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ASYMPTOTICS FOR EIGENVALUES OF STURM-LIOUVILLE OPERATOR WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. We employ the similar operators method for studying the spectral properties of the Sturm—Liouville operator generated by the differential expression $l(y) = -y'' - vy$ with a complex potential v and subject to periodic boundary conditions $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$. We obtain the results on the asymptotics for the spectrum of the operator.

Keywords: similar operators method, Sturm—Liouville operator, the spectrum of operator, asymptotics for the spectrum.

Mathematics Subject Classification: 34L20, 34L40, 47E05

1. Introduction

Let $L_2[0, 2\pi]$ be the Hilbert space of complex-valued square integrable functions defined on $[0, 2\pi]$ with the scalar product of the form

$$(x, y) = \frac{1}{2\pi} \int_0^{2\pi} x(\tau)\overline{y(\tau)}d\tau, \qquad x, y \in L_2[0, 2\pi].$$

By $W_2^2[0, 2\pi]$ we denote the Sobolev space $\{y \in L_2[0, 2\pi] : y'$ is absolutely continuous and $y'' \in L_2[0, 2\pi]\}$.

We consider a one-dimensional Strum-Liouville operator $L : D(L) \subset L_2[0, 2\pi] \to L_2[0, 2\pi]$ defined by the differential expression

$$l(y) = -y'' - vy$$

on the domain $y \in D(L) = \{y \in W_2^2[0, 2\pi] : y(0) = y(2\pi), y'(0) = y'(2\pi)\}$ introduced by periodic boundary conditions. It is assumed that potential v belongs to $L_2[0, 2\pi]$ and $v(t) = \sum\limits_{k\in\mathbb{Z}} v_k e^{ikt}$, $t \in [0, 2\pi]$, is its Fourier series.

Operator L can be represented as $L = A-B$, where operator $A : D(A) = D(L) \subset L_2[0, 2\pi] \to L_2[0, 2\pi]$ is defined by the differential expression

$$l_0(y) = -y'',$$

and operator B is the multiplication by potential v. It is well-defined by the condition $D(B) \supset D(A)$. Operator B will play the role of a perturbation.

Operator A is self-adjoint and it has a compact resolvent. Its spectrum $\sigma(A)$ reads as $\sigma(A) = \{n^2, n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\}$, $E_n^0 = \mathrm{Span}\,\{e_n^{(1)}, e_n^{(2)}\}$ is the eigenspace for an eigenvalue n^2 , $n \neq 0$, where $e_n^{(1)}(t) = e_n(t) = e^{int}, e_n^{(2)}(t) = e_{-n}(t) = e^{-int}$; $E_0^0 = \{\alpha\}, \alpha \in \mathbb{C}$.

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To study the spectral properties of the Sturm-Liouville operator, in the present paper we employ the similar operators method developed in [1]–[6]. The matter of this method is the similarity trasformation of the studied operator into an operator whose spectral properties are close to the spectral properties of the unperturbed operator. In this way the study of operator L is essentially simplified.

One of the main results of this paper is Theorem 1, where we obtain the specified asymptotics for the eigenvalues of operator L. In the proof of this theorem we employ the following two-sided sequences of complex numbers:

$$c_{n,n} = \sum_{\substack{j\in\mathbb{Z}\\ |j|\neq|n|}} v_{n-j}\frac{v_{j-n}}{j^2-n^2}, \qquad c_{-n,-n} = \sum_{\substack{j\in\mathbb{Z}\\ |j|\neq|n|}} v_{-(n+j)}\frac{v_{j+n}}{j^2-n^2},$$

$$c_{-n,n} = \sum_{\substack{j\in\mathbb{Z}\\ |j|\neq|n|}} v_{-(n+j)}\frac{v_{j-n}}{j^2-n^2}, \qquad c_{n,-n} = \sum_{\substack{j\in\mathbb{Z}\\ |j|\neq|n|}} v_{n-j}\frac{v_{j+n}}{j^2-n^2}, n\in\mathbb{Z}.$$

We observe that $c_{n,n} = c_{-n,-n}$.

Theorem 1. *There exists a number $m\in\mathbb{Z}_+$ such that the spectrum of operator L is represented as*

$$\sigma(L) = \sigma_{(m)} \bigcup \left(\bigcup_{n\geqslant m+1} \sigma_n \right), \tag{1}$$

where $\sigma_{(m)}$ is a finite set with total amount of elements not exceeding $2m+1$, while the sets $\sigma_n = \{\lambda_n^+, \lambda_n^-\}$, $n\geqslant m+1$, contain at most two points and are defined by the identities

$$\lambda_n^\pm = n^2 + v_0 - \frac{1}{2n}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{v_k v_{-k}}{k} \pm \sqrt{v_{2n}v_{-2n}} + \frac{\beta_n^\pm}{\sqrt{n}}, \qquad n\geqslant m+1, \tag{2}$$

where sequence $\beta_n^\pm$ possesses the property $\sum\limits_{n\geqslant m+1} |\beta_n^\pm|^{\frac{4}{3}} < \infty$.

We mention that in paper by V. Tkachenko [7, Thm. 3.6] there was given an asymptotics for the spectrum of operator L:

$$\lambda_n^\pm = n^2 + v_0 + \alpha_n^\pm, \qquad n\to\infty, \tag{3}$$

where $v_0 = \frac{1}{2\pi}\int\limits_0^{2\pi} v(t)dt$ is the mean of potential v, and $\sum\limits_{n=0}^{\infty} |\alpha_n^\pm|^2 < \infty$.

The asymptotics for the spectrum in Theorem 1 is more precise by the order in comparison with the asymptotics in formula (3). Indeed, we write one more calculated approximation that increases the order of the error term.

In the case of real potential v the asymptotics for the spectrum of operator L was given in monogrpaph by V.A. Marchenko [8, Thm.1.5.2]. If potential v is real, we have the following statement.

Theorem 2. *There exists a number $m\in\mathbb{Z}_+$ such that the spectrum of operator L is represented as*

$$\sigma(L) = \sigma_{(m)} \bigcup \left(\bigcup_{n\geqslant m+1} \sigma_n \right), \tag{4}$$

where $\sigma_{(m)}$ is a finite set with the total amount of the elements not exceepding $2m+1$, and the sets $\sigma_n = \{\lambda_n^+, \lambda_n^-\}$, $n\geqslant m+1$, contain at most two points and are defined by the identities

$$\lambda_n^\pm = n^2 + v_0 - \frac{1}{2n}\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{|v_k|^2}{k} \pm |v_{2n}| + \frac{\beta_n^\pm}{n}, \qquad n\geqslant m+1, \tag{5}$$

where sequence $\beta_n^{\pm}$ *possesses the property* $\sum\limits_{n\geqslant m+1}|\beta_n^{\pm}|<\infty$.

2. Preliminary similarity transformations

Let $\mathcal{H}$ be a separable Hilbert space. By $\mathrm{End}\,\mathcal{H}$ we indicate the Banach algebra of linear bounded operators in $\mathcal{H}$. A compact operator $X\in\mathrm{End}\,\mathcal{H}$ is called Hilbert-Schmidt operator (see[9]), if the trace of the self-adjoint operator XX^* is finite, i.e., $\mathrm{tr}\,(XX^*)<\infty$. The set of Hilbert-Schmidt operators is a two-sided ideal $\mathfrak{S}_2(\mathcal{H})$ (see [9]) in algebra $\mathrm{End}\,\mathcal{H}$. Ideal $\mathfrak{S}_2(\mathcal{H})$ is a Hilbert space with the scalar product $<X,Y>=\mathrm{tr}\,(X,Y^*)$, $X,Y\in\mathfrak{S}_2(\mathcal{H})$.

The symbol $\|X\|_2$ indicates the Hilbert-Schmidt norm of operator $X\in\mathfrak{S}_2(\mathcal{H})$, i.e., $\|X\|_2^2=tr(XX^*)$. We note that if $e_1,e_2,\dots,e_n,\dots$ is an arbitrary orthonormalized basis in $\mathcal{H}$, operator $X\in\mathrm{End}\mathcal{H}$ is a Sturm-Liouville operator if and only if $\|X\|_2^2=\sum\limits_{i,j\geqslant1}|(Xe_j,e_i)|^2<\infty$ (see [9]). Here we can introduce the ideal $\mathfrak{S}_1(\mathcal{H})$ of nuclear operators.

Definition 1. *Two linear operators* $\mathcal{A}_i:D(\mathcal{A}_i)\subset\mathcal{H}\to\mathcal{H}$, $i=1,2$, *are called* similar, *if there exists a continuously invertible operator* $U\in End\mathcal{H}$ *such that* $UD(\mathcal{A}_2)=D(\mathcal{A}_1)$ *and* $\mathcal{A}_1Ux=U\mathcal{A}_2x$, $x\in D(\mathcal{A}_2)$. *Operator* U *is called* transformation operator *of operator* $\mathcal{A}_1$ *into* $\mathcal{A}_2$.

It is important to note that similar operator have the same spectrum. This fact is permanently used here in the employed similarity transformations.

We return back to considering differential operator $L=A-B$. In what follows we consider Hilbert space $\mathcal{H}=L_2[0,2\pi]$ and the system of orthoprojectors $P_n:L_2[0,2\pi]\to L_2[0,2\pi]$, $n\in\mathbb{Z}_+$, of the form:

$$P_nx=(x,e_n)e_n+(x,e_{-n})e_{-n},n\in\mathbb{N},\qquad P_0x=(x,e_0)e_0. \tag{6}$$

We note that $AP_n=\lambda_nP_n$, $n\geqslant0$.

By the symbol ΓB we denote the Hilbert-Schmidt operator

$$((\Gamma B)x)(s)=\frac{1}{2\pi}\int\limits_0^{2\pi}G(s,\tau)x(\tau)d\tau,\qquad x\in\mathcal{H},$$

where

$$G(s,\tau)=\begin{cases}\frac14(-u(\frac{s+\tau}{2})+2(\frac{s-\tau}{2\pi})u_2(\frac{s+\tau}{2})), & \tau\leqslant s,\\ \frac14(u(\frac{s+\tau}{2})+2(\frac{s-\tau}{2\pi})u_2(\frac{s+\tau}{2})), & \tau>s,\end{cases} \tag{7}$$

$$u(s)=u_1(s)+u_2(s),\quad u_1(s)=\sum_{\substack{k\in2\mathbb{Z}+1\\k\neq0}}\frac{v_k}{ik}e^{iks},\quad u_2(s)=\sum_{\substack{k\in2\mathbb{Z}\\k\neq0}}\frac{v_k}{ik}e^{iks}.$$

In what follows we make the assumption $v_0=\frac{1}{2\pi}\int\limits_0^{2\pi}v(t)dt=0$, which is not restrictive since shifting the potential by a constant makes the same for the spectrum and does not change the eigenfunctions. However, in the formulation of theorems on asymptotics for the eigenvalues this constant is taken into consideration.

Together with ΓB, in Lemma 1 we make use of the operator $JB\in\mathfrak{S}_2(\mathcal{H})$ being

$$((JB)x)(s)=\frac{1}{2\pi}\int\limits_0^{2\pi}v(s+\tau)x(\tau)d\tau,\qquad x\in\mathcal{H}.$$

Let $k \in \mathbb{Z}_+$. We define the operators

$$J_k B = JB - J(P_{(k)} B P_{(k)}) + P_{(k)} B P_{(k)}, \tag{8}$$

$$\Gamma_k B = \Gamma B - \Gamma(P_{(k)} B P_{(k)}), \tag{9}$$

where projector P_k is defined by identity (6) and $P_{(k)} = \sum\limits_{|j| \leqslant k} P_j$.

It is clear that $J_0 B = JB, \Gamma_0 B = \Gamma B$. By the definition of operators $J_k B$ and $\Gamma_k B$ we obtain the following representations

$$J_k B = JB - P_{(k)} JBP_{(k)} + P_{(k)} BP_{(k)}, \qquad \Gamma_k B = \Gamma B - (P_{(k)} \Gamma BP_{(k)}) \tag{10}$$

which imply $J_k B, \Gamma_k B \in \mathfrak{S}_2(\mathcal{H})$ for each $k \geqslant 0$.

The proof of the next lemma reproduces that of Lemma 7 in paper [6].

Lemma 1. *Operators $\Gamma B, JB, B$ satisfy the following conditions:*
*(**a**)$\Gamma B \in \operatorname{End} \mathcal{H}$ and $\|\Gamma B\| < 1$;*
*(**b**)$(\Gamma B) D(A) \subset D(A)$;*
*(**c**)$B\Gamma B, (\Gamma B) JB \in \mathfrak{S}_2(\mathcal{H})$;*
*(**d**) $A(\Gamma B)x - (\Gamma B)Ax = Bx - (JB)x$, $x \in D(A)$;*
*(**e**) for each $\varepsilon > 0$ there exists a number $\lambda_\varepsilon \in \rho(A)$ such that $\|B(A - \lambda_\varepsilon I)^{-1}\| < \varepsilon$.*

The proof of the next theorem can be made in the same way as for Theorem 2 in paper [6].

Theorem 3. *If number $k \in \mathbb{Z}_+$ is so that*

$$\|\Gamma_k B\|_2 < 1, \tag{11}$$

then operator $L = A - B$, where $A = L_0$, B is the multiplication by potential v, is similar to operator

$$\widetilde{L} = L_0 - \widetilde{B},$$

where

$$\widetilde{B} = \widetilde{B}_k = J_k B + (I + \Gamma_k B)^{-1} (B \Gamma_k B - (\Gamma_k B) J_k B),$$

and the identity

$$(A - B)(I + \Gamma_k B) = (I + \Gamma_k B)(A - \widetilde{B}) \tag{12}$$

holds true. Operators $J_k B, \Gamma_k B, B\Gamma_k B, (\Gamma_k B)(J_k B), \widetilde{B}, \widetilde{B}_k$ are Hilbert-Schmidt operators in $\mathfrak{S}_2(L_2[0, 2\pi])$. Operator $\widetilde{B}$ in (12) can be represented as

$$\widetilde{B} = JB + B\Gamma B - (\Gamma B) JB + C \in \mathfrak{S}_2(L_2[0, 2\pi]), \tag{13}$$

where operator C belongs to ideal $\mathfrak{S}_1(L_{2,\pi})$ of nuclear operators [9] defined on $L_2[0, 2\pi]$.

The result obtained in Theorem 3 allows us to reduce the study of operator $L = A - B$ to studying operator $A - \widetilde{B}$, where operator $\widetilde{B}$ is a Hilbert-Schmidt one. Thus, $\sigma(A - B) = \sigma(A - \widetilde{B})$.

To formulate Theorem 4, we introduce transformers (i.e., linear operators in a space of linear operators; this is a terminology of M.G. Krein) $J, \Gamma : \mathfrak{S}_2(L_2[0, 2\pi]) \to \mathfrak{S}_2(L_2[0, 2\pi])$ with the following properties:

1) J is a projector, $\|J\| = 1$, and it is represented as absolutely converging in uniform operator topology series

$$JX = \sum_{n=0}^{\infty} P_n X P_n = X_0, \qquad X \in \mathfrak{S}_2(L_2[0, 2\pi]). \tag{14}$$

2) Transformer Γ is well-defined on each operator $X \in \mathfrak{S}_2(L_2[0,2\pi])$ by the identity (cf. [6])

$$\Gamma X = \sum_{\substack{i,j\geqslant 0\\ i\neq j}} \frac{P_i X P_j}{\lambda_i - \lambda_j}. \tag{15}$$

It follows from (14) and (15) that

$$\|JX\|_2^2 = \sum_{n=0}^{\infty} \|P_n X P_n\| \leqslant \|X\|_2^2, \qquad \|\Gamma X\|_2^2 = \sum_{\substack{i,j\geqslant 0\\ i\neq j}} \frac{\|P_i X P_j\|_2^2}{|\lambda_i - \lambda_j|^2} \leqslant \gamma_0^{-1}\|X\|_2^2,$$

where $\gamma_0 = \inf\limits_{\substack{i\neq j\\ i,j\geqslant m}} |\lambda_i - \lambda_j|$.

We consider the sequences of transformers $(J_m), (\Gamma_m)$, $m \in \mathbb{Z}_+$, defined by the identities

$$J_m X = P_{(m)} X P_{(m)} + \sum_{|k|\geqslant m+1} P_k X P_k = J(X - P_{(m)} X P_{(m)}) + P_{(m)} X P_{(m)},$$
$$\Gamma_m X = \Gamma(X - P_{(m)} X P_{(m)}),$$

where $X \in \mathfrak{S}_2(\mathcal{H})$. We note that J_m is a projector. Since it is a self-adjoint operator, $\|J_m\| = 1$. Transformer Γ_m is anti-self-adjoint, i.e., $\Gamma_m^* = -\Gamma_m$ and $\|\Gamma_m\| = \gamma_0^{-1} = (\inf\limits_{\substack{i\neq j\\ i,j\geqslant m}} |\lambda_i - \lambda_j|)^{-1}$.

We note that in the proof of Theorem 1 we shall make use of the following properties of transformers J_k, Γ_k

$$J_k\left((\Gamma_k X)(J_k Y)\right) = 0, \qquad J_k\left((\Gamma_k X) J_k(Y \Gamma_k X)\right) = 0, \quad k \in \mathbb{Z}_+, \tag{16}$$

where $X, Y \in \mathfrak{S}_2(L_2[0,2\pi])$.

In what follows we shall make use of a compact self-adjoint operator A_0,

$$A_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} P_k + P_0.$$

Theorem 4 ([1],[3],[6])**.** *For each number $k \in \mathbb{Z}_+$ obeying the inequality*

$$\|\widetilde{B}\|_2 = \|\widetilde{B}_k\|_2 < \frac{2k+3}{4} \tag{17}$$

the operator $A - \widetilde{B}$ is similar to the operator $A - J_k\widetilde{X}$, where operator $\widetilde{X}$ solves the nonlinear equation

$$X = \widetilde{B}\Gamma_k X - (\Gamma_k X)(J_k \widetilde{B}) - (\Gamma_k X) J_k(\widetilde{B}\Gamma_k X) + \widetilde{B} = \Phi(X) \tag{18}$$

considered in $\mathfrak{S}_2(L_2[0,2\pi])$. Solution $\widetilde{X}$ can be represented as $X_0 A_0^{-\frac{1}{2}}$, where $X_0 \in \mathfrak{S}_2(L_2[0,2\pi])$ and it can be found by simple iteration methods. The similarity transformation of the operator $A - \widetilde{B}$ into the operator $A - J_k\widetilde{X}$ is made by the invertible operator $I + \Gamma_k\widetilde{X} \in \mathrm{End}(L_2[0,2\pi])$.

3. Proof of Theorem 1

The choice of number $k \in \mathbb{Z}_+$ is due to condition (17) in Theorem 4 notations of which we employ.

Applying transformer J_k to the both sides of equation (18) and employing property (16) of transformers J_k, Γ_k, we obtain:

$$\begin{aligned} J_k\widetilde{X} =& J_k(\widetilde{B}\Gamma_k\widetilde{X}) + J_k\widetilde{B} = J_k\widetilde{B} + J_k(\widetilde{B}\Gamma_k\widetilde{B}) + J_k(\widetilde{B}\Gamma_k(\widetilde{X} - \widetilde{B})) \\ =& J_k\widetilde{B} + J_k(\widetilde{B}\Gamma\widetilde{B}) + K = J_k B + J_k(B\Gamma B) + T_1 = JB + J(B\Gamma B) + T_2, \end{aligned}$$

where operators K, T_1, T_2 are represented as $K = K_0 A_0^{-\frac{1}{2}}, T_1 = T_{1,0} A_0^{-\frac{1}{2}}, T_2 = T_{2,0} A_0^{-\frac{1}{2}}$ and operators K_0, $T_{1,0}$, $T_{2,0}$ belong to the ideal of nuclear operators $\mathfrak{S}_1(L_2[0,2\pi])$. It is clear that $J_k T_j = T_j$, $j = 0,1$. While obtaining these identities we have also employed the following properties: the product of two Hilbert-Schmidt operators is a nuclear operator and the operators $J_k X - JX$, $\Gamma_k X - \Gamma X$, $X \in \mathfrak{S}_2(L_2[0,2\pi])$, $k \geqslant 0$, are of finite rank.

Thus, applying Theorems 3 and 4 to the considered operator $L = A - B$, we obtain that the operator $A - B$ is similar to the operator $A - (JB + J(B\Gamma B) + T_1) = A - B_0$ and $\sigma(A - B) = \sigma(A - B_0)$, where $B_0 = JB + J(B\Gamma B) + T_1, T_1 \in \mathfrak{S}_1(L_2[0,2\pi])$.

The matrix for restriction B_n of operator $P_n B_0 P_n$ on $\mathcal{H}_n$ in the basis e_n, e_{-n} reads as

$$\begin{pmatrix} 0 & v_{2n} \\ v_{-2n} & 0 \end{pmatrix} + \begin{pmatrix} c_{n,n} & c_{n,-n} \\ c_{-n,n} & c_{-n,-n} \end{pmatrix} + \frac{1}{n} \begin{pmatrix} f_1(n) & f_2(n) \\ f_3(n) & f_4(n) \end{pmatrix},$$

where f_1, f_2, f_3, f_4 are summable sequences.

The eigenvalues $\mu_n^{\pm}$ of operator B_n are

$$\begin{aligned}
\mu_n^{\pm} =& c_{n,n} + \frac{f_1(n) + f_4(n)}{2n} \pm \\
&\pm \frac{1}{2}\sqrt{\left(\frac{f_1(n) - f_4(n)}{n}\right)^2 + 4\left(v_{2n} + c_{n,-n} + \frac{f_2(n)}{n}\right)\left(v_{-2n} + c_{-n,n} + \frac{f_3(n)}{n}\right)} \\
=& c_{n,n} + \frac{f_1(n) + f_4(n)}{2n} \pm \frac{\sqrt{4(v_{2n} + c_{n,-n})(v_{-2n} + c_{-n,n})}}{2} \pm \\
&\pm (\frac{1}{2}\sqrt{\left(\frac{f_1(n) - f_4(n)}{n}\right)^2 + 4\left(v_{2n} + c_{n,-n} + \frac{f_2(n)}{n}\right)\left(v_{-2n} + c_{-n,n} + \frac{f_3(n)}{n}\right)} \\
&- \frac{\sqrt{4(v_{2n} + c_{n,-n})(v_{-2n} + c_{-n,n})}}{2}).
\end{aligned}$$

Then $\mu_n^{\pm} = c_{n,n} \pm \sqrt{\widetilde{\omega}_n} + \beta_n^{\pm}$, where $\beta_n^{\pm} = \frac{\alpha(n)}{\sqrt{n}}$, $\sum\limits_{|n| \geqslant m+1} |\alpha(n)|^{\frac{4}{3}} < \infty$.

Sequence $c_{n,n}$, $n = 0, 1, \ldots$, can be represented as

$$\begin{aligned}
c_{n,n} =& \sum_{\substack{j \in \mathbb{Z} \\ |j| \neq |n|}} v_{n-j} \frac{v_{j-n}}{j^2 - n^2} = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0 \\ k \neq -2n}} \frac{v_k v_{-k}}{k(k+2n)} \\
=& \frac{1}{2n} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0 \\ k \neq -2n}} \frac{v_k v_{-k}}{k} - \frac{1}{2n} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0 \\ k \neq -2n}} \frac{v_k v_{-k}}{k + 2n} \\
=& \frac{1}{2n} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{v_k v_{-k}}{k} - \frac{1}{2n} \frac{v_{2n} v_{-2n}}{-2n} - \frac{1}{2n} \sum_{\substack{k \in \mathbb{Z} \\ k \neq -2n}} \frac{v_k v_{-k}}{k + 2n} = \frac{1}{2n} \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{v_k v_{-k}}{k} + \omega_n' + \frac{\alpha_n'}{n^2},
\end{aligned}$$

where $\omega_n' = -\frac{1}{2n} \sum\limits_{\substack{k \in \mathbb{Z} \\ k \neq -2n}} \frac{v_k v_{-k}}{k + 2n}$, (α_n') is a summable sequence.

Let us prove that $\sum\limits_{\substack{k \in \mathbb{Z} \\ k \neq -2n}} \left|\frac{v_k v_{-k}}{k+2n}\right|^2 < \infty$. In order to do it, we consider the convolution

$$(\omega * \gamma)(n) = \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \omega(k) \gamma(n - k), n \in \mathbb{Z},$$

of the sequence $\omega : \mathbb{Z} \to \mathbb{C}$, $w(k) = v_k v_{-k}$, $k \in \mathbb{Z}$, with the property $\sum\limits_{k\in\mathbb{Z}} |\omega(k)| < \infty$, with the sequence $\gamma : \mathbb{Z} \to \mathbb{R}$, $\gamma(k) = \begin{cases} \frac{1}{k}, & k \neq 0 \\ 0, & k = 0 \end{cases}$ satisfying $\sum\limits_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\gamma(k)|^2 < \infty$. Then sequence $\omega'_n = -(\omega * \gamma)(-2n)$, $n \in \mathbb{Z}$, as a convolution of a summable sequence with a square integrable sequence is square integrable. Thus, we obtain representation (2).

In the case of real potential v sequence α is summable and we arrive at the statement of Theorem 2.

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