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UNIFORM ASYMPTOTICS FOR EIGENVALUES OF MODEL SCHRÖDINGER OPERATOR WITH SMALL TRANSLATION

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Abstract. We consider a model Schrödinger operator with a constant coefficient on the unit segment and the Dirichlet and Neumann condition on opposite ends with a small translation in the free term. The value of the translation is small parameter, which can be both positive and negative. The main result is the spectral asymptotics for the eigenvalues and eigenfunctions with an estimate for the error term, which is uniform in the small parameter. For finitely many first eigenvalues and associated eigenfunctions we provide asymptotics in the small parameter. We prove that each eigenvalue is simple, and the system of eigenfunctions forms a basis in the space $L_2(0,1)$.

Key words: Schrödinger operator on a segment, small translation, uniform spectral asymptotics.

Mathematics Subject Classification: 34B24, 34L20, 34E18

1. Introduction

Differential–difference equations are one of important examples of nonlocal operators, and nowadays they are actively studied. An interest to studying these equations is due to the fact that the corresponding boundary and initial boundary equations possess non–standard properties, which are absent in classical formulations. The qualitative theory of elliptic differential–difference and functional–difference equations is actively developed, see [1]–[5] and the references therein, but it is still far from being completed. We also mention works on qualitative theory of evolutionary differential–difference equations, see [6], [7]. At the same time we know just a single work [3], in which spectral properties of corresponding operators were studied.

The study of asymptotics of eigenvalues in an index for the Sturm–Liouville operator was made in a huge number of books and papers. Not pretending for the completeness, we mention only two classical monographs [8], [9], see also the references in these books. The known classical works provide spectral asymptotics for many classes of elliptic operators. At the same time, if one considers families of operators depending on a parameter, then the question on spectral asymptotics becomes open. The reason is that in this case the entire asymptotics, including the error term, become dependent on the parameter and such dependence can destroy the error term. The classical perturbation turns out to be inappropriate here since the statements on the convergence of the resolvent allow one to conclude only on the behavior of the spectra in compact sets, that is, for finitely many eigenvalues and for the entire set.

Spectral asymptotics uniform in a small parameter were earlier known only for a series of particular models. In [10], [11], there was studied a behavior of the eigenvalues for the spectral problem on an interval (a,b)

$$i\varepsilon u'' + qu = \lambda u, \qquad u(a) = u(b) = 0,$$

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where q is a given function and $\varepsilon > 0$ is a small parameter. Two types of functions were considered in the mentioned papers, $q(x) = x$ and $q(x) = (x-a)^2$. This problem is related with the Orr–Sommerfeld equation, which is well–known in the hydrodynamics, while the function q serves as the velocity of the stationary profile of the liquid in the channel $a \leqslant x \leqslant b$. It was shown in work [10] that the eigenvalues localize along certain set, the shape of which resembles a tie. Similar results were also obtained in [12]–[15]. A similar set appeared also in the second case for $q(x) = (x-a)^2$. For such problems, an asymptotics was established, in which the error term is uniform both in the small parameter ε and the index. Later in papers [16]–[18] these results were extended for other classes of the function q. We also note that the questions on norm resolvent convergence of elliptic operators of order $2m$ with small variable translations in lower order terms were considered in [19].

In the present work we study uniform spectral asymptotics for a model Schrödinger operator with a constant complex potential perturbed by an operator of small translation. The quantity of the translation is a small parameter, which can be both positive and negative. The domain of this operator is defined by the Dirichlet and Neumann condition on the opposite ends. Our main result is a uniform in the small parameter spectral asymptotics for the eigenvalues. We calculate fourth leadings terms in the asymptotics as well as an error term in form $O(n^{-3})$. The structure of the obtained uniform spectral asymptotics demonstrates a non–trivial high–frequency phenomenon in the behavior of large eigenvalues. In the work we also find the asymptotics for the associated eigenfunctions uniform in the small parameter. For the first eigenvalues and associated eigenfunctions we write the asymptotics in the small parameter. We show that all eigenvalues are simple and the associated eigenfunctions form a basis in $L_2(0,1)$.

2. Formulation of problem and main results

On the interval $(0,1)$ we consider an operator with the differential expression

$$\hat{\mathcal{H}} := -\frac{d^2}{dx^2}$$

and the boundary conditions

$$u(0) = 0, \qquad u'(1) = 0. \tag{2.1}$$

We denote this operator by $\mathcal{H}$ and treat it as an unbounded in the space $L_2(0,1)$ on the domain

$$\mathfrak{D}(\mathcal{H}) := \big\{u \in W_2^2(0,1) : \text{ the boundary conditions (2.1) are satisfied}\big\}. \tag{2.2}$$

The operator $\mathcal{H}$ is obviously m–sectorial and has a compact resolvent. Its spectrum is pure discrete and consists of the eigenvalues

$$\lambda_n = \kappa_n^2, \qquad \kappa_n := \frac{\pi}{2} + \pi n, \qquad n \in \mathbb{Z}_+. \tag{2.3}$$

The main object of the study in the present work is the perturbation of the operator $\mathcal{H}$ by an operator of small translation. Namely, let $\mathcal{L}$ be the operator of continuation by zero outside the interval $(0,1)$, which is regarded as acting from $L_2(0,1)$ into $L_2(\mathbb{R})$, and $\mathcal{R}$ the operator of restriction to $(0,1)$, which acts from $L_2(\mathbb{R})$ into $L_2(0,1)$. These operators are introduced by the formulas

$$\mathcal{L}y = \begin{cases} y & \text{in} \quad (0,1), \\ 0 & \text{outside} \quad (0,1), \end{cases} \qquad \mathcal{R}y = y \quad \text{on} \quad (0,1).$$

In the space $L_2(\mathbb{R})$ we define a pair of operators of small translations $\mathcal{T}^\varepsilon$ acting by the rule $(\mathcal{T}^\varepsilon y)(x) = y(x+\varepsilon)$, $x \in (0,1)$, where ε is a small parameter, which can be both positive and negative.

We define the perturbing operator $\mathcal{P}^\varepsilon$ in the space $L_2(0,1)$ as

$$\mathcal{P}^\varepsilon y = a\big(\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}y - y\big), \tag{2.4}$$

where $a\in\mathbb{C}$ is some constant. The action of such operator is described by the identity

$$(\mathcal{P}^\varepsilon y)(x)=a\big(y(x+\varepsilon)-y(x)\big). \tag{2.5}$$

Here the function y is continued by zero outside the segment $[0,1]$, and the result of the action is restricted to this section. We also note that, as $\varepsilon=0$, the operator $\mathcal{P}^\varepsilon$ becomes the zero operator.

The perturbed operator is defined by the identity $\mathcal{H}^\varepsilon=\mathcal{H}+\mathcal{P}^\varepsilon$ in the space $L_2(0,1)$ on the domain $\mathfrak{D}(\mathcal{H}^\varepsilon):=\mathfrak{D}(\mathcal{H})$. Under the assumption of the identity (2.5), the action of the perturbed operator is described by the formula

$$(\mathcal{H}^\varepsilon y)(x)=-\frac{d^2y}{dx^2}(x)+a\big(y(x+\varepsilon)-y(x)\big),\qquad x\in(0,1).$$

Our first result is auxiliary, and it describes basic properties of the operator $\mathcal{H}^\varepsilon$.

Theorem 2.1. *The operator $\mathcal{H}^\varepsilon$ is m–sectorial, and the associated closed sectorial form in the space $L_2(0,1)$ is given by the identity*

$$\mathfrak{h}^\varepsilon(u,v)=(u',v')_{L_2(0,1)}+a(\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}u-u,v)_{L_2(0,1)} \tag{2.6}$$

on the domain

$$\mathfrak{D}(\mathfrak{h}^\varepsilon):=\big\{u\in W_2^1(0,1):\,u(0)=0\big\}.$$

There exists a number λ_0 independent of ε such that the half–plane $\operatorname{Re}\lambda\leqslant\lambda_0$ *is in the resolvent set of the operator $\mathcal{H}^\varepsilon$ for each $\varepsilon\neq0$. The operator $\mathcal{H}^\varepsilon$ has a compact resolvent, and its spectrum consists of countably many eigenvalues with the only accumulation point at infinity. As $\varepsilon\to0$, the operator $\mathcal{H}^\varepsilon$ converges to $\mathcal{H}$ in the sense of the norm resolvent convergence, namely, the estimate*

$$\|\mathcal{H}^\varepsilon-\mathcal{H}\|_{L_2(0,1)\to W_2^1(0,1)}\leqslant C\varepsilon^{\frac12} \tag{2.7}$$

holds, where $\|\cdot\|_{L_2(0,1)\to W_2^1(0,1)}$ is a norm of bounded operators acting from $L_2(0,1)$ into $W_2^1(0,1)$, and C is some constant independent of ε. The eigenvalues of the operator $\mathcal{H}^\varepsilon$ converge to the eigenvalues of the operator $\mathcal{H}$.

In what follows we arrange the eigenvalues of the operator $\mathcal{H}^\varepsilon$ in the ascending order of their absolute values counting the multiplicities, and they are denoted by λ_n^ε, $n\geqslant0$.

Our first main result describes the asymptotics of the eigenvalues λ_n^ε as $n\to+\infty$ uniformly in the small parameter ε.

Theorem 2.2. *The eigenavlues of the operator $\mathcal{H}^\varepsilon$ are simple. As $n\to+\infty$, the asymptotics of the eigenvalues λ_n^ε read as*

$$\lambda_n^\varepsilon=\kappa_n^2+\Lambda_{n,0}^\varepsilon+\frac{\Lambda_{n,1}^\varepsilon}{\pi n}+\frac{\Lambda_{n,2}^\varepsilon}{\pi^2n^2}+O(n^{-3}), \tag{2.8}$$

where the estimate for the error term is uniform in ε and

$$\begin{aligned}
\Lambda_{n,0}^\varepsilon:=&a(1-|\varepsilon|)\cos\kappa_n\varepsilon-a,\qquad \Lambda_{n,1}^\varepsilon:=\frac{a^2\varepsilon(1-\varepsilon^2)}{4}\sin2\kappa_n\varepsilon,\\
\Lambda_{n,2}^\varepsilon:=&-\frac{3a^3}{32}\varepsilon^2(1-|\varepsilon|)(1+\varepsilon)^2\cos3\kappa_n\varepsilon+\frac{a^2}{8}(1-2\varepsilon-\varepsilon^2)\cos2\kappa_n\varepsilon\\
&+\frac{a^3}{32}\varepsilon^2(3+|\varepsilon|)(1-\varepsilon)^2\cos\kappa_n\varepsilon-\frac{\pi a^2}{16}\varepsilon(1-\varepsilon^2)\sin2\kappa_n\varepsilon\\
&-\frac{a^2}{8}(1+\varepsilon^2)-\frac{a^2}{4}(\varepsilon+|\varepsilon|).
\end{aligned} \tag{2.9}$$

This theorem provides a description for the eigenvalues for sufficiently large n, namely, for $n\geqslant N$ with some fixed and sufficiently large N chosen independently of ε. The behavior of finitely many eigenvalues with the indices $n<N$ is described by the next theorem.

Theorem 2.3. *For each fixed N, for $n < N$, the eigenvalues λ_n^ε are holomorphic in ε, and the leading terms of their Taylor series in ε read as*

$$\lambda_n^\varepsilon = \kappa_n^2 + \varepsilon\Upsilon_{n,1} + \varepsilon^2\Upsilon_{n,2} + O(\varepsilon^3), \tag{2.10}$$

where the estimate for the error term is, generally speaking, not uniform in ε, and

$$\Upsilon_{n,1} := -3a, \qquad \Upsilon_{n,2} := \frac{a\kappa_n^2}{2} + \frac{a^2}{4} - \frac{9a^2}{4\kappa_n^2} \qquad \text{for} \qquad \varepsilon > 0, \tag{2.11}$$

$$\Upsilon_{n,1} := -a, \qquad \Upsilon_{n,2} := \frac{a\kappa_n^2}{2} + \frac{a^2}{4} - \frac{a^2}{4\kappa_n^2} \qquad \text{for} \qquad \varepsilon < 0. \tag{2.12}$$

Our second result describes the behavior of the associated eigenfunctions.

Theorem 2.4. *The eigenfunctions $\psi_n^\varepsilon = \psi_n^\varepsilon(x)$ of the operator $\mathcal{H}^\varepsilon$ associated with the eigenvalues λ_n^ε form a basis in $L_2(0,1)$. As $n \to \infty$, the asymptotics*

$$\psi_n^\varepsilon(x) = \sqrt{2}\sin\pi\kappa_n x + \frac{\Psi_n^\varepsilon(x)}{\pi n} + O(n^{-2}) \tag{2.13}$$

hold in $C[0,1]$, where the estimate for the error term is uniform in ε, and

$$\Psi_n^\varepsilon(x) := \frac{a}{\sqrt{2}}\begin{cases} -(1-\varepsilon)(1-x)\cos\kappa_n x\cos\kappa_n\varepsilon, & x \in (0, 1-\varepsilon), \\ \frac{1}{2}\big(((1-\varepsilon)x - 1 + \varepsilon)\cos\kappa_n(x-\varepsilon) & \\ \quad - (x(1+\varepsilon) - 1 + \varepsilon)\cos\kappa_n(x+\varepsilon)\big), & x \in (1-\varepsilon, 1), \end{cases} \tag{2.14}$$

for $\varepsilon > 0$, and

$$\Psi_n^\varepsilon(x) := \frac{a}{\sqrt{2}}\begin{cases} (1+\varepsilon)x\cos\kappa_n\varepsilon\cos\kappa_n x, & x \in (0, |\varepsilon|), \\ -\frac{1}{2}\big((x(1-\varepsilon) + 2\varepsilon)\cos\kappa_n(x+\varepsilon) & \\ \qquad - x(1+\varepsilon)\cos\kappa_n(x-\varepsilon)\big), & x \in (|\varepsilon|, 1), \end{cases} \tag{2.15}$$

for $\varepsilon < 0$. For each fixed N, for $n \leqslant N$, the eigenfunctions ψ_n^ε possess asymptotics

$$\psi_n^\varepsilon(x) = \Phi_{n,0}^\varepsilon(x) + \varepsilon\,\Phi_{n,1}^\varepsilon(x) + O(\varepsilon^2) \tag{2.16}$$

in the norm of $C[0,1]$, where the estimates for the error term is, generally speaking, not uniform in n and

$$\begin{aligned} &\Phi_{n,0}^\varepsilon(x) := \sqrt{2}\sin\kappa_n x, && x \in [0, 1-\varepsilon], \\ &\Phi_{n,0}^\varepsilon(x) := (-1)^n\sqrt{2}\cos\sqrt{\kappa_n^2 + a}(x-1), && x \in [1-\varepsilon, 1], \\ &\Phi_{n,1}^\varepsilon(x) := -\frac{ax}{2\sqrt{2}\kappa_n}\big(\pi\sin\kappa_n x + 6\cos\kappa_n x\big), && x \in [0, 1-\varepsilon], \\ &\Phi_{n,1}^\varepsilon(x) := (-1)^n a\bigg(-\frac{1}{\sqrt{2}}\cos\sqrt{\kappa_n^2 + a}(x-1) && \\ &\qquad + \frac{3\sqrt{2}(x-1)}{\sqrt{\kappa_n^2 + a}}\sin\sqrt{\kappa_n^2 + a}(x-1)\bigg), && x \in [1-\varepsilon, 1], \end{aligned} \tag{2.17}$$

for $\varepsilon > 0$, and

$$
\begin{aligned}
&\Phi^\varepsilon_{n,0}(x) :=\frac{\sqrt{2}\kappa_n}{\sqrt{\kappa_n^2+a}}\sin\sqrt{\kappa_n^2+a}x, && x\in[0,|\varepsilon|],\\
&\Phi^\varepsilon_{n,0}(x) :=\sqrt{2}\sin\kappa_n x, && x\in[|\varepsilon|,1],\\
&\Phi^\varepsilon_{n,1}(x) :=\sqrt{2}a\Bigg(\bigg(\frac{\kappa_n}{\sqrt{\kappa_n^2+a}}-\frac{\mathrm{i}}{\kappa_n}-\frac{3a}{(\kappa_n^2+a)^{\frac{3}{2}}}\bigg)\sin\sqrt{\kappa_n^2+a}x\\
&\qquad\qquad -\frac{3a\kappa_n}{2(\kappa_n^2+a)}x\cos\sqrt{\kappa_n^2+a}x\Bigg), && x\in[0,|\varepsilon|],\\
&\Phi^\varepsilon_{n,1}(x) :=-\frac{a}{\sqrt{2}}\left(\frac{3x-2}{\kappa_n}\cos\kappa_n x+(x-1)\sin\kappa_n x\right), && x\in[|\varepsilon|,1].
\end{aligned}
\tag{2.18}
$$

for $\varepsilon < 0$.

Let us briefly describe the model and main results. The main feature of the operator $\mathcal{H}^\varepsilon$ is the small translation in the perturbation, which makes the operator $\mathcal{H}^\varepsilon$ nonlocal. The translation is described by the small parameter ε, which can be both positive and negative. For $\varepsilon > 0$ the operator $\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}$ describes the translation to the right, while for $\varepsilon < 0$ it does to the left. Then the function $\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}y$ turns out to be zero on the integral $(0,\varepsilon)$ for $\varepsilon > 0$ and on the interval $(1-|\varepsilon|,1)$ for $\varepsilon < 0$. In the case $\varepsilon > 0$ this small interval is attached to the point $x = 0$ with the Dirichlet condition, while in the case $\varepsilon < 0$ it is attached to the point $x = 1$ with the Neumann condition. This difference is reflect in the formulas for the leading terms in the asymptotics for the eigenvalues and eigenfunctions of the operator $\mathcal{H}^\varepsilon$, which are the main results of the work, see Theorems 2.2, 2.3, 2.4.

Theorem 2.2 describes four leading terms in the asymptotics for the eigenvalues of the operator $\mathcal{H}^\varepsilon$ as $n \to +\infty$ with the error of order $O(n^{-3})$ uniform in ε. This is a principally different result in comparison with classical spectral asymptotics, see, for instance, [8], [9], since the asymptotic depends on the small parameter and the uniform estimate for the error is provided. In the asymptotics (2.8) we explicitly find four leading terms, and this is the main difference from similar results in our recent works [22], [23], in which the method of similar operators allowed us to construct the asymptotics only with the error of order $O(n^{-2})$. The corresponding coefficients are described by the formulas (2.9). These relations contain the functions $\sin p\kappa_n\varepsilon$, $\cos p\kappa_n\varepsilon$, $p = 1, 2, 3$. They weakly oscillate in the index n. For small n they are close respectively to 0 and 1. For n of order $O(\varepsilon^{-1})$ the functions $\sin p\kappa_n\varepsilon$ and $\cos p\kappa_n\varepsilon$ smoothly vary from -1 to $+1$, while for n exceeding essentially $O(\varepsilon^{-1})$ these functions fast oscillate. The presence of such functions in the leadings terms of the asymptotics for the eigenvalues demonstrates an interesting high–frequency phenomenon, in which the perturbation (small translation) non–trivially interacts with large values of the index n.

If we fix the index n and lessen the parameter ε, then the eigenvalue λ^ε_n turns out to be holomorphic in sufficiently small ε and as $\varepsilon \to 0$, convergence to the eigenvalue λ_n of the limiting operator. This convergence is non–uniform, as the asymptotics (2.8) shows. At the same, if we fix a sufficiently large N, then for $n \geqslant N$ the asymptotics (2.8) describes in detail the behavior of the eigenvalues λ^ε_n. Then for $n \leqslant N$ we choose sufficiently small ε, and for the eigenvalues λ^ε_n one can write asymptotics in the small parameter ε, see the formulas (2.10), (2.11), (2.12) in Theorem 2.3. The leading terms in in asymptotics are written explicitly, but the error term in not uniform in n.

We also succeed to write the leading terms of the asymptotic expansions for the eigenfunction, and this result is formulated in Theorem 2.4. The relations (2.13), (2.14), (2.15) describe the asymptotics in large $n \geqslant N$, while the identities (2.16), (2.17), (2.18) provide the asymptotics in the parameter ε for $n \leqslant N$.

Theorems 2.2, 2.3, 2.4 describe rather explicitly the global behavior of the total ensemble of eigenvalues of operator $\mathcal{H}^\varepsilon$ and the associated eigenfunctions. Outside some rather large circle in the complex plane, that is, for $n \geqslant N$, we use asymptotics as $n \to +\infty$, while inside the circles we use the asymptotics as $\varepsilon \to +0$. In addition, we show that the eigenfunctions form a (non–orthonormalized) basis in $L_2(0,1)$. This is a stronger result in comparison with similar statements in papers [22], [23], where the basicity of the system of eigenfunctions and generalized eigenfunctions was stated. In the present work we succeed to exclude the existence of the generalized eigenfunctions.

It should be noted that the leading terms of the asymptotics both as $n \to +\infty$ and $\varepsilon \to 0$ depend on the sign of ε. This is shown by the formulas (2.9). The terms $\Lambda^\varepsilon_{n,0}$ and $\Lambda^\varepsilon_{n,1}$ turn out to be dependent on the absolute value of the parameter ε, while the term $\Lambda^\varepsilon_{n,2}$ involves an additional summand $-\frac{a^2}{4}(\varepsilon + |\varepsilon|)$, which is equal to $-\frac{a^2\varepsilon}{2}$ for $\varepsilon > 0$ and vanishes identically for $\varepsilon < 0$. The differences in terms of the asymptotics for the eigenvalues in ε arise from the coefficients at ε, see (2.11), (2.12). Moreover, the formulas (2.14), (2.15), (2.17), (2.18) show essential differences of the terms of the asymptotics as $\varepsilon > 0$ and $\varepsilon < 0$.

Let us dwell on the technique of the present work. Since the limiting operator is very simple and the coefficients a in the perturbing operator is constant, we succeed to solve rather explicitly the eigenvalue equation for the operator $\mathcal{H}^\varepsilon$. The employed explicit solution allows us to obtain all above described asymptotics. However, the nonlocality generated by the small translation does not ensure that the constructed solution is general and the constructed system of the eigenvalues exhausts the entire spectrum. In order to establish this fact, we make an additional analysis of the associated eigenfunctions and show that they form a basis in $L_2(0,1)$, and hence, the constructed system of the eigenvalues exhausts the entire spectrum of the operator $\mathcal{H}^\varepsilon$. We stress that the method of similar operators used in [22] and [23] has no such a disadvantage and provides immediately the results on the entire set of the eigenvalues. At the same, the advantage of our explicit constructions in a possibility to construct further terms in the asymptotics. Moreover, our technique allows us to construct any prescribed number of the terms in the asymptotics of the eigenvalues and eigenfunctions, but the calculations turn out to be very and very bulky.

3. Form and resolvent of perturbed operator

In the present section we prove Theorem 2.1. The scheme follows the lines of the proof of Theorem 1 in paper [22], but for the reader's convenience we briefly describe main milestones.

The application of general results from the proof of Theorem 3 in [19, Sect. 5] allows us to conclude immediately that the form $\mathfrak{h}^\varepsilon$ is sectorial and closed, and its numerical domain is located in the sector $\{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant C_0(\operatorname{Re} z - C_1)\}$, where C_0 and C_1 are some constants independent of ε and $C_0 > 0$. According to the first representation theorem [20, Ch. VI, Sect. 2.1, Thm. 2.1], there exists an associated m–sectorial operator. We denote this operator by $\widetilde{\mathcal{H}}^\varepsilon$ and we are going to show that it coincides with $\mathcal{H}^\varepsilon$. Due to the same first representation theorem the domain of $\widetilde{\mathcal{H}}^\varepsilon$ consists of the functions $u \in \mathfrak{D}(\mathfrak{h}^\varepsilon)$ obeying the identity

$$\mathfrak{h}^\varepsilon(u,v) = (h,v)_{L_2(0,1)} \qquad \text{for all} \qquad v \in \mathfrak{D}(\mathfrak{h}^\varepsilon) \tag{3.1}$$

with some function $h \in L_2(0,1)$. This implies immediately that the function u is a generalized solution of the boundary value problem

$$\begin{gathered} -u'' = \widetilde{h} \quad \text{on} \quad (0,1), \qquad u(0) = u'(1) = 0, \\ \widetilde{h} := h - a\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}u + au \in L_2(0,1), \end{gathered}$$

and by the standard smoothness improving theorems we see that $u \in \mathfrak{D}(\mathcal{H}^\varepsilon)$. Hence, $\mathfrak{D}(\widetilde{\mathcal{H}}^\varepsilon) \subseteq \mathfrak{D}(\mathcal{H}^\varepsilon)$ and the operator $\widetilde{\mathcal{H}}^\varepsilon$ is a continuation of the operator $\mathcal{H}^\varepsilon$. For all $u \in \mathfrak{D}(\mathcal{H}^\varepsilon)$, $v \in \mathfrak{D}(\mathfrak{h}^\varepsilon)$

by an integration by parts we verify the identity

$$(\mathcal{H}^\varepsilon u, v)_{L_2(0,1)} = \mathfrak{h}^\varepsilon(u,v),$$

which implies immediately that the operator $\mathcal{H}^\varepsilon$ is a continuation of the operator $\widetilde{\mathcal{H}}^\varepsilon$. Therefore, the form $\mathfrak{h}^\varepsilon$ corresponds to the operator $\mathcal{H}^\varepsilon$, and the statements of the theorem on the m–sectoriality of the operator $\mathcal{H}^\varepsilon$ and the location of its spectrum are true.

The definition of the operator $\mathcal{H}^\varepsilon$ and the compact embedding of the space $W_2^1(0,1)$ into $L_2(0,1)$ implies that the resolvent of the operator $\mathcal{H}^\varepsilon$ is compact, and this is why its spectrum consists of countably many isolated eigenvalues, which can accumulate at the infinity only.

Other statements of the theorem follows from the general results of Theorems 4, 5 in work [19] applied to the operator $\mathcal{H}^\varepsilon$. The proof of Theorem 2.1 is complete.

4. Transcendental equation for eigenvalues

In the present section we begin proving Theorem 2.2. Let λ be an eigenvalue of the operator $\mathcal{H}^\varepsilon$. Then the associated normalized in $L_2(0,1)$ eigenfunction ψ should satisfy an integral identity, namely,

$$\lambda = \mathfrak{h}^\varepsilon(\psi,\psi) = \|\psi'\|^2_{L_2(0,1)} + a(\mathcal{R}\mathcal{T}^\varepsilon\mathcal{L}\psi - \psi, \psi)_{L_2(0,1)}.$$

Taking real and imaginary parts of this identity, in view of the normalization of the function ψ we immediately obtain the apriori estimates

$$\operatorname{Re}\lambda \geqslant -c_1, \qquad |\operatorname{Im}\lambda| \leqslant c_1, \tag{4.1}$$

where c_1 is some positive constant independent of ε and λ. This is why all eigenvalues of the operator $\mathcal{H}^\varepsilon$ lie in some fixed semi–strip along the real semi–axis.

According to Theorem 2.1, the eigenvalues of the operator $\mathcal{H}^\varepsilon$ located in a fixed circle in the complex plane converge to the eigenvalues of the operator $\mathcal{H}$, which can be found explicitly and read as $\lambda_n^0 := \kappa_n^2$. This is why in view of the estimate (4.1), we seek the eigenvalues of the operator $\mathcal{H}^\varepsilon$ in the form

$$\lambda = k^2, \tag{4.2}$$

where $k \in \mathbb{C}$ is a new complex parameter ranging in the domain

$$\Omega := \left\{ k \in \mathbb{C} :\ \operatorname{Re} k \geqslant \frac{\pi}{8},\ |\operatorname{Im} k| \leqslant \frac{c_2}{\operatorname{Re} k} \right\},$$

with some positive constant c_2 independent of k and ε.

Due to the definition of the operator $\mathcal{H}^\varepsilon$, the eigenvalue equation $\mathcal{H}^\varepsilon\psi = \lambda\psi$ with the spectral parameter defined by the formula (4.2) is equivalent to the boundary value problems

$$\begin{aligned} &-\psi''(x) + a\big(\psi(x+\varepsilon) - \psi(x)\big) - k^2\psi(x) = 0, && x \in (0, 1-\varepsilon),\\ &-\psi''(x) - (k^2+a)\psi(x) = 0, && x \in (1-\varepsilon, 1),\\ &\psi(0) = 0, \qquad \psi'(1) = 0, \qquad [\psi]_{1-\varepsilon} = 0, && [\psi']_{1-\varepsilon} = 0, \end{aligned} \tag{4.3}$$

for $\varepsilon > 0$, and

$$\begin{aligned} &-\psi''(x) - (k^2+a)\psi(x) = 0, && x \in (0, |\varepsilon|),\\ &-\psi''(x) + a\big(\psi(x+\varepsilon) - \psi(x)\big) - k^2\psi(x) = 0, && x \in (|\varepsilon|, 1),\\ &\psi(0) = 0, \qquad \psi'(1) = 0, \qquad [\psi]_{|\varepsilon|} = 0, && [\psi']_{|\varepsilon|} = 0, \end{aligned} \tag{4.4}$$

for $\varepsilon < 0$. Here $[u]_{x_0}$ is the jump of a function u at the point $x = x_0$:

$$[u]_{x_0} := u(x_0+0) - u(x_0-0).$$

We seek a solution to this pair of boundary value problems separately on the intervals $(0, 1-\varepsilon)$, $(1-\varepsilon, 1)$, and $(0, |\varepsilon|)$, $(|\varepsilon|, 1)$ taking into consideration the boundary conditions at the points $x = 0$ and $x = 1$. Then we substitute the obtained solution into the matching conditions at the points $x=1-\varepsilon$ and, respectively, $x=|\varepsilon|$, and this finally will give a transcendental equation for k.

In order to construct solutions to the nonlocal equations on the intervals $(0,1-\varepsilon)$ and $(0,|\varepsilon|)$ in the boundary value problems (4.3), (4.4), we consider an auxiliary nonlocal equation on the axis

$$-y''(x)+a\big(y(x+\varepsilon)-y(x)\big)-k^2y(x)=0,\qquad x\in\mathbb{R}. \tag{4.5}$$

We seek its partial solution in the form

$$y(x)=e^{\mathrm{i}\tau x}. \tag{4.6}$$

Then for τ we immediately obtain the characteristic equation

$$\tau^2+a(e^{\mathrm{i}\tau\varepsilon}-1)-k^2=0. \tag{4.7}$$

The left hand side of this equation is an entire function of an exponential type for $\varepsilon\neq0$, $a\neq0$. The function $\tau\mapsto e^{\mathrm{i}\tau\varepsilon}-1$ has infinitely many zeroes, and the same is true for the function in the left hand side of Equation (4.7). Thus, Equation (4.5) has *infinitely many linearly independent* solutions of the form (4.6) once $\varepsilon\neq0$. Among this set of zeroes, in what follows we employ just a certain pair, the existence of which is described in the next lemma.

We denote

$$\begin{aligned}
&\Omega:=\left\{k\in\mathbb{C}:\ \operatorname{Re}k\geqslant\frac{\pi}{8},\ |\operatorname{Im}k|\leqslant\frac{c_2}{\operatorname{Re}k}\right\},\\
&\Omega_0:=\left\{k\in\mathbb{C}:\ \frac{\pi}{8}\leqslant\operatorname{Re}k\leqslant R_1+\frac{\pi}{3},\ |\operatorname{Im}k|\leqslant\frac{c_2}{\operatorname{Re}k}\right\},\\
&\Omega_1:=\left\{k\in\mathbb{C}:\ \operatorname{Re}k\geqslant R_1,\ |\operatorname{Im}k|\leqslant\frac{c_2}{\operatorname{Re}k}\right\},
\end{aligned} \tag{4.8}$$

where a constant R_1 is large enough, fixed, and independent of ε. Let $B_r(k)$ be an open ball of a radius r centered at a point k.

Lemma 4.1. *For a sufficiently large R_1 there exist fixed numbers $R_2\in(R_1-1,R_1)$ and c_3 independent of ε such that for all $k\in\Omega$ each of the sets*

$$\Pi_\pm:=\big\{\tau\in\mathbb{C}:\ |\operatorname{Im}\tau|\leqslant c_3,\ \pm\operatorname{Re}\tau>R_2\big\}$$

contains exactly a single root $\tau^\pm=\tau^\pm(\varepsilon,k)$ of Equation (4.7). These roots satisfy the estimates

$$|\tau^\pm\mp k|\leqslant r, \tag{4.9}$$

where r is a sufficiently small number independent of k and ε. The roots $\tau^\pm$ are holomorphic in $k\in\Omega$ for each fixed small ε.

Proof. For $\tau\in\overline{\Pi_+}$, $k\in\overline{\Omega}$ and sufficiently small ε the estimates

$$|\tau^2-k^2|\geqslant|\tau-k|\operatorname{Re}(\tau+k)\geqslant2R_2|\tau-k|,\qquad |a(e^{\mathrm{i}\tau\varepsilon}-1)|\leqslant|a|(1+e^{c_3\varepsilon})\leqslant3|a| \tag{4.10}$$

hold. We choose large enough R_2 and a sufficiently small but fixed $r>0$ so that the identity and embedding

$$2R_2>3|a|r+1,\qquad \overline{B_r(k)}\subset\Pi_+,\qquad k\in\overline{\Omega}, \tag{4.11}$$

are guaranteed. It is clear that such a choice is possible. These estimates immediately imply that

$$|\tau^2-k^2|>|a(e^{\mathrm{i}\tau\varepsilon}-1)|\quad\text{for}\quad\tau\in\overline{\Pi_+}\setminus B_r(k).$$

Therefore, Equation (4.7) has no roots in $\tau\in\overline{\Pi_+}\setminus B_r(k)$, while the application of Rouché theorem ensures that Equation (4.7) contains in $B_r(k)$ as many zeroes counting the multiplicities as the function $\tau\mapsto\tau^2-k^2$ does, that is, exactly a single root. We denote this root by $\tau^+(\varepsilon,k)$, and since it belongs to the circle $B_r(k)$, it obeys the estimate (4.9).

If the parameter τ ranges over the domain Π_-, then $-\tau \in \Pi_+$. The change τ by $-\tau$ transforms Equation (4.7) into a similar one:

$$\tau^2 + a(e^{-\mathrm{i}\tau\varepsilon} - 1) - k^2 = 0, \qquad \tau \in \Pi_+, \qquad k \in \Omega. \tag{4.12}$$

This equation can be studied exactly in the same way as it has been done above, and it also has exactly one root in Π_+. Returning back to the domain Π_-, we conclude that this domain contains exactly on root $\tau^-(\varepsilon, k)$ of Equation (4.7), and it satisfies the estimate (4.9).

The application of the inverse function theorem [21, Thm. 1.3.5, Rem. 1.3.6] to Equation (4.7) shows immediately that the roots $\tau^\pm$ are holomorphic in k for each sufficiently small ε. The proof is complete. □

Lemma 4.2. *For each fixed R_1, for $k \in \Omega_0$ and sufficiently small complex ε Equation (4.7) has exactly one root $\tau^\pm = \tau^\pm(\varepsilon, k)$ in the circles $B_{\frac{\pi}{8}}(\pm k)$. These roots are holomorphic in ε and k.*

Proof. The set Ω_0 is bounded and this is why for $k \in \Omega_0$ we can apply the inverse function theorem [21, Thm. 1.3.5, Rem. 1.3.6] to Equation (4.7) and this implies the statement of the lemma. □

We proceed to constructing solutions to the equations in (4.3), (4.4) satisfying the boundary conditions at the points $x = 0$ and $x = 1$. We seek a solution to Equation (4.3) on the interval $(0, 1-\varepsilon)$ obeying the Dirichlet condition at the point $x = 0$ in the form

$$\psi(x) = C_1\big(e^{\mathrm{i}\tau^+(\varepsilon,k)x} - e^{\mathrm{i}\tau^-(\varepsilon,k)x}\big), \qquad x \in (0, 1-\varepsilon), \tag{4.13}$$

where $\tau^\pm = \tau^\pm(\varepsilon, k)$ are the roots of Equation (4.7) found in Lemma 4.1, and C_1 is an arbitrary constant. The general solution to Equation (4.3) on the interval $(1-\varepsilon, 1)$ satisfying the Neumann condition at the point $x = 1$ is obviously reads as

$$\psi(x) = C_2 \cos\sqrt{k^2 + a}(x - 1), \qquad x \in (1-\varepsilon, 1), \tag{4.14}$$

where C_2 is an arbitrary constant, and the branch of the root is chosen by the condition $\sqrt{1} = 1$ with the cut along the negative real semi–axis. Under such a choice the identity $\sqrt{k^2 + a} = k\sqrt{1 + ak^{-2}}$ is obviously true.

The found solutions should satisfy the matching conditions at the point $x = 1-\varepsilon$ from the problem (4.3), and this gives a system of linear equations for the constants C_1 and C_2

$$\begin{aligned} &C_1\big(e^{\mathrm{i}(1-\varepsilon)\tau^+} - e^{\mathrm{i}(1-\varepsilon)\tau^-}\big) - C_2\cos\sqrt{k^2+a}\,\varepsilon = 0,\\ &C_1\mathrm{i}\big(\tau^+e^{\mathrm{i}(1-\varepsilon)\tau^+} - \tau^-e^{\mathrm{i}(1-\varepsilon)\tau^-}\big) - C_2\sqrt{k^2+a}\sin\sqrt{k^2+a}\,\varepsilon = 0. \end{aligned} \tag{4.15}$$

By the Cramer's rule this system has a nontrivial solution and hence, the boundary value problem (4.3) has a nontrivial solution if and only if the parameter k satisfies the equation

$$\big(e^{\mathrm{i}(1-\varepsilon)\tau^+} - e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\sqrt{k^2+a}\sin\sqrt{k^2+a}\,\varepsilon - \mathrm{i}\big(\tau^+e^{\mathrm{i}(1-\varepsilon)\tau^+} - \tau^-e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\cos\sqrt{k^2+a}\,\varepsilon = 0,$$

and it is convenient to rewrite this equation as

$$\big(e^{\mathrm{i}(1-\varepsilon)\tau^+} - e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\sin\sqrt{k^2+a}\,\varepsilon - \mathrm{i}\big(\tau^+e^{\mathrm{i}(1-\varepsilon)\tau^+} - \tau^-e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\frac{\cos\sqrt{k^2+a}\,\varepsilon}{\sqrt{k^2+a}} = 0. \tag{4.16}$$

This is exactly the transcendental equation for the parameter k, which determines the eigenvalues of the operator $\mathcal{H}^\varepsilon$ in the case $\varepsilon > 0$.

In the same way we construct solutions to the equation in the problem (4.4) obeying the same boundary conditions at the points $x=0$ and $x=1$. We have

$$
\begin{aligned}
&\psi(x)=C_1\sin\sqrt{k^2+a}x, && x\in(0,|\varepsilon|),\\
&\psi(x)=C_2\sqrt{k^2+a}\left(\frac{e^{\mathrm{i}\tau^+(\varepsilon,k)(x-1)}}{\tau^+(\varepsilon,k)}-\frac{e^{\mathrm{i}\tau^-(\varepsilon,k)(x-1)}}{\tau^-(\varepsilon,k)}\right), && x\in(|\varepsilon|,1).
\end{aligned}
\tag{4.17}
$$

The matching conditions at the point $x=|\varepsilon|=-\varepsilon$ give

$$
\begin{aligned}
&C_1\sin\sqrt{k^2+a}\varepsilon+C_2\sqrt{k^2+a}\left(\frac{e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)}}{\tau^+(\varepsilon,k)}-\frac{e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}}{\tau^-(\varepsilon,k)}\right)=0,\\
&C_1\cos\sqrt{k^2+a}\varepsilon-\mathrm{i}C_2\big(e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)}-e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}\big)=0.
\end{aligned}
\tag{4.18}
$$

The application of the Cramer's rule gives the equation for k

$$
\begin{aligned}
&\left(\frac{e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)}}{\tau^+(\varepsilon,k)}-\frac{e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}}{\tau^-(\varepsilon,k)}\right)\sqrt{k^2+a}\cos\sqrt{k^2+a}\varepsilon\\
&\quad+\mathrm{i}\big(e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)}-e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}\big)\sin\sqrt{k^2+a}\varepsilon=0,
\end{aligned}
\tag{4.19}
$$

which determines the eigenvalues of the operator $\mathcal{H}^\varepsilon$ for $\varepsilon<0$.

Our next step is a detailed study of the obtained transcendental Equations (4.16), (4.19). This will be done in the next section.

5. Solvability of transcendental equations and asymptotics for roots

In order to study the solvability of Equations (4.16), (4.19), we need to know the structure of the dependence of the roots $\tau^\pm(\varepsilon,k)$ of Equation (4.7) on ε and k. We shall consider separately two cases: $k\in\Omega_0$ and $k\in\Omega_1$. At the same time the number R_1 in the definition of these sets can be arbitrarily large (but fixed!), and the choice will be made later.

We first consider the case $k\in\Omega_1$. We choose R_1 large enough so that k also turns out to be large, and we are going to describe the asymptotic behavior of $\tau^\pm(\varepsilon,k)$ for large k.

Lemma 5.1. *The parameter R_1 in the definition (4.8) of the set Ω_1 can be chosen so that for $k\in\Omega_1$ the roots $\tau^\pm(\varepsilon,k)$ satisfy the relations*

$$
\tau^\pm(\varepsilon,k)=\pm k+\frac{\xi_1^\pm(\varepsilon,k)}{k}+\frac{\xi_2^\pm(\varepsilon,k)}{k^2}+\frac{\xi_3^\pm(\varepsilon,k)}{k^3}+O(k^{-4}),
\tag{5.1}
$$

where the estimates for the error terms are uniform in ε and

$$
\begin{aligned}
&\xi_1^\pm(\varepsilon,k):=\mp\frac{a}{2}(e^{\pm\mathrm{i}\varepsilon k}-1), \qquad \xi_2^\pm(\varepsilon,k):=\mp\frac{\mathrm{i}a\varepsilon}{2}e^{\pm\mathrm{i}k\varepsilon}\xi_1^\pm(\varepsilon,k),\\
&\xi_3^\pm(\varepsilon,k):=\mp\frac{(\xi_1^\pm(\varepsilon,k))^2}{4}(2-a\varepsilon^2e^{\pm\mathrm{i}k\varepsilon})\mp\frac{\mathrm{i}a\varepsilon}{2}e^{\pm\mathrm{i}k\varepsilon}\xi_2^\pm(\varepsilon,k).
\end{aligned}
\tag{5.2}
$$

Proof. As it has been established in the proof of Lemma 4.1, the root τ^+ is located inside the circle of the radius r centered at the point k. This is why this root can be represented as

$$
\tau=k+\tau_0,\qquad |\tau_0|\leqslant r,
\tag{5.3}
$$

where $\tau_0=\tau_0(\varepsilon,k)$ is some function. Substituting this representation into Equation (4.7), we immediately obtain equations for τ_0

$$
2\tau_0+\frac{\tau_0^2}{k}+\frac{a}{k}(e^{\mathrm{i}\varepsilon k}e^{\mathrm{i}\varepsilon r}-1)=0.
$$

Due to the apriori boundedness of τ_0, see (5.3), and the belonging $k\in\Omega$, it follows from the obtained equation that $|\tau_0|\leqslant Ck^{-1}$ with some constant C independent of ε and k. This is why the representation (5.3) can be specified

$$\tau^+=k+\frac{\tau_1^+}{k},\qquad |\tau_1^+|\leqslant C. \tag{5.4}$$

We substitute the specified representation into (4.7), then for τ_1^+ we obtain the equation

$$2\tau_1^+ + \frac{(\tau_1^+)^2}{k^2}+a(e^{\mathrm{i}\varepsilon k}-1)+ae^{\mathrm{i}\varepsilon k}(e^{\mathrm{i}\varepsilon\frac{\tau_1^+}{k}}-1)=0.$$

This equation, estimate for τ_1^+ from (5.4), and the obvious inequality

$$|ae^{\mathrm{i}\varepsilon k}(e^{\mathrm{i}\varepsilon\frac{\tau_1^+}{k}}-1)|\leqslant C\varepsilon k^{-1}$$

with one more constant C imply that

$$\tau_1^+(\varepsilon,k)=\xi_1^+(\varepsilon,k)+\frac{\tau_2^+(\varepsilon,k)}{k},\qquad |\tau_2^+(\varepsilon,k)|\leqslant C, \tag{5.5}$$

where $\tau_2^+=\tau_2^+(\varepsilon,k)$ is some function, and C is some constant independent of ε and k. We substitute the obtained representation for τ_1^+ in (5.4) and similarly to the above calculations we write an equation for τ_2^+. Taking then into consideration that

$$e^{\mathrm{i}\varepsilon\frac{\tau_1^+}{k}}=1-\frac{a\varepsilon}{2k}(e^{\mathrm{i}\varepsilon k}-1)+O(k^{-2}),$$

where the estimate for error term is uniform in ε, we obtain the identity

$$\tau_2^+=\xi_2^+(\varepsilon,k)+\frac{\xi_3^+(\varepsilon,k)}{k}, \tag{5.6}$$

Substituting this identity, (5.4), and (5.5) into Equation (4.16) and extracting the terms of order up to $O(k^{-2})$, we arrive at the identity

$$\tau_3^+=\xi_3^+(\varepsilon,k)+\frac{\xi_4^+(\varepsilon,k)}{k},$$

where the function $\xi_4^+(\varepsilon,k)$ is bounded uniformly in ε and k. Substituting this relation, (5.5), and (5.6) into (5.4), we get the formula (5.1) for τ^+. The same identity for τ^- can be proved similarly. The proof is complete. ☐

Lemma 5.2. *For each fixed R_1, for $k\in\Omega_0$, and sufficiently small ε the leading terms of the Taylor series of the roots $\tau^\pm(\varepsilon,k)$ read as*

$$\tau^\pm(\varepsilon,k)=\pm k+\frac{\mathrm{i}a\varepsilon}{2}\mp\frac{a(2k^2+a)}{8k}\varepsilon^2+O(\varepsilon^3). \tag{5.7}$$

Proof. Due to Lemma 4.2, for $k\in\Omega_0$ the roots $\tau^\pm(\varepsilon,k)$ are holomorphic in ε. We write the Taylor series with the first three terms with unknown coefficients and substitute these formulas into Equation (4.7). We expand the obtained equation into the Taylor series up to the order $O(\varepsilon^2)$ and equate the coefficients at ε and ε^2 to zero. Solving the obtained equations, we arrive at the relations (5.7). The proof is complete. ☐

Let $k\in\Omega_1$ and R_1 be chosen according to Lemma 5.1. The identity (5.1) allows us to obtain similar representations for separate expressions in the left hand side of (4.16)

$$\begin{aligned} e^{\mathrm{i}(1-\varepsilon)\tau^\pm}=e^{\pm\mathrm{i}(1-\varepsilon)k}\Bigg(1&+\frac{\mathrm{i}(1-\varepsilon)\xi_1^\pm}{k}+\frac{2\mathrm{i}(1-\varepsilon)\xi_2^\pm-(1-\varepsilon)^2(\xi_1^\pm)^2}{2k^2}\\ &+\frac{6\mathrm{i}(1-\varepsilon)\xi_3^\pm-6(1-\varepsilon)^2\xi_2^\pm\xi_1^\pm-\mathrm{i}(1-\varepsilon)^3(\xi_1^\pm)^3}{6k^3}\Bigg)+O(k^{-4}),\end{aligned}\tag{5.8}$$

where the estimate for the error term is uniform ε. The obvious identities

$$\begin{aligned} &\sin\sqrt{k^2+a}\varepsilon=\sin k\varepsilon+\frac{a\varepsilon}{2k}\cos k\varepsilon-\frac{a^2\varepsilon^2}{8k^2}\sin k\varepsilon-\frac{a^2\varepsilon(a\varepsilon^2+6)}{48k^3}\cos k\varepsilon+O(k^{-4}),\\ &\frac{\cos\sqrt{k^2+a}\varepsilon}{\sqrt{k^2+a}}=\frac{\cos k\varepsilon}{k}-\frac{a\varepsilon\sin k\varepsilon}{2k^2}-\frac{a(4+a\varepsilon^2)\cos k\varepsilon}{8k^3}+\frac{a^2\varepsilon(a\varepsilon^2+18)\sin k\varepsilon}{48k^4}+O(k^{-5}),\\ &\sqrt{k^2+a}\cos\sqrt{k^2+a}\varepsilon=k\cos k\varepsilon-\frac{a\varepsilon}{2}\sin k\varepsilon+\frac{a(4-a\varepsilon^2)\cos k\varepsilon}{8k}+\frac{a^2\varepsilon(a\varepsilon^2-6)\sin k\varepsilon}{48k^2}\\ &\qquad\qquad+\frac{a^2(a^2\varepsilon^2-48)}{384}\cos k\varepsilon+O(k^{-4})\end{aligned}\tag{5.9}$$

are also true.

We substitute the obtained relations into Equation (4.16) and collect the terms up to order $O(k^{-3})$. Then the equation is rewritten in the form

$$K_0(k)+\sum_{j=1}^{4}\frac{K_j^+(\varepsilon,k)}{k^j}=0,\tag{5.10}$$

where K_4^+ is some uniformly bounded in ε and $k\in\overline{\Omega}$ function holomorphic in $k\in\Omega$ for each sufficiently small ε, and the functions K_0, K_1^+, K_2^+ are given by the formulas

$$\begin{aligned} &K_0(k):=\cos k,\qquad K_1^+(\varepsilon,k):=-\frac{a}{2}\sin k+\frac{a}{2}(1-\varepsilon)\sin k(1+\varepsilon),\\ &K_2^+(\varepsilon,k):=-\frac{a}{8}\big(1-2(a(1-\varepsilon^2))\cos k(1+\varepsilon)+a(1-\varepsilon^2)\cos k(1+2\varepsilon)\\ &\qquad\qquad+2\cos k(1-\varepsilon)+a\cos k\big),\\ &K_3^+(\varepsilon,k):=\frac{a^2}{48}\Big(3(a+2(a-1)\varepsilon-a\varepsilon^2-2a\varepsilon^3)\sin k(1+2\varepsilon)+6(1-\varepsilon)\sin k(1-\varepsilon)\\ &\qquad\qquad-a(1-\varepsilon)(1+2\varepsilon)^2\sin k(1+3\varepsilon)+a\sin k\\ &\qquad\qquad-3(a+2-(a-6)\varepsilon+a\varepsilon^2+a\varepsilon^3)\sin k(1+\varepsilon)\Big).\end{aligned}$$

It is clear that the functions K_i^+, $i=1,2,3$, are bounded uniformly in $k\in\overline{\Omega}$ and sufficiently small ε.

The function K_0 has zeroes at the points $k=\kappa_n$, $n\in\mathbb{N}$, and obvious estimates

$$|K_0(k)|\geqslant C_1\min_{n\in\mathbb{N}}|k-\kappa_n|,\qquad k\in\overline{\Omega},\tag{5.11}$$

hold true, where C_1 is some constant independent of k, the choice R_1 and n, and the index n is chosen by the condition $n\geqslant n_0$ with some fixed n_0, which ensures the embedding of the circles $B_{\frac{\pi}{4}}(\kappa_n)$ in the domain Ω. Equation (5.10) can be rewritten in the form

$$K_0^+(k)+\frac{f^+(\varepsilon,k)}{k}=0,\qquad f^+(\varepsilon,k):=\sum_{j=1}^{4}\frac{K_j^+(\varepsilon,k)}{k^{j-1}}.\tag{5.12}$$

The function f is holomorphic in $k \in \Omega$ and bounded uniformly in ε and k, and the estimate

$$\left|\frac{f^+(\varepsilon,k)}{k}\right| \leqslant \frac{C_2}{|k|} \leqslant \frac{C_2}{R_1} \tag{5.13}$$

holds, where C_2 is some constant independent of ε, k, and the choice of R_1. We choose the number R_1 by the condition

$$\frac{\pi}{4}C_1 > \frac{C_2}{R_1}.$$

Then it follows from (5.11), (5.13) that

$$|K_0(k)| > \left|\frac{f^+(\varepsilon,k)}{k}\right| \quad \text{for} \quad k \in \overline{\Omega} \setminus \bigcup_{n\in\mathbb{N}} B_{\frac{\pi}{4}}(\kappa_n). \tag{5.14}$$

This inequality means that Equation (5.10) has no roots outside the circles $B_{\frac{\pi}{4}}(\kappa_n)$, $n \geqslant n_0$, and by the Rouché theorem it has exactly one root in each of these circles. We denote these roots by k_n^ε.

In view of (5.13), the estimate (5.14) can be specified, namely,

$$|K_0^+(k)| > \left|\frac{f^+(\varepsilon,k)}{k}\right| \quad \text{for} \quad k \in B_{\frac{C_3}{n}}(\kappa_n), \quad n \geqslant n_0, \tag{5.15}$$

with some constant C_3 independent of n and ε. This is why the roots k_n^ε satisfy the estimates

$$|k_n^\varepsilon - \kappa_n| \leqslant \frac{C_3}{n}. \tag{5.16}$$

By this inequality for k_n^ε the representation

$$k_n^\varepsilon = \kappa_n + \frac{\xi_{n,1}^\varepsilon}{\kappa_n} \tag{5.17}$$

holds, where $\xi_{n,1}^\varepsilon$ are some quantities bounded uniformly in ε and n. Substituting this relation into (5.10) and writing the terms up to the order $O(n^{-1})$, for large n we get

$$(-1)^{n+1}\xi_{n,1}^\varepsilon - \frac{(-1)^n a}{2}\big(1-(1-\varepsilon)\cos\kappa_n\varepsilon\big) + O(n^{-1}) = 0.$$

This implies

$$\xi_{n,1}^\varepsilon = \zeta_{n,1}^\varepsilon + \frac{\xi_{n,2}^\varepsilon}{\kappa_n}, \qquad \zeta_{n,1}^\varepsilon := \frac{a}{2}(1-\varepsilon)\cos\kappa_n\varepsilon - \frac{a}{2}, \tag{5.18}$$

where $\xi_{n,2}^\varepsilon$ are some quantities bounded uniformly in ε and n. We substitute this representation into (5.17), and the result is substituted into Equation (5.10). Then we extract the terms up to the order $O(n^{-2})$. This gives the relation

$$\xi_{n,2}^\varepsilon = \zeta_{n,2} + \frac{\xi_{n,3}^\varepsilon}{\kappa_n}, \qquad \zeta_{n,2}^\varepsilon := \frac{a^2}{8}\varepsilon^2(1-\varepsilon^2)\sin 2\kappa_n\varepsilon, \tag{5.19}$$

where $\xi_{n,3}^\varepsilon$ is some uniformly bounded quantity. It is determined by the same scheme as $\xi_{n,2}^\varepsilon$, and after routine technical calculations we obtain

$$\xi_{n,3}^\varepsilon = \zeta_{n,3}^\varepsilon + O(n^{-1}), \tag{5.20}$$

where the error term is uniform in the small parameter ε, and the quantity $\zeta_{n,3}^\varepsilon$ is given by the formula

$$\begin{aligned}\zeta_{n,3}^\varepsilon := &-\frac{3a^3\varepsilon^2}{64}(1-\varepsilon)(1+\varepsilon)^2\cos 3\kappa_n\varepsilon - \frac{a^2\varepsilon^2}{8}\cos 2\kappa_n\varepsilon \\ &+ \frac{a^2}{64}(1-\varepsilon)(16+3a\varepsilon^2-2a\varepsilon^3-a\varepsilon^4)\cos\kappa_n\varepsilon - \frac{a^2}{8}(2-\varepsilon+\varepsilon^2).\end{aligned} \tag{5.21}$$

This formula and (5.17), (5.18), (5.19) finally imply

$$k_n^\varepsilon=\kappa_n+\frac{\zeta_{n,1}^\varepsilon}{\kappa_n}+\frac{\zeta_{n,2}^\varepsilon}{\kappa_n^2}+\frac{\zeta_{n,3}^\varepsilon}{\kappa_n^3}+O(n^{-4}) \tag{5.22}$$

for $\varepsilon>0$, where the error term is uniform in ε.

The study of Equation (4.19) follows the same lines. Here the identities (5.8) are replaced by

$$\begin{aligned}e^{-\mathrm{i}(1+\varepsilon)\tau^\pm}={}&e^{\mp\mathrm{i}(1+\varepsilon)k}\Bigg(1-\frac{\mathrm{i}(1+\varepsilon)\xi_1^\pm}{k}-\frac{2\mathrm{i}(1+\varepsilon)\xi_2^\pm+(1+\varepsilon)^2(\xi_1^\pm)^2}{2k^2}\\&+\frac{\mathrm{i}(1+\varepsilon)^3(\xi_1^\pm)^3-6(1+\varepsilon)^2\xi_1^\pm\xi_2^\pm-\mathrm{i}(1+\varepsilon)\xi_3^\pm}{6k^3}\Bigg)+O(k^{-4}),\end{aligned}$$

where the estimate for the error term is uniform in ε. We substitute these relations and (5.9) into Equation (4.19) and extract the terms up to the order $O(k^{-2})$. As a result we obtain an analogue of Equation (5.10)

$$K_0(k)+\frac{K_1^-(\varepsilon,k)}{k}+\frac{K_2^-(\varepsilon,k)}{k^2}+\frac{K_3^-(\varepsilon,k)}{k^3}=0, \tag{5.23}$$

where K_3^- is some function bounded uniformly in ε and $k\in\overline{\Omega}$, holomorphic in $k\in\Omega$ for each sufficiently small ε, while the functions K_1^- and K_2^- are given by the formulas

$$\begin{aligned}K_1^-(\varepsilon,k):={}&\frac{a}{2}(1+\varepsilon)\sin(1-\varepsilon)k-\frac{a}{2}\sin k,\\K_2^-(\varepsilon,k):={}&\frac{a}{8}\big(2(a+1-\varepsilon^2)\cos k(1-\varepsilon)-a(1-\varepsilon^2)\cos k(1-2\varepsilon)-a\cos k+2\cos k(1+\varepsilon)\big),\\K_3^-(\varepsilon,k):={}&\frac{a^2}{48}\Big((12+3a+6(1-a)\varepsilon-3a\varepsilon^2+6a\varepsilon^3)\sin k(1-2\varepsilon)+6(1+\varepsilon)\sin k(1+\varepsilon)\\&-a(1-\varepsilon)(2\varepsilon-1)^2\sin k(1-3\varepsilon)+(a+12)\sin k\\&-3(a+6-(a-2)\varepsilon-a\varepsilon^2+a\varepsilon^3)\sin k(1-\varepsilon)\Big).\end{aligned}$$

The calculations from (5.11)–(5.16) are repeated almost literally, and we again arrive at the representation (5.17). The substitution of this formula into Equation (4.19) and writing the terms up to $O(n^{-1})$ gives the analogue of the identity (5.18)

$$\xi_{n,1}^\varepsilon=\zeta_{n,1}+\frac{\xi_{n,2}^\varepsilon}{\pi n},\qquad \zeta_{n,1}^\varepsilon:=\frac{a}{2}(1+\varepsilon)\cos\kappa_n\varepsilon-\frac{a}{2}, \tag{5.24}$$

where $\xi_{n,1}^\varepsilon$ are some quantities bounded uniformly in ε and n. Further calculations are similar to (5.18)–(5.20) and lead to the representation (5.22), but for $\varepsilon<0$ with a uniform in ε estimate for the error term and the coefficients

$$\begin{aligned}\zeta_{n,2}^\varepsilon:={}&\frac{a^2\varepsilon(1-\varepsilon^2)}{8}\sin 2\kappa_n\varepsilon,\\\zeta_{n,3}^\varepsilon:={}&-\frac{3a^3\varepsilon^2}{64}(1+\varepsilon)(1-\varepsilon)^2\cos 3\kappa_n\varepsilon-\frac{a^2\varepsilon(2+\varepsilon)}{8}\cos 2\kappa_n\varepsilon\\&+\frac{a^2}{64}(1+\varepsilon)(16+3a\varepsilon^2+2a\varepsilon^3-a\varepsilon^4)\cos\kappa_n\varepsilon-\frac{a^2}{8}(2+3\varepsilon+\varepsilon^2).\end{aligned} \tag{5.25}$$

These formulas and (5.22), (4.2) imply the asymptotics (2.8), (2.9).

The above calculations have fixed a choice of a sufficiently large number R_1 in the definition (4.8) of the set Ω_0. Now let $k\in\Omega_1$ with the same R_1. We rewrite Equations (4.16) and (4.19) in the form

$$F_+(\varepsilon,k)=0\quad\text{for}\quad\varepsilon>0,\qquad F_-(\varepsilon,k)=0\quad\text{for}\quad\varepsilon<0, \tag{5.26}$$

where

$$
\begin{aligned}
F_+(\varepsilon,k) :=& \big(e^{\mathrm{i}(1-\varepsilon)\tau^+} - e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\sqrt{k^2+a}\sin\sqrt{k^2+a}\varepsilon \\
&- \mathrm{i}\big(\tau^+ e^{\mathrm{i}(1-\varepsilon)\tau^+} - \tau^- e^{\mathrm{i}(1-\varepsilon)\tau^-}\big)\cos\sqrt{k^2+a}\varepsilon = 0,
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
F_-(\varepsilon,k) :=& \left(\frac{e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)}}{\tau^+(\varepsilon,k)} - \frac{e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}}{\tau^-(\varepsilon,k)}\right)\cos\sqrt{k^2+a}\varepsilon \\
&+ \mathrm{i}\big(e^{-\mathrm{i}\tau^+(\varepsilon,k)(1+\varepsilon)} - e^{-\mathrm{i}\tau^-(\varepsilon,k)(1+\varepsilon)}\big)\frac{\sin\sqrt{k^2+a}\varepsilon}{\sqrt{k^2+a}} = 0.
\end{aligned}
\tag{5.28}
$$

Due to Lemma 4.2, the functions $F_\pm(\varepsilon,k)$ are holomorphic in sufficiently small complex ε and $k\in\Omega_0$, the left hand sides of these equations are holomorphic in k and ε functions, and

$$
F_+(\varepsilon,k) = -2\mathrm{i}k\cos k + O(\varepsilon), \qquad F_-(\varepsilon,k) = \frac{2\cos k}{k} + O(\varepsilon).
$$

This is why by the inverse function theorem [21, Thm. 1.3.5, Rem. 1.3.6], for sufficiently small ε Equations (5.26) possess exactly one root k_n^ε in the neighbourhoods of the points κ_n belonging to the domain Ω_0, and these roots are holomorphic in ε. The leading terms in the Taylor series of the roots of these equations are easily found similarly to the proof of Lemma 5.2:

$$
k_n^\varepsilon = \kappa_n - \frac{6a\varepsilon}{\kappa_n} + \frac{a\varepsilon^2}{4}\left(\kappa_n + \frac{2a}{\kappa_n} - \frac{576a}{\kappa_n^3}\right) + O(\varepsilon^3) \tag{5.29}
$$

for $\varepsilon>0$, and

$$
k_n^\varepsilon = \kappa_n - \frac{2a\varepsilon}{\kappa_n} + \frac{a\varepsilon^2}{8}\left(\kappa_n + \frac{2a}{\kappa_n} - \frac{64a}{\kappa_n^3}\right) + O(\varepsilon^3) \tag{5.30}
$$

for $\varepsilon<0$. This implies the relations (2.10)–(2.12).

The above calculations do not prove completely Theorem 2.2. The reason is that the solutions to the equation in (4.3) on the interval $(0,1-\varepsilon)$ and to the equation in (4.4) on the interval $(|\varepsilon|,1)$ are sought in the form (4.13) and (4.17) without having a statement that this is a general solution. Moreover, this statement is wrong; we have already mentioned before Lemma 4.1 that Equation (4.5) has a much richer family of solutions. This is why to complete the proof of Theorem 2.2 we need to prove that for sufficiently large n the operator $\mathcal{H}^\varepsilon$ has no eigenvalues except for the found ones. In order to do this, we need to study the behavior of the eigenfunctions associated with the found eigenvalues, and this will also prove Theorem 2.4. Such a study will be performed in the next section.

6. Eigenfunctions

Let k_n^ε be one of the roots of Equation (4.16). Then system of the linear equations (4.15) has a nontrivial solution. For large $n\geqslant N$ the asymptotics (5.8), (5.9), (5.22) yield that

$$
\begin{aligned}
&\big(e^{\mathrm{i}(1-\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)} - e^{\mathrm{i}(1-\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}\big) = 2(-1)^n\mathrm{i}\cos\kappa_n\varepsilon + O(n^{-1}),\\
&\mathrm{i}\frac{\tau^+ e^{\mathrm{i}(1-\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)} - \tau^- e^{\mathrm{i}(1-\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}}{\sqrt{(k_n^\varepsilon)^2+a}} = 2(-1)^n\mathrm{i}\sin\kappa_n\varepsilon + O(n^{-1}),\\
&\sin\sqrt{(k_n^\varepsilon)^2+a}\varepsilon = \sin\kappa_n\varepsilon + O(n^{-1}), \qquad \cos\sqrt{(k_n^\varepsilon)^2+a}\varepsilon = \cos\kappa_n\varepsilon + O(n^{-1}).
\end{aligned}
$$

This is why the rank of matrix of the linear system (4.15) is equal to one, and it has a unique linearly independent solution. This solution generates a unique eigenfunction $\psi_n^\varepsilon(x)$ associated with the eigenvalue λ_n^ε by the formulas (4.14), (4.15).

We multiply the second equation in the system by $\frac{\mathrm{i}}{\sqrt{k^2+a}}$ and sum it with the first equation. In the obtained equation we let $C_2=(-1)^n\sqrt{2}$ and express C_1. Then we have the solution

$$\begin{aligned}C_1=&(-1)^n\sqrt{2}e^{\mathrm{i}\sqrt{(k_n^\varepsilon)^2+a}\varepsilon}\\ &\cdot\left(\left(1-\frac{\tau^+(k_n^\varepsilon,\varepsilon)}{\sqrt{(k_n^\varepsilon)^2+a}}\right)e^{\mathrm{i}(1-\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)}-\left(1-\frac{\tau^-(k_n^\varepsilon,\varepsilon)}{\sqrt{(k_n^\varepsilon)^2+a}}\right)e^{\mathrm{i}(1-\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}\right)^{-1}.\end{aligned}\tag{6.1}$$

The expansions (5.8), (5.9), (5.22) allow us to obtain similar expansions for C_1 as $n\to+\infty$

$$C_1=-\frac{\mathrm{i}}{\sqrt{2}}+\frac{\mathrm{i}a}{2\sqrt{2}\pi n}(1-\varepsilon)\sin\kappa_n\varepsilon+O(n^{-2}),\tag{6.2}$$

where the estimate for the error term is uniform in ε. We substitute this formula and the identity $C_2=(-1)^n$ into (4.14), (4.15), and in view of the asymptotics (5.22) with the coefficients in (5.18), (5.19), (5.21) we write the leading terms of the asymptotics of the eigenfunctions associated with the eigenvalues λ_n^ε. After technical and long but simple calculations we get

$$\psi_n^\varepsilon(x)=\sqrt{2}\sin\kappa_n x-\frac{a(1-\varepsilon)}{\sqrt{2}\pi n}(1-x)\cos\kappa_n x\cos\kappa_n\varepsilon+O(n^{-2}),$$

in the norm of $C^2[1-\varepsilon,1]$, and

$$\begin{aligned}\psi_n^\varepsilon(x)=\sqrt{2}\sin\kappa_n x+\frac{a}{2\sqrt{2}\pi n}\big(&((1-\varepsilon)x-1+\varepsilon)\cos\kappa_n(x-\varepsilon)\\ &-(x(1+\varepsilon)-1+\varepsilon)\cos\kappa_n(x+\varepsilon)\big)+O(n^{-2})\end{aligned}$$

in the norm of $C^2[0,1-\varepsilon]$, where the estimates for the error terms are uniform in ε. This implies the asymptotics (2.13), (2.14).

In the case $\varepsilon<0$ and the system (4.18) the calculations are similar. The rank of the matrix of this system is again one, and to obtain the required nontrivial solution, we multiply the first equation by i and sum it with the second equation. Then we let $C_1=\sqrt{2}$ and express C_2 :

$$\begin{aligned}C_2=&-\mathrm{i}\sqrt{2}e^{\mathrm{i}\sqrt{(k_n^\varepsilon)^2+a}\varepsilon}\\ &\cdot\left(\left(1-\frac{\sqrt{(k_n^\varepsilon)^2+a}}{\tau^+(k_n^\varepsilon,\varepsilon)}\right)e^{-\mathrm{i}(1+\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)}-\left(1-\frac{\sqrt{(k_n^\varepsilon)^2+a}}{\tau^-(k_n^\varepsilon,\varepsilon)}\right)e^{-\mathrm{i}(1+\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}\right)^{-1}.\end{aligned}\tag{6.3}$$

The asymptotics of this coefficient as $n\to+\infty$ turns out to be

$$C_2=\frac{(-1)^n}{\sqrt{2}}+\frac{(-1)^n a}{2\sqrt{2}\pi n}(1+\varepsilon)\sin\kappa_n\varepsilon+O(n^{-2}).\tag{6.4}$$

We substitute $C_1=\sqrt{2}$ and the formula (6.3) into (4.17), and in view of the asymptotics (6.4) and (5.22) with the coefficients in (5.24), (5.25) we write the leading terms in the asymptotics of the eigenfunctions associated with the eigenvalues λ_n^ε. As a result we obtain

$$\psi_n^\varepsilon(x)=\sqrt{2}\sin\kappa_n x+\frac{a}{\sqrt{2}\pi n}(1+\varepsilon)x\cos\kappa_n\varepsilon\cos\kappa_n x+O(n^{-2})\tag{6.5}$$

in the norm of $C^2[0,|\varepsilon|]$, and

$$\begin{aligned}\psi_n^\varepsilon(x)=\sqrt{2}\sin\kappa_n x-\frac{a}{2\sqrt{2}\pi n}\big(&(x(1-\varepsilon)+2\varepsilon)\cos\kappa_n(x+\varepsilon)\\ &-x(1+\varepsilon)\cos\kappa_n(x-\varepsilon)\big)+O(n^{-2})\end{aligned}\tag{6.6}$$

in the norm of $C^2[|\varepsilon|,1]$, where the estimates for the error terms are uniform in ε. This implies the asymptotics (2.13), (2.15).

Now let $n \leqslant N$. Then the expansions (5.29), (5.30) hold for the roots k_n^ε, and each of the systems (4.15), (4.18) again has one nontrivial solution. We choose these solutions as above taking the aforementioned linear combinations of the equations in the systems (4.15), (4.18). At the same time, we choose a nontrivial solution to the first system letting $C_1 = 1$ and

$$\begin{aligned}
C_1 &= -\frac{\mathrm{i}}{\sqrt{2}},\\
C_2 &= -\frac{\mathrm{i}}{\sqrt{2}} e^{-\mathrm{i}\sqrt{(k_n^\varepsilon)^2+a}\varepsilon}\Bigg(\bigg(1-\frac{\tau^+(k_n^\varepsilon,\varepsilon)}{\sqrt{(k_n^\varepsilon)^2+a}}\bigg)e^{\mathrm{i}(1-\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)}\\
&\qquad -\bigg(1-\frac{\tau^-(k_n^\varepsilon,\varepsilon)}{\sqrt{(k_n^\varepsilon)^2+a}}\bigg)e^{\mathrm{i}(1-\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}\Bigg).
\end{aligned}$$

For the system (4.18) we let

$$\begin{aligned}
C_2 &= \frac{(-1)^n\sqrt{2}k_n^\varepsilon}{\sqrt{(k_n^\varepsilon)^2+a}},\\
C_1 &= \mathrm{i}C_2 e^{-\mathrm{i}\sqrt{(k_n^\varepsilon)^2+a}\varepsilon}\Bigg(\bigg(1-\frac{\sqrt{(k_n^\varepsilon)^2+a}}{\tau^+(k_n^\varepsilon,\varepsilon)}\bigg)e^{-\mathrm{i}(1+\varepsilon)\tau^+(k_n^\varepsilon,\varepsilon)}\\
&\qquad -\bigg(1-\frac{\sqrt{(k_n^\varepsilon)^2+a}}{\tau^-(k_n^\varepsilon,\varepsilon)}\bigg)e^{-\mathrm{i}(1+\varepsilon)\tau^-(k_n^\varepsilon,\varepsilon)}\Bigg).
\end{aligned}$$

We substitute these formulas into (4.13), (4.14), (4.17), and in view of the expansions (5.29) and (5.30) we write similar expansions for the eigenfunctions as $\varepsilon \to +0$. As a result, for $\varepsilon > 0$ we obtain

$$\psi_n^\varepsilon(x) = \sqrt{2}\sin\varkappa_n x - \frac{\varepsilon a x}{2\sqrt{2}\varkappa_n}\big(\pi\sin\varkappa_n x + 6\cos\varkappa_n x\big) + O(\varepsilon^2)$$

in the norm of $C[0,1-\varepsilon]$, and

$$\begin{aligned}
\psi_n^\varepsilon(x) =& (-1)^n\sqrt{2}\cos\sqrt{\varkappa_n^2+a}(x-1)\\
&+(-1)^n\varepsilon a\Bigg(-\frac{1}{\sqrt{2}}\cos\sqrt{\varkappa_n^2+a}(x-1)+\frac{3\sqrt{2}(x-1)}{\sqrt{\varkappa_n^2+a}}\sin\sqrt{\varkappa_n^2+a}(x-1)\Bigg)+O(\varepsilon^2)
\end{aligned}$$

in the norm of $C[1-\varepsilon,1]$, where the estimates for the error terms are, generally speaking, non–uniform in ε. In the case $\varepsilon < 0$ similar formulas read as

$$\begin{aligned}
\psi_n^\varepsilon(x) =& \frac{\sqrt{2}\varkappa_n}{\sqrt{\varkappa_n^2+a}}\sin\sqrt{\varkappa_n^2+a}x\\
&+\sqrt{2}\varepsilon a\Bigg(\bigg(\frac{\varkappa_n}{\sqrt{\varkappa_n^2+a}}-\frac{\mathrm{i}}{\varkappa_n}-\frac{3a}{(\varkappa_n^2+a)^{\frac{3}{2}}}\bigg)\sin\sqrt{\varkappa_n^2+a}x\\
&\qquad -\frac{3a\varkappa_n}{2(\varkappa_n^2+a)}x\cos\sqrt{\varkappa_n^2+a}x\Bigg)+O(\varepsilon^2)
\end{aligned}$$

in the norm of $C[0,|\varepsilon|]$, and

$$\psi_n^\varepsilon(x) = \sqrt{2}\sin\varkappa_n x - \frac{\varepsilon a}{\sqrt{2}}\bigg(\frac{3x-2}{\varkappa_n}\cos\varkappa_n x + (x-1)\sin\varkappa_n x\bigg) + O(\varepsilon^2)$$

in the norm of $C[|\varepsilon|,1]$, where the estimates for the error terms are, generally speaking, non–uniform in ε. The obtained relations prove the asymptotics (2.16)–(2.18).

Now we are going to show that for sufficiently small ε the eigenfunctions ψ_n^ε, $n \in \mathbb{Z}_+$, form a basis in $L_2(0,1)$. We first note that the functions $\psi_n^0(x) := \sqrt{2}\sin\varkappa_n x$ form an orthonormalized basis in $L_2(0,1)$. We represent each of the functions $\psi_n^\varepsilon(x)$ as

$$\psi_n^\varepsilon(x) = \psi_n^0(x) + \phi_n^\varepsilon(x), \tag{6.7}$$

and we note that the asymptotics (2.13), (2.16) immediately imply the estimates

$$\|\phi_n^\varepsilon\|_{L_2(0,1)} \leqslant \frac{c_4}{n}, \qquad n \geqslant N, \tag{6.8}$$

$$\|\phi_n^\varepsilon\|_{L_2(0,1)} \leqslant c_5|\varepsilon|^{\frac{1}{2}}, \qquad n \leqslant N. \tag{6.9}$$

Here the choice of the number N is determined by the asymptotics (2.13), namely, this is a number independent of ε such that as $n \geqslant N$, the asymptotic identity (2.13) and the estimate (6.8) hold with a constant c_4 independent on ε, n, and the choice of N. Having fixed N, then we choose a sufficiently small $\varepsilon_0 = \varepsilon_0(N)$ so that for $|\varepsilon| < \varepsilon_0$ the asymptotics (2.16) hold for all $n \leqslant N$ as well as the estimates (6.9) with a constant c_5 independent of ε and n.

Since the functions ψ_n^0 form a basis in $L_2(0,1)$, each of the functions ϕ_n^ε can be expanded over this basis

$$\phi_n^\varepsilon = \sum_{m=0}^{\infty} \alpha_{mn}^\varepsilon \psi_m^0, \qquad \alpha_{mn}^\varepsilon := (\phi_n^\varepsilon, \psi_m^0)_{L_2(0,1)}. \tag{6.10}$$

In the space $L_2(0,1)$ we introduce an operator acting by the rule

$$\mathcal{A}u = \sum_{m=0}^{\infty} \psi_m^0 \sum_{n=0}^{\infty} \alpha_{mn}^\varepsilon u_n, \tag{6.11}$$

where the coefficients u_n are determined by the expansion of the function u over the basis $\{\psi_n^0\}$

$$u = \sum_{n=0}^{\infty} u_n \psi_n^0. \tag{6.12}$$

Let us show that the operator $\mathcal{A}$ is well–defined and its norm is small for small ε.

We let

$$v_m^\varepsilon := \sum_{n=0}^{\infty} \alpha_{mn}^\varepsilon u_n. \tag{6.13}$$

Since u_n and α_{mn}^ε the coefficients of expansion of the functions u, $\phi_n^\varepsilon \in L_2(0,1)$ over the basis $\{\psi_n^0\}$, then the series in the definition of the numbers v_m^ε converges and the sequence $\{v_m^\varepsilon\}$ is well–defined. Cauchy–Schwarz inequality and the estimates (6.8), (6.9) immediately imply the inequality

$$|v_m^\varepsilon| \leqslant \left(\sum_{n=0}^{\infty} |u_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |\alpha_{mn}^\varepsilon|^2\right)^{\frac{1}{2}} \leqslant \|u\|_{L_2(0,1)} \|\phi_m^\varepsilon\|_{C[0,1]} \leqslant \|u\|_{L_2(0,1)} \begin{cases} c_4 n^{-1}, & n \geqslant N, \\ c_5|\varepsilon|^{\frac{1}{2}}, & n \leqslant N. \end{cases}$$

This yields

$$\sum_{m=0}^{\infty} |v_m^\varepsilon|^2 \leqslant \left(c_5^2|\varepsilon|N + c_4^2 \sum_{m=N}^{\infty} n^{-2}\right) \|u\|_{L_2(0,1)}^2. \tag{6.14}$$

Since the series $\sum\limits_{m=0}^{\infty} n^{-2}$ converges, we choose and fix sufficiently large N to ensure the inequality

$$c_4^2 \sum_{n=N}^{\infty} n^{-2} \leqslant \frac{1}{8}.$$

Then we choose a sufficiently small ε_0 so that for $|\varepsilon| < \varepsilon_0$ one more inequality

$$c_5^2|\varepsilon|N \leqslant \frac{1}{8}$$

is satisfied. These two inequalities and (6.14) give

$$\sum_{m=0}^{\infty} |v_m^\varepsilon|^2 \leqslant \frac{1}{4}\|u\|^2_{L_2(0,1)}.$$

Due to (6.11), (6.13) this implies immediately that the series in the definition of the operator $\mathcal{A}$ converges in $L_2(0,1)$, and this means that the operator $\mathcal{A}$ is well–defined, and the estimate

$$\|\mathcal{A}u\|_{L_2(0,1)} \leqslant \frac{1}{2}\|u\|_{L_2(0,1)} \tag{6.15}$$

holds.

It follows from the definition of the operator $\mathcal{A}$ and the formula (6.7) that

$$\psi_n^\varepsilon = (\mathcal{I}+\mathcal{A})\psi_n^0,$$

where $\mathcal{I}$ is the unit operator in $L_2(0,1)$. The estimate (6.15) allows us to state the existence of the inverse bounded operator $(\mathcal{I}+\mathcal{A})^{-1}$ on the space $L_2(0,1)$. Thus, we have a bounded operator possessing a bounded inverse, which maps the basis $\{\psi_n^0\}$ into the system of functions $\{\psi_n^\varepsilon\}$. Therefore, the second system of functions is also a basis in $L_2(0,1)$. This implies that the operator $\mathcal{H}^\varepsilon$ has no other eigenvalues except for the above constructed functions ψ_n^ε. Therefore, the set of the eigenvalues λ_n^ε described in Theorem 2.2 exhausts the entire spectrum of the operator $\mathcal{H}^\varepsilon$. The proof of Theorems 2.2, 2.3, 2.4 are complete.

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