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ON LEVEL SETS OF NORM OF GENERALIZED RESOLVENT OF OPERATORS PENCILS

M.A. MANSOURI, A. KHELLAF, H. GUEBBAI

Abstract. We prove that the generalized resolvent operator defined in a Hilbert space cannot remain constant on any open subset of the resolvent set. Under certain conditions we also prove the same result for a complex uniformly convex Banach space. These results extend the known ones.

Keywords: ε–pseudospectrum, ε–pseudospectrum of operators pencils, generalized spectrum approximation, operator pencil.

Mathematics Subject Classification: 35P15; 47A75; 35J10

1. Introduction

We consider a bounded operator T in a Banach space X. The symbol $\sigma(T)$ denotes the spectrum of operator T. The ε–pseudospectrum of T is defined as

$$\sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| > \varepsilon^{-1}\} \cup \sigma(T)$$

or as

$$\Sigma_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| \geqslant \varepsilon^{-1}\} \cup \sigma(T)$$

where $\varepsilon > 0$. For more details on this concept, see [4], [12], [15], [17]. The difference between $\Sigma_\varepsilon(T)$ and $\sigma_\varepsilon(T)$ is characterized by the ε–level set of T given as

$$L_\varepsilon(T) = \{z \in \mathbb{C} : \|(T - zI)^{-1}\| = \varepsilon^{-1}\}. \tag{1.1}$$

A pertinent question is whether the set $L_\varepsilon(T)$ can contain an open subset. If so, $\Sigma_\varepsilon(T)$ would be significantly larger than the closure of $\sigma_\varepsilon(T)$. This issue remained unresolved for some time, see [6], and was resolved in [14], [5], [3].

For $T, S \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the space of linear bounded operators in a Banach space X, the generalized eigenvalue problem is $Tu = \lambda Su$, where $\lambda \in \mathbb{C}$ and $u \in X \setminus \{0\}$. The generalized resolvent set is defined by

$$\rho(T, S) = \big\{z \in \mathbb{C} : (T - zS)^{-1} \in \mathcal{B}(X)\big\}.$$

The generalized spectrum is defined as $\sigma(T, S) = \mathbb{C} \setminus \rho(T, S)$. The pair (T, S) is generally called regular if $\rho(T, S) \neq \emptyset$, a condition that is always met in this work. For a more detailed explanation of these definitions see [13], [16], [8], [10], [1], [2].

The ε–pseudospectrum of operator pencils of $T, S \in \mathcal{B}(X)$ is defined as

$$\sigma_\varepsilon(T, S) = \{z \in \mathbb{C} : \|(T - zS)^{-1}S\| > \varepsilon^{-1}\} \cup \sigma(T, S) \tag{1.2}$$

or as

$$\Sigma_\varepsilon(T, S) = \{z \in \mathbb{C} : \|(T - zS)^{-1}S\| \geqslant \varepsilon^{-1}\} \cup \sigma(T, S)$$

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where $\varepsilon > 0$. This definition is borrowed from [11], where it was proved that it is a natural generalization of the case $S = I$. It shows that the set defined in (1.2) remains consistent and preserves fundamental properties of the ε–pseudospectrum, see [11, Thms. 2.1, 2.3, 2.4]. For other definitions of the ε–pseudospectrum see [4], [1], [2], [17].

The difference between $\Sigma_\varepsilon(T, S)$ and $\sigma_\varepsilon(T, S)$ is the level set $L_\varepsilon(T, S)$

$$L_\varepsilon(T, S) = \{\lambda \in \mathbb{C} : \|(T - \lambda S)^{-1} S\| = \varepsilon^{-1}\}. \tag{1.3}$$

We address the issue about a condition for the set $L_\varepsilon(T, S)$ ensuring that it contains no open set.

In this paper we prove that the set defined in (1.3) contains no open set when $X = H$ is a Hilbert space, see Theorem 2.1. This result is established under the condition that S is a compact injective operator. Our second main result demonstrate that given a pair (T, S) acting in a complex uniformly convex Banach space, if the generalized resolvent operator defined as $(T - zS)^{-1}S$, $z \in \rho(T, S)$, has a constant norm on an open set, then this constant represents the global minimum, see Theorem 2.2. Theorems 2.3, 2.4 establish the same for a complex uniformly convex Banach space X, namely, the set defined in (1.3) contains no open set provided S is either invertible with $S^{-1} \in \mathcal{B}(X)$, or is compact and injective.

It was shown in [5, Thm. 2.2] that if A is an unbounded operator with a compact resolvent defined on a uniformly convex Banach space, then the set (1.1) contains no open set. Let $\alpha \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. We consider the operators $S = (A - \alpha I)^{-1}$ and $T = (A - \alpha I)^{-1}A$. It was shown in [9, Thms. 2.3, 4.5] that $T, S \in \mathcal{B}(X)$ and

$$\sigma(T, S) = \sigma(A), \quad \sigma_\varepsilon(T, S) = \sigma_\varepsilon(A)$$

for $\varepsilon > 0$. It is important to note that the assumption that S is compact and injective represents a correct generalization and contributes significantly to the existing studies in the literature. This extension promotes further exploration and understanding of the established concepts in the field of operator pencils.

2. Main results

Let $T, S \in \mathcal{B}(X)$. In what follows, if we write $X = H$, then H is a Hilbert space. We begin with providing an example, in which the difference between two definitions of ε–pseudospectrum

$$\{z \in \mathbb{C} : \|(T - zS)^{-1}\| > \varepsilon^{-1}\} \cup \sigma(T, S) \tag{2.1}$$

and

$$\{z \in \mathbb{C} : \|(T - zS)^{-1}\| \geqslant \varepsilon^{-1}\} \cup \sigma(T, S) \tag{2.2}$$

contains open subset. For more details on these definitions see [17]. We introduce the generalized spectral problem as

$$Tu = \lambda S u,$$

where, $H = \mathbb{R}^3$ and

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

It is clear that T and S are degenerate. By elementary matrix calculations we get

$$\|(T - zS)^{-1}\| = \max\left\{1, \frac{4}{|7z+1|}, \frac{3}{|7z+1|} + \frac{|2z-1|}{|7z^2+z|}\right\}, \quad z \in \mathbb{C} \setminus \left\{-\frac{1}{7}, 0\right\}$$

and

$$\|(T - zS)^{-1}S\| = \frac{4|2z-1|}{|7z^2+z|} + \frac{15}{|7z+1|}, \quad z \in \mathbb{C} \setminus \left\{-\frac{1}{7}, 0\right\}.$$

It is clear that, for any open set in $\mathbb{C}$ obeyin the properties $\operatorname{Re} z > 1$ and $\operatorname{Im} z = 0$, we have

$$\|(T - zS)^{-1}\| = 1.$$

Hence, the difference between the sets (2.1) and (2.2) contain an open subsets.

Our first result describes that the set defined in (1.3) contains no open set when $X = H$ is a Hilbert space of infinite dimension. This result is established under the condition that S is a compact injective operator. In what follows, we use the notation $\operatorname{Re}(z, T, S) = (T - zS)^{-1}$ for all $z \in \rho(T, S)$.

Theorem 2.1. *Let $T, S \in \mathcal{B}(H)$, where S is compact and injective operator. Let U be an open subset of $\rho(T, S)$. If*

$$\|\operatorname{Re}(\lambda, T, S)S\| \leqslant M \quad \lambda \in U,$$

then

$$\|\operatorname{Re}(\lambda, T, S)S\| < M \quad \lambda \in U.$$

Let us now study the situation where X is a uniformly convex Banach space, see [7]. In the references [3], [5], [14], [15], this situation was studied in the case $S = I$. Here we generalize these results for the operator pencils. The next theorem states that, for a pair (T, S) acting in a complex uniformly convex Banach space, if the generalized resolvent operator $\operatorname{Re}(z, T, S)S$, $z \in \rho(T, S)$, has a norm that remains constant over an open set, then this constant value represents the global minimum.

Theorem 2.2. *Let T and S belong to $\mathcal{B}(X)$, where X is a complex uniformly convex Banach space. Assume that there exist an open subset $U \subset \rho(T, S)$ and constant $M > 0$ such that*

$$\|\operatorname{Re}(\lambda, T, S)\, S\| = M, \quad \lambda \in U.$$

Then

$$\|\operatorname{Re}(\lambda, T, S)\, S\| \geqslant M \quad \text{for all} \quad \lambda \in \rho(T, S).$$

The next theorem establishes that if X is a uniformly convex Banach space, then the set described by (1.3) contains no open subsets under the condition that S is invertible operator and $S^{-1} \in \mathcal{B}(X)$.

Theorem 2.3. *Let T and S belong to $\mathcal{B}(X)$, where X is a complex uniformly convex Banach space. If S is an invertible operator such that $S^{-1} \in \mathcal{B}(X)$, then there is no open subset in $\rho(T, S)$ such that the function $\|\operatorname{Re}(\,\cdot\,, T, S)S\|$ is constant on it.*

The next theorem establishes that if X is a uniformly convex Banach space, then the set described by (1.3) contains no open set under the condition that S is a compact and injective operator.

Theorem 2.4. *Let X be a complex uniformly convex Banach space and $T, S \in \mathcal{B}(X)$. Assume that the operator S is injective and compact. Then there is no open subset in $\rho(T, S)$ such that the function $\|\operatorname{Re}(\cdot, T, S)S\|$ is constant on it.*

Let us discuss the main results of the work. An example obeying the assumption of the Theorem 2.1 reads as follows: $H = L^2(\Omega)$,, where $\Omega \subset \mathbb{R}^n$ and $n \geqslant 1$. The operators T and S are defined as

$$Tu(t) = u(t) + \int_{\Omega} k_1(t, s)u(s)\, ds, \qquad Su(t) = \int_{\Omega} k_2(t, s)u(s)\, ds,$$

where the functions k_1 and k_2 are the kernels of the integral operators. This case agrees with the results in [8], [11]. Let A be an unbounded operator in X and $\alpha \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. Consider the operators $S = (A - \alpha I)^{-1}$ and $T = (A - \alpha I)^{-1}A$. It was shown in [9, Thms. 2.3, 4.5] that $T, S \in \mathcal{B}(X)$, and additionally, we have

$$\mathrm{Re}(\lambda, T, S)S = \mathrm{Re}(\lambda, A)$$

for all $\lambda \in \rho(A)$. Consequently, if X is a complex uniformly convex Banach space or a Hilbert space, and S is compact operator, then according to Theorems 2.1 and 2.4 there is no subset of $\rho(A)$, on which $\|\mathrm{Re}(\,\cdot\,, A)\|$ remains constant.

3. Case of Hilbert space

In this section we prove Theorem 2.1 and the following lemmas will play a crucial role.

Lemma 3.1. *Given $A \in \mathcal{B}(X)$, let $\|A\| < 1$. Then $(I - A)$ possesses a bounded inverse in X, which is represented by the Neumann series*

$$(I - A)^{-1} = \sum_{k=0}^{+\infty} A^k.$$

Lemma 3.2. *Let $T, S \in \mathcal{B}(X)$ and $\lambda_0 \in \rho(T, S)$. If there exists a $\lambda \in \mathbb{C}$ such that*

$$|\lambda - \lambda_0| < \|\mathrm{Re}(\lambda_0, T, S)S\|^{-1}, \tag{3.1}$$

then $\lambda \in \rho(T, S)$ and

$$\mathrm{Re}(\lambda, T, S)S = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k [\mathrm{Re}(\lambda_0, T, S)S]^{k+1}.$$

Proof. Let $\lambda \in \rho(T, S)$ satisfy the relation (3.1). Then

$$T - \lambda S = (T - \lambda_0 S)\left(I - (\lambda - \lambda_0)\,\mathrm{Re}(\lambda_0, T, S)S\right).$$

Using Lemma 3.1, we arrive at the desired result. The proof is complete. □

Here we prove Theorem 2.1. We argue by contradiction. Let $\lambda_0 \in U$ such that $\|\mathrm{Re}(\lambda_0, T, S)S\| = M$. Since the set U is open, we can choose $r > 0$ such that

$$\|(\lambda - \lambda_0)\,\mathrm{Re}(\lambda_0, T, S)S\| < 1, \quad \lambda \in B(\lambda_0, r),$$

where B is of radius r centered at λ_0. By Lemma 3.2 we have

$$\mathrm{Re}(\lambda, T, S)S = \sum_{k=0}^{+\infty} (\lambda - \lambda_0)^k (\mathrm{Re}(\lambda_0, T, S)S)^{k+1}, \quad \lambda \in B(\lambda_0, r).$$

For each $f \in H$ we get

$$\|\mathrm{Re}(\lambda, T, S)Sf\|^2 = \sum_{k,m=0}^{+\infty} (\lambda - \lambda_0)^k \overline{(\lambda - \lambda_0)}^m \left\langle (\mathrm{Re}(\lambda_0, T, S)S)^{k+1} f, (\mathrm{Re}(\lambda_0, T, S)S)^{m+1} f \right\rangle. \tag{3.2}$$

Integrating Equation (3.2) along the circle $|\lambda - \lambda_0| = r$, where $\lambda = \lambda_0 + re^{i\theta}$, we find

$$\frac{1}{2\pi} \int_0^{2\pi} \|\mathrm{Re}(\lambda_0 + re^{i\theta}, T, S)Sf\|^2 d\theta$$
$$= \sum_{k,m=0}^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} (re^{i\theta})^k (re^{-i\theta})^m d\theta \left\langle (\mathrm{Re}(\lambda_0, T, S)S)^{k+1} f, (\mathrm{Re}(\lambda_0, T, S)S)^{m+1} f \right\rangle.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}(re^{i\theta})^k(re^{-i\theta})^m d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^{k+m}e^{i\theta(k-m)}\,d\theta = \begin{cases} r^{2k} & \text{if} \quad k=m,\\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\|^2 d\theta = \sum_{k=0}^{+\infty} r^{2k}\|(\operatorname{Re}(\lambda_0,T,S)S)^{k+1}f\|^2.$$

It is clear that

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant \sum_{k=0}^{+\infty} r^{2k}\|(\operatorname{Re}(\lambda_0,T,S)S)^{k+1}f\|^2$$

and therefore

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi}\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\|^2 d\theta.$$

Using

$$\|\operatorname{Re}(\lambda_0+re^{i\theta},T,S)Sf\| \leqslant M\|f\|,$$

we find

$$\|\operatorname{Re}(\lambda_0,T,S)Sf\|^2 + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f\|^2 \leqslant M^2\|f\|^2. \tag{3.3}$$

We choose an arbitrary $\varepsilon > 0$. Since $\|\operatorname{Re}(\lambda_0,T,S)S\| = M$, there exists $f_\varepsilon \in H$ such that

$$\|f_\varepsilon\| = 1 \qquad \text{and} \qquad \|\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon\|^2 > M^2-\varepsilon.$$

Therefore, due to (3.3),

$$M^2-\varepsilon + r^2\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 < M^2.$$

Then

$$\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 < \frac{\varepsilon}{r^2},$$

which implies

$$\lim_{\varepsilon\to 0}\|(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon\|^2 = 0,$$

and hence

$$\lim_{\varepsilon\to 0}(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon = 0. \tag{3.4}$$

Since S is compact operator and the sequence $(\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon)_{\varepsilon>0}$ is bounded, there exists an infinite subset $I \subset \mathbb{R}^+$ and $y_0 \in H$ such that

$$\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon = y_0 \tag{3.5}$$

for all $\varepsilon \in I$. We also find

$$\operatorname{Re}(\lambda_0,T,S)S\left(\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon\right) = \operatorname{Re}(\lambda_0,T,S)Sy_0.$$

According to the continuity of $\operatorname{Re}(\lambda_0,T,S)S$,

$$\lim_{\varepsilon\to 0}(\operatorname{Re}(\lambda_0,T,S)S)^2 f_\varepsilon = \operatorname{Re}(\lambda_0,T,S)Sy_0$$

for all $\varepsilon \in I$. In view of Equation (3.4) this gives

$$\operatorname{Re}(\lambda_0,T,S)Sy_0 = 0.$$

Since S is injective, we conclude $y_0 = 0$. By (3.5)

$$\lim_{\varepsilon\to 0}\operatorname{Re}(\lambda_0,T,S)Sf_\varepsilon = 0$$

for all $\varepsilon \in I$ and this contradicts to

$$\|\operatorname{Re}(\lambda_0, T, S) S f_\varepsilon\|^2 > M^2 - \varepsilon.$$

The proof is complete.

4. Case of complex uniformly convex banach space

In this section we prove Theorems 2.2, 2.3 and 2.4.

The next theorem plays a crucial role in proving Theorems 2.2, 2.3 and 2.4.

Theorem 4.1. *Let T and S belong to $\mathcal{B}(X)$, were X is a complex uniformly convex Banach space. Assume that there exists an open subset $U \subset \rho(T,S)$ and a constant $M > 0$ such that*

$$\|\operatorname{Re}(\lambda, T, S)\, S\| = M, \quad \forall \lambda \in U.$$

Then there exists $(e_n)_{n \geqslant 0} \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$, where

$$\lim_{n\to\infty} \|\operatorname{Re}(\lambda_0, T, S) S e_n\| = M$$

and

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S) S)^2 e_n\| = 0$$

for all $\lambda_0 \in U$.

In the proof of Theorem 4.1 we employ the following lemma.

Lemma 4.1. *Let*

$$\lambda \mapsto f(\lambda) = \sum_{k=0}^{+\infty} a_k (\lambda - \lambda_0)^k$$

be a function with values in a complex Banach space X, defined and analytic in a neighborhood of the point λ_0. If $\|f(\lambda)\| = \|a_0\|$ in a neighborhood of the point λ_0, then for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that

$$\|a_0 + (\lambda - \lambda_0) a_k\| \leqslant \|a_0\|, \qquad |\lambda - \lambda_0| \leqslant r_k.$$

The lemma is implied by [7, Lm. 1.1].

Proof of Theorem 4.1. The proof is partially based on the proof of [3, Thm. 3.2] in the case $S = I$. Let $\lambda_0 \in U$, we choose $r > 0$ such that $\|\operatorname{Re}(\lambda_0, T, S) S\|^{-1} > r$. According to Lemma 3.2, the function $\operatorname{Re}(\cdot, T, S) S$ is analytic in the ball $B(\lambda_0, r)$ and

$$\operatorname{Re}(\lambda, T, S) S = \sum_{k=0}^{+\infty} (\lambda - \lambda_0)^k (\operatorname{Re}(\lambda_0, T, S) S)^{k+1}, \quad \text{for all} \quad \lambda \in B(r, \lambda_0).$$

Since $\|\operatorname{Re}(\lambda, T, S) S\| = M$ for all $\lambda \in U$, we have

$$\|\operatorname{Re}(\lambda, T, S) S\| = \|\operatorname{Re}(\lambda_0, T, S) S\| = M \quad \text{for all} \quad \lambda \in U.$$

By Lemma (4.1), for each $k \in \mathbb{N}^*$ there exists $r_k > 0$ such that

$$\|\operatorname{Re}(\lambda_0, T, S) S + (\lambda - \lambda_0)(\operatorname{Re}(\lambda_0, T, S) S)^{k+1}\| \leqslant M, \quad |\lambda - \lambda_0| \leqslant r_k$$

for all $\lambda \in U$. This implies

$$\|\operatorname{Re}(\lambda_0, T, S) S x + (\lambda - \lambda_0) \operatorname{Re}(\lambda_0, T, S) S)^{k+1} x\| \leqslant M, \quad \lambda \in B(\lambda_0, r_k)$$

for each $x \in X$ with $\|x\| = 1$. Therefore,

$$\|\frac{1}{M} \operatorname{Re}(\lambda_0, T, S) S x + \frac{(\lambda - \lambda_0) r_k}{M} (\operatorname{Re}(\lambda_0, T, S) S)^{k+1} x\| \leqslant 1 \quad \lambda \in B(\lambda_0, 1).$$

Since $\|\operatorname{Re}(\lambda_0, T, S)S\| = M$, there exists $(e_n)_{n\geqslant 0} \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\operatorname{Re}(\lambda_0, T, S)Se_n\| = M.$$

We define the sequence

$$x_n = \frac{1}{M}\operatorname{Re}(\lambda_0, T, S)Se_n, \quad n \in \mathbb{N},$$

and then $\|x_n\| \to 1$. We let

$$y_n = \frac{r_1}{M}(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n, \quad n \in \mathbb{N}.$$

We are going to show that $\|y_n\| \to 0$. We suppose the opposite, that is, there exist $\varepsilon > 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that $\|y_n\| > \varepsilon$ for all $n \in I$. Then

$$\|x_n + (\lambda - \lambda_0)y_n\| \leqslant 1, \quad \lambda \in B(\lambda_0, 1).$$

Applying the complex uniform convexity of X, we get the existence of some $\delta > 0$ such that $\|x_n\| < 1 - \delta$ for all $n \in \mathbb{N}$. This contradicts to $\|x_n\| \to 1$. Then $\|y_n\| \to 0$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0,$$

and this completes the proof. □

Proof of Theorem 2.2. Let $\lambda_0 \in U$. For an arbitrary $\lambda \in \rho(T, S)$, we use twice the first resolvent identity to find

$$\begin{aligned}\operatorname{Re}(\lambda, T, S)S - \operatorname{Re}(\lambda_0, T, S)S =& (\lambda - \lambda_0)\operatorname{Re}(\lambda, T, S)S\operatorname{Re}(\lambda_0, T, S)S \\ =& (\lambda - \lambda_0)\left((\lambda - \lambda_0)\operatorname{Re}(\lambda, T, S)S + I\right)\left(\operatorname{Re}(\lambda_0, T, S)S\right)^2.\end{aligned}$$

Hence,

$$\|\operatorname{Re}(\lambda, T, S)S\| \geqslant \left|\|\operatorname{Re}(\lambda_0, T, S)S\| - |\lambda - \lambda_0|\,\|(\lambda - \lambda_0)\operatorname{Re}(\lambda, T, S)S + I\|\,\|\left(\operatorname{Re}(\lambda_0, T, S)S\right)^2\|\right|.$$

Due to Theorem 4.1, there exists $(e_n)_{n\geqslant 0} \subset X$ such that

$$\lim_{n\to\infty} \|\operatorname{Re}(\lambda_0, T, S)Se_n\| = M,$$

and

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0,$$

and $\|e_n\| = 1$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\|\operatorname{Re}(\lambda, T, S)S\| \geqslant& \Big|\|\operatorname{Re}(\lambda_0, T, S)Se_n\| \\ &- |\lambda - \lambda_0|\,\|(\lambda - \lambda_0)\operatorname{Re}(\lambda, T, S)S + I\|\,\|\left(\operatorname{Re}(\lambda_0, T, S)S\right)^2 e_n\|\Big|.\end{aligned}$$

Then as $n \to +\infty$ we have

$$\|\operatorname{Re}(\lambda, T, S)S\| \geqslant M, \quad \lambda \in \rho(T, S).$$

The proof is complete. □

Proof of Theorem 2.3. Assume that there exists an open set $U \subset \rho(T, S)$ such that

$$\|\operatorname{Re}(\lambda, T, S)S\| = M, \quad \lambda \in U.$$

By Theorem 4.1 there exists $\lambda_0 \in U$ and $(e_n)_n \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\operatorname{Re}(\lambda_0, T, S)Se_n\| = M,$$

and

$$\lim_{n\to\infty} \|(\operatorname{Re}(\lambda_0, T, S)S)^2 e_n\| = 0.$$

Since S is invertible, we find

$$\lim_{n\to\infty}\|e_n\| \leqslant \lim_{n\to\infty}\|S^{-1}(T-\lambda_0 S)S^{-1}(T-\lambda_0 S)\|\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\|$$
$$\leqslant \|S^{-1}(T-\lambda_0 S)S^{-1}(T-\lambda_0 S)\| \lim_{n\to\infty}\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\| = 0.$$

This contradicts to $\|e_n\| = 1$ for all $n \in \mathbb{N}$. The proof is complete. □

Proof Theorem 2.4. Let T and S belong to $\mathcal{B}(X)$, where X is a complex uniformly convex Banach space. Assume that there exists an open set $U \subset \rho(T,S)$ such that

$$\|\mathrm{Re}(\lambda,T,S)S\| = M, \quad \lambda \in U.$$

By Theorem 4.1 there exists $(e_n)_n \subset X$ such that $\|e_n\| = 1$ for all $n \in \mathbb{N}$, where

$$\lim_{n\to\infty}\|\mathrm{Re}(\lambda_0,T,S)Se_n\| = M.$$

and

$$\lim_{n\to\infty}\|(\mathrm{Re}(\lambda_0,T,S)S)^2 e_n\| = 0.$$

Since S is compact operator, there exists infinite subset $I \subseteq \mathbb{N}$ and $y \in X$ such that

$$\lim_{n\to\infty}\mathrm{Re}(\lambda_0,T,S)Se_n = y, \quad n \in I.$$

We have

$$\mathrm{Re}(\lambda_0,T,S)S(\lim_{n\to\infty}\mathrm{Re}(\lambda_0,T,S)Se_n) = \mathrm{Re}(\lambda,T,S)Sy$$

for all $n \in I$. Thus, $\mathrm{Re}(\lambda_0,T,S)Sy = 0$. Since $\mathrm{Re}(\lambda_0,T,S)S$ is injective operator, this implies $y = 0$. The latter contradicts to $M > 0$. The proof is complete. □

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Mohammed Abderrazak Mansouri
Laboratoire des Mathématiques Appliquées
et Modélisation,
Department of Mathematics
E-mail: mansouri.mohammedabderrazak@univ-guelma.dz.

Ammar Khellaf,
Ecole Nationale Polytechniques de Constantine,
Nouvelle Ville of Ali Mendjeli, BP 75,
25000, Constantine, Algeria
Laboratoire des Mathématiques Appliquées
et Modélisation, Université 8 Mai 1945,
BP.401, Guelma, 24000, Algeria
E-mail: amarlasix@gmail.com

Hamza Guebbai
Laboratoire des Mathématiques Appliquées
et Modélisation,
Department of Mathematics,
Université 8 Mai 1945,
BP.401, Guelma, 24000, Algeria
E-mail: guebbai.hamza@univ-guelma.dz