpolation sequence in $\tilde{\Pi}_q$, then the series (2.1) converges and there exists an infinitesimal sequence $\{\varepsilon(n)\}_1^\infty$ such that

$$|\pi'_\beta(z_n, z_k)| \geqslant \exp\frac{-\varepsilon(n)}{(1-|z_n|)^{\frac{2}{q}}}, \tag{4.1}$$

for all $\beta > \frac{2}{q} - 2$.

And vice versa, if the integral (2.2) converges and the condition (4.1) is satisfied, then the sequence $\{z_k\}_1^\infty$ is interpolation in the class $\tilde{\Pi}_q$.

To prove the main result of this part of work, in the spaces $\tilde{\Pi}_q$, $0 < q < 1$, and $\tilde{l}^q(z_n)$ we introduce the metrics by the rules

$$\rho_{\tilde{\Pi}_q}(f,g) = \int_0^1 \int_{-\pi}^{\pi} \ln^q\left(1 + |f(re^{i\theta}) - g(re^{i\theta})|\right) d\theta dr,$$

for all $f, g \in \tilde{\Pi}_q$, and

$$\rho_{\tilde{l}^q}(a,b) = \sup_{n\geqslant 1}\left\{(1-|z_n|)^{\frac{2}{q}} \ln(1+|a_n - b_n|)\right\},$$

for all $a = \{a_n\}_1^\infty$, $b = \{b_n\}_1^\infty \in \tilde{l}^q(z_n)$.

It is easy to verify that the mentioned spaces with the introduced metrics are complete metric spaces, and moreover, the space $\tilde{\Pi}_q$ is F–space, see [5].

The next lemma holds.

Lemma 4.1. *If the operator $R(f) = (f(z_1), \ldots, f(z_n), \ldots)$ maps the space $\tilde{\Pi}_q$ onto the space $\tilde{l}^q$, then there exists a sequence of functions $\{g_n(z)\}_1^\infty \in \tilde{\Pi}_q$ such that*

$$\sup_{n\geqslant 1} \rho_{\tilde{\Pi}_q}(g_n, 0) \leqslant C, \qquad C > 0$$

and

$$g_n(z_k) = w_k^{(n)}, \qquad \text{where} \qquad w_k^{(n)} = \begin{cases} 0, & \text{for} \quad k \neq n, \\ \exp\dfrac{\delta(k)}{(1-|z_k|)^{\frac{2}{q}}} & \text{for} \quad k = n, \end{cases}$$

for all $k, n = 1, 2, \ldots$, $\delta(k) = o(1)$, $k \to +\infty$.

The proof of this lemma is similar to that of Lemma 2.1 in [18].

For all $0 < \alpha < \frac{2}{q}$ we consider the function

$$h(z) = h_k(z) = \exp \sum_{s=1}^{n} \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-z\rho_m e^{-i\theta_s})^{\alpha+\frac{2}{q}}}, \qquad z \in D, \tag{4.2}$$

where $\{u_k\}_1^\infty$ is the infinitesimal sequence associated with the interpolation nodes $\{z_k\}_1^\infty$, $0 < u_k < 1$, $k = 1, 2, \ldots$, $\{1-\rho_m\}_1^\infty$ is a positive infinitesimal sequence such that for each fixed k the series

$$\sum_{m=1}^{+\infty} \frac{u_k^m}{(1-\rho_m)^{\frac{2}{q}}} < +\infty \tag{4.3}$$

converges.

Lemma 4.2. *If the points of sequence $\{z_k\}_1^\infty$ are located in finitely many Stolz angles, that is,*

$$\{z_k\}_1^\infty \subset \bigcup_{s=1}^{n} \Gamma_\delta(\theta_s),$$

then the function $h(z)$ defined by the identity (4.2) satisfies the estimate

$$|h(z_k)| \geqslant \exp \frac{\mu_0(k)}{(1-|z_k|)^{\frac{2}{q}}}, \tag{4.4}$$

where $\mu_0(k)$ is some positive infinitesimal sequence.

Proof. Without loss of generality we can suppose that the interpolation nodes are contained in the Stolz angle $\Gamma_\delta(\theta)$. It is clear that the series

$$\sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-z\rho_m e^{-i\theta})^{\alpha+\frac{2}{q}}}$$

converges for each fixed $k \in \mathbb{N}$ and all $z \in D$ in view of the condition (4.3) and hence $h \in H(D)$.

Let us show that $h \in \tilde{\Pi}_q$. For the sake of brevity we denote $\alpha' = \alpha + \frac{2}{q}$, then

$$h(z) = h_k(z) = \exp \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-z\rho_m e^{-i\theta})^{\alpha'}}, \qquad z \in D.$$

We fix $k \in \mathbb{N}$ and proceed to estimating the function $h_k(z)$

$$\begin{aligned}\int_0^1 \int_{-\pi}^{\pi} (\ln^+ |h_k(re^{i\varphi})|)^q d\varphi dr &= \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ \left| \exp \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-re^{i\varphi}\rho_m e^{-i\theta})^{\alpha'}} \right| \right)^q d\varphi dr \\ &\leqslant \int_0^1 \int_{-\pi}^{\pi} \left(\sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{|1-r\rho_m e^{i(\varphi-\theta)}|^{\alpha'}} \right)^q d\varphi dr.\end{aligned}$$

We continue estimating

$$\int_0^1\int_{-\pi}^{\pi}(\ln^+|h(re^{i\varphi})|)^q d\varphi dr \leqslant \int_0^1\sum_{m=1}^{+\infty}\int_{-\pi}^{\pi}\left(u_k^m\frac{(1-{\rho_m}^2)^\alpha}{|1-r\rho_m e^{i(\varphi-\theta)}|^{\alpha'}}\right)^q d\varphi dr$$
$$=\int_0^1\sum_{m=1}^{+\infty}\int_{-\pi}^{\pi}u_k^{mq}\frac{(1-{\rho_m}^2)^{\alpha q}}{|1-r\rho_m e^{i(\varphi-\theta)}|^{\alpha' q}}d\varphi dr$$
$$\leqslant\int_0^1\sum_{m=1}^{+\infty}\frac{u_k^{mq}(1-{\rho_m}^2)^{\alpha q}}{(1-r\rho_m)^{(\alpha' q-1)}}dr$$
$$=\sum_{m=1}^{+\infty}\frac{u_k^{mq}(1-{\rho_m}^2)^{\alpha q}}{(1-r\rho_m)^{(\alpha' q-2)}}$$
$$=\sum_{m=1}^{+\infty}\frac{u_k^{mq}(1-{\rho_m}^2)^{\alpha q}}{(1-r\rho_m)^{\alpha q}}$$
$$\leqslant 2^{\alpha q}\sum_{m=1}^{+\infty}u_k^{mq}=2^{\alpha q}\frac{u_k^q}{1-u_k^q}<+\infty.$$

Thus, we conclude that $h_k\in\tilde{\Pi}_q$ for each $k=1,2,\dots$

Now we estimate $|h(z_k)|$ from below in the angle $\Gamma_\delta(\theta)$

$$|h(z_k)|=\exp\sum_{m=1}^{+\infty}u_k^m\operatorname{Re}\frac{(1-{\rho_m}^2)^\alpha}{(1-z_k\rho_m e^{-i\theta})^{\alpha'}}=\exp\sum_{m=1}^{+\infty}u_k^m(1-{\rho_m}^2)^\alpha\frac{\operatorname{Re}(1-\overline{z_k}\rho_m e^{i\theta})^{\alpha'}}{|1-z_k\rho_m e^{-i\theta}|^{2\alpha'}}.$$

We have

$$\operatorname{Re}(1-\overline{z_k}\rho_m e^{i\theta})^{\alpha'}=\operatorname{Re}\left(1-r_k\rho_m e^{-i(\varphi_k-\theta)}\right)^{\alpha'}$$
$$=\operatorname{Re}\left(1-\rho_m r_k+\rho_m r_k(1-e^{-i(\varphi_k-\theta)})\right)^{\alpha'}$$
$$=\operatorname{Re}\left(1-\rho_m r_k+\rho_m r_k(1-e^{-i(\varphi_k-\theta)})\right)^{\alpha'}$$
$$=(\rho_m r_k\rho)^{\alpha'}\operatorname{Re}\left(\frac{1-\rho_m r_k}{\rho_m r_k\rho}+e^{-i\varphi}\right)^{\alpha'},$$

where $z_k=r_ke^{i\varphi_k}$, $(1-e^{-i(\varphi_k-\theta)})=\rho e^{-i\varphi}$, $|\varphi|<\frac{\pi}{2\alpha'}$. Using Lemma 1.3 from [7], we obtain the estimate

$$\operatorname{Re}(1-\overline{z_k}\rho_m e^{i\theta})^{\alpha'}\gtrsim(\rho_m r_k\rho)^{\alpha'},\qquad c_1>0.$$

On the other hand,

$$|1-e^{-i(\varphi_k-\theta)}|^{\alpha'}=2^{\alpha'}\sin^{\alpha'}\left(\frac{\theta-\varphi_k}{2}\right),$$

and this yields

$$\operatorname{Re}\frac{1}{(1-\alpha_k\rho_m e^{-i\theta})^{\alpha'}}\gtrsim\frac{(\rho_m r_k)^{\alpha'}2^{\alpha'}\sin^{\alpha'}\left(\frac{\theta-\varphi_k}{2}\right)}{\left((1-\rho_m r_k)^2+4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)\rho_m r_k\right)^{\alpha'}}$$
$$\geqslant\frac{(\rho_m r_k)^{\alpha'}2^{\alpha'}\sin^{\alpha'}\left(\frac{\theta-\varphi_k}{2}\right)}{\left((1-\rho_m r_k)^2+4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)\right)^{\alpha'}}\gtrsim\frac{2^{\alpha'}\sin^{\alpha'}\left(\frac{\theta-\varphi_k}{2}\right)}{(1-\rho_m r_k)^{2\alpha'}\left(1+\frac{4\sin^2\left(\frac{\theta-\varphi_k}{2}\right)}{(1-\rho_m r_k)^2}\right)^{\alpha'}}.$$

Since $\{z_k\}_1^\infty \subset \Gamma_\delta(\theta)$, we have

$$\sup_k \frac{\left|\sin\left(\frac{\theta-\varphi_k}{2}\right)\right|}{(1-r_k)} \leqslant C.$$

We obtain

$$\mathrm{Re}\,\frac{1}{(1-z_k\rho_m e^{-i\theta})^{\alpha'}} \geqslant \frac{c(\alpha')}{(1-\rho_m r_k)^{\alpha'}}.$$

Thus, for $|h(z_k)|$ in the angle $\Gamma_\delta(\theta)$ the estimate

$$|h(z_k)| \geqslant \exp c(\alpha') \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}}, \qquad k=1,2,\ldots,$$

holds. We observe that the series in the right hand of obtained inequality converges. Indeed,

$$u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}} \leqslant u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-\rho_m)^{\alpha'}} \leqslant u_k^m \frac{2^\alpha}{(1-\rho_m)^{\frac{2}{q}}}.$$

It remains to employ the condition (4.3).

We continue estimating $|h(z_k)|$ from below. In order to do this, we split the sum

$$\begin{aligned} S &= \sum_{m=1}^{+\infty} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}} \\ &= \sum_{(1-\rho_m)=(1-r_k)} (\ldots) + \sum_{(1-\rho_m)>(1-r_k)} (\ldots) + \sum_{(1-\rho_m)<(1-r_k)} (\ldots) \\ &= S_0(k) + S_1(k) + S_2(k). \end{aligned}$$

We are going to estimate separately each part of the sum. Let $m_0 = \inf\limits_{\rho_m = r_k} m$, then

$$S_0(k) = \sum_{(1-\rho_m)=(1-r_k)} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}} = \sum_{\rho_m=r_k} u_k^m \frac{1}{(1-r_k^2)^{\frac{2}{q}}} \geqslant \frac{u_k^{m_0}}{(1-r_k^2)^{\frac{2}{q}}}.$$

We estimate $S_1(k)$

$$\begin{aligned} S_1(k) &= \sum_{(1-\rho_m)>(1-r_k)} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}} \geqslant \sum_{\rho_m<r_k} u_k^m \frac{1}{(1-\rho_m^2)^{\alpha'-\alpha}} \\ &\geqslant \left(\frac{1}{2}\right)^{\frac{2}{q}} \sum_{\rho_m<r_k} u_k^m \frac{1}{(1-\rho_m)^{\frac{2}{q}}} \geqslant \frac{u_k^{m_1}}{(1-\rho_{m_1})^{\frac{2}{q}}}, \end{aligned}$$

where m_1 is the index, for which $\rho_m < r_k$.

Now we find the lower bound for $S_2(k)$

$$\begin{aligned} S_2(k) &= \sum_{(1-\rho_m)<(1-r_k)} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\alpha'}} = \sum_{\rho_m>r_k} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^{\frac{2}{q}}(1-r_k\rho_m)^\alpha} \\ &\geqslant \frac{1}{(1-r_k^2)^{\frac{2}{q}}} \sum_{\rho_m>r_k} u_k^m \frac{(1-{\rho_m}^2)^\alpha}{(1-r_k\rho_m)^\alpha} \geqslant \frac{1}{(1-r_k^2)^{\frac{2}{q}}} \frac{u_k^{m_2}(1-\rho_{m_2}^2)^\alpha}{(1-r_k^2)^\alpha}, \end{aligned}$$

here $m_2 = \inf\limits_{\rho_m>r_k} m$.

The estimates for S_0, S_1, S_2 imply

$$S(k) \geqslant \frac{u_k^{m_2}(1-\rho_{m_2}^2)^\alpha}{(1-r_k^2)^{\frac{2}{q}+\alpha}} + \frac{u_k^{m_1}}{(1-\rho_{m_1})^{\frac{2}{q}}} + \frac{u_k^{m_0}}{(1-r_k^2)^{\frac{2}{q}}},$$

and hence,

$$S(k) \geqslant \frac{u_k^{m_0}}{(1-r_k^2)^{\frac{2}{q}}}$$

for all $k = 1, 2, \ldots$.

We thus obtain

$$|h(z_k)| \geqslant \exp \frac{\mu_0(k)}{(1-r_k)^{\frac{2}{q}}}, \tag{4.5}$$

where

$$0 < \mu_0(k) \leqslant \frac{u_k^{m_0}}{2^{\frac{2}{q}}} = o(1), \qquad k \to +\infty.$$

The proof is complete. □

Proof of Theorem 4.1. We first prove the necessity. Suppose that $\{z_k\}_1^\infty \subset D$ is an interpolation sequence in the class $\tilde{\Pi}_q$, that is, for each sequence $\{w_k\}_1^\infty \in \tilde{l}_q$ there exists a function $f \in \tilde{\Pi}_q$ such that $f(z_k) = w_k$, $k = 1, 2, \ldots$

We consider the sequence $\{w_k\}_1^{+\infty}$ defined as $w_1 = 1$, $w_2 = w_3 = \ldots = 0$. It is clear that $\{w_k\}_1^{+\infty} \in \tilde{l}_q$. Since $\{z_k\}_2^\infty$ is the zero sequence of the function $f \in \tilde{\Pi}_q$, Theorem 2.1 implies the estimate (2.1).

In order to establish (4.1), we fix an index $n \in \mathbb{N}$ and construct a sequence $\{w_k^{(n)}\}_1^\infty$ as follows:

$$w_k^{(n)} = 0, \qquad k \neq n, \qquad w_n^{(n)} = \exp \frac{\delta(n)}{(1-|z_n|)^{\frac{2}{q}}}, \qquad \text{where} \qquad \delta(n) \to 0, \qquad n \to +\infty.$$

According to Lemma 4.1, there exists a function $g_n \in \tilde{\Pi}_q$ such that

$$\rho_{\tilde{\Pi}_q}(g_n, 0) \leqslant C \quad \text{and} \quad g_n(z_k) = w_k^{(n)} \quad \text{for all} \quad k = 1, 2, \ldots,$$

where the constant $C > 0$ is independent of the index n. In particular, $g_n(z_n) = w_n^{(n)}$.

We consider the function

$$G_n = \frac{g_n}{\pi_{\beta,n}},$$

where $\pi_{\beta,n} = \pi_{\beta,n}(z, z_k)$ is the Dzhrbashyan product constructed by the zeros of the function g_n located in Stolz angles without the nth factor, β is an arbitrary number such that $\beta > \frac{2}{q} - 2$. It is obvious that under such choice of the parameter β the product $\pi_{\beta,n}$ converges and belongs to the class $\tilde{\Pi}_q$.

Now we are going to prove that $G_n \in \tilde{\Pi}_q$. We multiply both sides of latter identity by $\pi_{\beta+1,n}$. It is obvious that $G_n \in \Pi_q$ if and only if $\frac{\pi_{\beta+1,n}}{\pi_{\beta,n}} \in \tilde{\Pi}_q$. As it was established in the proof of Theorem 3.8 in [11],

$$\frac{\pi_{\beta+1}(z, z_k)}{\pi_\beta(z, z_k)} = \exp\left\{\sum_{k=1}^{+\infty} \left(\frac{1-|z_k|^2}{1-\bar{z}_k z}\right)^{\beta+2}\right\}.$$

This representation and the proof of sufficiency in Theorem 2.1 imply that $\frac{\pi_{\beta+1,n}}{\pi_{\beta,n}} \in \tilde{\Pi}_q$. Hence, $G_n = \frac{g_n}{\pi_{\beta,n}} \in \tilde{\Pi}_q$.

By Theorem 2.2

$$\ln^+ M(r, G_n) = o((1-r)^{-\frac{2}{q}}), \qquad r \to 1-0, \tag{4.6}$$

and therefore,

$$|G_n(z_n)| = \frac{|g_n(z_n)|}{|\pi_{\beta,n}(z_n, z_k)|} = \frac{|w_n^{(n)}|}{|\pi_{\beta,n}(z_n, z_k)|} = \exp \frac{\delta(n)}{(1-|z_n|)^{\frac{2}{q}}} \frac{1}{|\pi_{\beta,n}(z_n, z_k)|} \leqslant C \exp \frac{\varepsilon(n)}{(1-|z_n|)^{\frac{2}{q}}},$$

where $\varepsilon(n) = o(1)$, $n \to +\infty$, $\varepsilon(n) > 0$, $n = 1, 2, \ldots$ and C is independent of n by Lemma 4.1. Letting $\delta(n) = \frac{\varepsilon(n)}{2}$, we obtain

$$|\pi_{\beta,n}(z_n, z_k)| \geqslant \exp \frac{-\varepsilon(n)}{2(1 - |z_k|)^{\frac{2}{q}}},$$

for all $\beta \geqslant \frac{2}{q} - 2$, and this implies the condition (4.1), see the proof of Theorem 2.1 in [4]. This completes the proof of first part of Theorem 4.1.

We proceed to proving the inverse statement. Suppose that $\{z_k\}_1^{+\infty}$ is an arbitrary sequence of distinct points in D contained in finitely many Stolz angles, and the conditions (2.2), (4.1) are satisfied. We are going to show that there exists a function $f \in \tilde{\Pi}_q$ such that $f(z_k) = w_k$, $k = 1, 2, \ldots$, where $\{w_k\}_1^{+\infty} \in \tilde{l}_q$, that is,

$$w_k = \exp \frac{\delta(k)}{(1 - |z_k|)^{\frac{2}{q}}}, \tag{4.7}$$

$\delta(k) \to 0$, $k \to +\infty$.

We construct the function $f(z)$ as

$$f(z) = \sum_{k=1}^{+\infty} w_k \frac{\pi_\beta(z, z_j)}{(z - z_k)} \frac{1}{\pi'_\beta(z_k, z_j)} \left(\frac{1 - |z_k|}{1 - \overline{z_k} z}\right)^{\beta+2} \frac{h(z)}{h(z_k)}, \tag{4.8}$$

where $\pi_\beta(z, z_k)$ is the Dzhrbashyan product with zeros at the interpolation nodes

$$\{z_k\}_1^{+\infty}, \qquad \beta \geqslant \frac{2}{q} - 2,$$

$h(z)$ is defined by the identity (4.2), the sequence $\{u_k\}$ is chosen so that

$$\varepsilon(k) + \delta(k) - \mu_0(k) \leqslant 0,$$

where $\varepsilon(k)$, $\delta(k)$, $\mu_0(k)$ are the infinitesimal sequence from the estimates (4.1), (4.7), (4.5), respectively.

It is obvious that $f(z_n) = w_n$, $n = 1, 2, \ldots$

The function $f(z)$ is analytic in the circle D due to the convergence of series (4.8). Indeed, the convergence of (4.8) is equivalent to the convergence of the series

$$\sum_{k=1}^{+\infty} (1 - |z_k|)^{\beta+2} < +\infty. \tag{4.9}$$

But this series converges for all $\beta + 2 > \frac{2}{q}$ due to the condition (2.2).

We show that $f \in \tilde{\Pi}_q$

$$\begin{aligned}\int_0^1 \int_{-\pi}^{\pi} (\ln^+ |f(re^{i\varphi})|)^q d\varphi dr \leqslant& \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ \left|\pi_\beta(re^{i\varphi}, z_j)\right|\right)^q d\varphi dr + \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ \left|h(re^{i\varphi})\right|\right)^q d\varphi dr \\ &+ \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ \sum_{k=1}^{+\infty} \exp \frac{\varepsilon(k) + \delta(k) - \mu_0(k)}{(1 - |z_k|)^{\frac{2}{q}}} \cdot \frac{(1 - |z_k|)^{\beta+2}}{|1 - \overline{z_k} re^{i\varphi}|^{\beta+3}}\right)^q d\varphi dr \\ =& I_1 + I_2 + I_3.\end{aligned}$$

The convergence of the integral I_1 is implied by the proof of sufficiency in Theorem 2.1, while the convergence of I_2 was proved in Lemma 4.2.

Let us prove the convergence I_3 taking into consideration the above remark

$$\varepsilon(k) + \delta(k) - \mu_0(k) \leqslant 0.$$

We have

$$I_3 \leqslant \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ \frac{\sum\limits_{k=1}^{+\infty} (1-|z_k|)^{\beta+2}}{(1-r)^{\beta+3}} \right)^q d\varphi dr \leqslant \int_0^1 \left(\ln^+ \frac{c(\beta)}{(1-r)^{\beta+3}} \right)^q dr < +\infty.$$

While estimating I_3, we have taken into consideration the convergence of series (4.9).

Thus, $f \in \tilde{\Pi}_q$ indeed solves the interpolation problem in the class $\tilde{\Pi}_q$ for all $0 < q < 1$.

The proof is complete. □

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Evgenia Gennadievna Rodikova,
Bryansk State University,
Bezhitskaya str. 14,
241050, Bryansk, Russia
E-mail: evheny@yandex.ru