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ON VECTOR DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We propose a sequence of Lax pairs, the compatibility conditions of which are integrable vector nonlinear equations. The first equations in this hierarchy are vector Kaup — Newell, Chen — Lee — Liu, Gerdjikov — Ivanov integrable nonlinear equations. The type of vector equation depends on an additional parameter α. The proposed form of the vector Kaup — Newell equation has slight differences in comparison with the classical form. We show that the evolution of simplest nontrivial solutions of these equations is a composition of the evolutions of length and orientations of solution. We study properties of spectral curves of simplest nontrivial solutions the vector equations in the constructed hierarchy.

Keywords: integrable nonlinear equation, Kaup — Newell equation, Chen — Lee — Liu equation, Gerdjikov — Ivanov equation, multiphase equation, spectral curve.

Mathematics Subject Classification: 35Q51, 35Q55

1. Introduction

Recently a lot of attention was paid to vector variants of the nonlinear Schrödinger equations (see, for instance, [1]– [8]). This is motivated by the aim to double the amount of information transmitted by optic channels [9]– [13]. Of course, vectors forms of derivative nonlinear Schrödinger equations are also actively studied (see, for instance, [14]– [23]). It should be noted that the Lax pairs used in these works often differ one from another and this seems to be incorrect. This is why we propose a sequence of Lax pairs, which depend on functional parameters s and s_k ($\partial_{t_k} s = \partial_x s_k$). The compatibility conditions of these pairs are an hierarchy of integrable vector nonlinear equations. For $s = \alpha(\mathbf{p}^t\mathbf{q})$ the first equation in this hierarchy is a vector form of derivative nonlinear Schrödinger equation. If $\alpha = 0$, then this equation is Gerdjikov — Ivanov equation [23]

$$i\mathbf{p}_{t_1} - \mathbf{p}_{xx} + 2i(\mathbf{p}^t\mathbf{q}_x)\mathbf{p} - 2(\mathbf{p}^t\mathbf{q})^2\mathbf{p} = 0,$$
$$i\mathbf{q}_{t_1} + \mathbf{q}_{xx} + 2i(\mathbf{q}^t\mathbf{p}_x)\mathbf{q} + 2(\mathbf{p}^t\mathbf{q})^2\mathbf{q} = 0.$$

For $\alpha = 1$ the first equation in the constructed hierarchy is the vector Chen — Lee — Liu equation

$$i\mathbf{p}_{t_1} - \mathbf{p}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{p}_x = 0,$$
$$i\mathbf{q}_{t_1} + \mathbf{q}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{q}_x = 0.$$

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We note that the found vector form of Kaup — Newell equation ($\alpha = 2$)

$$i\mathbf{p}_{t_1} - \mathbf{p}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{p}_x - 2i(\mathbf{p}\mathbf{q}^t)_x\mathbf{p} = 0,$$
$$i\mathbf{q}_{t_1} + \mathbf{q}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{q}_x - 2i(\mathbf{p}\mathbf{q}^t)_x\mathbf{q} = 0,$$

differs from its classical version [14]. At the same time, in the scalar case there is no difference between $(\mathbf{p}^t\mathbf{q})$ and $(\mathbf{p}\mathbf{q}^t)$ and this equation is Kaup — Newell one. This is why it is one of its integrable vector forms. For other values of the parameter α equation has a more complicated form. We note that choosing other values of the functional parameter s, we can obtain vector derivative analogues of Kundu equation [24]– [26].

The work consists of Introduction, five section and concluding remarks. In Section 2 we define the Lax operator

$$i\Psi_x = U\Psi, \qquad U = -\lambda^2 J + \lambda Q + R + sJ,$$

which depends on the functional parameter $s \in \mathbb{R}$, and we find the structure of the corresponding monodromy matrix M [27] by the equation

$$iM_x + MU - UM = \mathbf{0}. \tag{1.1}$$

As usually, we seek matrix M as a polynomial in the spectral parameter λ

$$M = \sum_{k=0}^{N} m_k(x)\lambda^k. \tag{1.2}$$

The structure of the matrix U gives rise to differences in the structure of the coefficients m_k for even and odd indices k. Apart of the structure of the matrix M, by Equation (1.1) we also find recurrent relations for the entries of the coefficients $m_k(x)$. In Section 3 we propose a sequence of the second operators for the Lax pairs. The compatibility conditions of these Lax pairs are evolutionary integrable nonlinear equations, which are rather simply written in terms of the entries of matrix M introduced in Section 2. The first equations in this hierarchy are vector forms of the derivative variants of nonlinear Schrödinger equation given above. In the next section we briefly discuss stationary equations, which are satisfied by multi–phase solutions. As in other cases (see, for instance, [23], [26], [28]), the stationary equations are divided into two groups. The first consists of two matrix equations, which are restrictions of Equation (1.1) to the zeroth and first power of the spectral parameter λ. Since the structure of coefficients m_k depends on the parity of k, the scalar form of these stationary equations depends on the parity of the highest power N of the polynomial (1.2). The second group of stationary equations follows from the constancy of the coefficients in the equation of the corresponding spectral curve. We recall that the equation of spectral curve is the characteristic equations of the matrix M [27]. In simplest cases the system of these stationary equations can be resolved. We note that as in the case of usual vector equations [1], [26] and scalar derivative equations [28], the number of the phases of solutions is less than the genus of its spectral curve.

In Section 5 we consider in detail the case $N = 3$ (or $n = 1$, where $n = N - 2$), when the stationary equations have no solutions of form plane waves, but they can be still solved analytically. The number n can be regarded as the complexity level of solution. If $n = 0$, then solutions of stationary equations are planar waves. If $n = 1$, then, as a rule, the solutions are expressed in terms of elliptic functions and their degenerations. In this section we show that for $n = 1$, a natural geometric interpretation of functional parameters of the solution vector arise, namely, its length and direction. We note that the length and direction of solution vector are independent of the functional parameter s. Moreover, in this case the stationary equations are reduced to a first order autonomous differential equation for the length and to an equation relating the variation of the direction with its length. Also in this section we study the dependence of the behavior of simple nontrivial solutions and their spectral curves on the parameters. We show that in the general case for $n = 1$ the solution is two–phase, while the genus of the corresponding curve is equal to $g = 3$.

In Section 6 we consider evolutionary integrable nonlinear equations from the geometric point of view. We show that if we do not employ stationary equations, the geometric approach has no advantages. If we consider the case $n = 1$ and use the formulas implied by the stationary equations, then the evolution of the length and direction of the solution is described by rather simple equations. There is still an open question whether the geometric approach remain useful as the complexity level n increases. We note that from the applied point of view the length and the direction of solution are theoretical objects since in practice the polarized beams passes through the splitter, while the optical fiber has a rather complicated structure, see, for instance, [29], [30] and the references therein.

2. Monodromy matrix

Let the Lax operator be of form

$$i\Psi_x = U\Psi, \tag{2.1}$$

where

$$U = -\lambda^2 J + \lambda Q + R + sJ, \tag{2.2}$$

$$J = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & \mathbf{p}^t \\ -\mathbf{q} & \mathbf{0} \end{pmatrix}, \quad R = \begin{pmatrix} -\mathbf{p}^t\mathbf{q} & \mathbf{0}^t \\ \mathbf{0} & \mathbf{q}\mathbf{p}^t \end{pmatrix}, \tag{2.3}$$

where $\mathbf{p}^t = (p_1, p_2)$, $\mathbf{q}^t = (q_1, q_2)$.

We consider Equations (2.1), (2.2) with the matrices (2.3). Following the works [27], [28], we seek the monodromy matrix M of the function Ψ as a polynomial in the spectral parameter λ. Then Equation (1.1) implies the following structure for the matrix M:

$$M_n = V_n + \sum_{k=1}^{n-1} c_k V_{n-k} + c_n U_0 + c_{n+1} V_{-1} + J_n,$$

where

$$V_{-1} = -\lambda J + Q, \quad U_0 = \lambda V_{-1} + R, \quad V_1 = \lambda U_0 + V_1^0, \quad V_{j+1} = \lambda V_j + V_{j+1}^0,$$

$$V_{2k-1}^0 = \begin{pmatrix} 0 & \mathbf{H}_k^t \\ \mathbf{G}_k & \mathbf{0} \end{pmatrix}, \quad V_{2k}^0 = \begin{pmatrix} -\mathcal{F}_k & \mathbf{0}^t \\ \mathbf{0} & F_k \end{pmatrix}, \quad \mathcal{F}_k = \operatorname{Tr} F_k, \quad k \geqslant 1,$$

$$J_n = \begin{pmatrix} -2c_{n+2} & 0 & 0 \\ 0 & c_{n+2} + c_{n+3} & c_{n+4} \\ 0 & c_{n+5} & c_{n+2} - c_{n+3} \end{pmatrix}.$$

Here $c_k \in \mathbb{R}$ are some constants parameterizing the solution.

The entries of the matrix V_k^0 satisfy the following recurrent relations

$$\begin{aligned}
&\mathbf{H}_1 = -i\mathbf{p}_x + s\mathbf{p}, \quad \mathbf{G}_1 = -i\mathbf{q}_x - s\mathbf{q}, \\
&\mathbf{H}_{k+1} = \left(F_k^t + \mathcal{F}_k I\right)\mathbf{p} - \left(\mathbf{p}\mathbf{q}^t + (\mathbf{p}^t\mathbf{q})I\right)\mathbf{H}_k + s\mathbf{H}_k - i\partial_x \mathbf{H}_k, \\
&\mathbf{G}_{k+1} = -\left(F_k + \mathcal{F}_k I\right)\mathbf{q} - \left(\mathbf{q}\mathbf{p}^t + (\mathbf{q}^t\mathbf{p})I\right)\mathbf{G}_k + s\mathbf{G}_k + i\partial_x \mathbf{G}_k, \\
&\partial_x F_k = \mathbf{q}\partial_x \mathbf{H}_k^t - \partial_x \mathbf{G}_k \mathbf{p}^t - i\left(\mathbf{q}\mathbf{p}^t + (\mathbf{q}^t\mathbf{p})I\right)\mathbf{G}_k\mathbf{p}^t \\
&\qquad - i\mathbf{q}\mathbf{H}_k^t\left(\mathbf{q}\mathbf{p}^t + (\mathbf{q}^t\mathbf{p})I\right) + is(\mathbf{G}_k\mathbf{p}^t + \mathbf{q}\mathbf{H}_k^t).
\end{aligned} \tag{2.4}$$

In particular,

$$
\begin{aligned}
F_1 =& i\left(\mathbf{q}_x\mathbf{p}^t - \mathbf{q}\mathbf{p}_x^t\right) - (\mathbf{q}\mathbf{p}^t)^2 + 2s\mathbf{q}\mathbf{p}^t,\\
\mathcal{F}_1 =& i(\mathbf{p}^t\mathbf{q}_x - \mathbf{q}^t\mathbf{p}_x) - (\mathbf{p}^t\mathbf{q})^2 + 2s(\mathbf{p}^t\mathbf{q}),\\
H_2 =& -\mathbf{p}_{xx} + 2i(\mathbf{p}^t\mathbf{q}_x)\mathbf{p} - 2(\mathbf{p}^t\mathbf{q})^2\mathbf{p} + 2s((\mathbf{p}^t\mathbf{q})\mathbf{p} - i\mathbf{p}_x) + (s^2 - is_x)\mathbf{p},\\
G_2 =& \mathbf{q}_{xx} + 2i(\mathbf{q}^t\mathbf{p}_x)\mathbf{q} + 2(\mathbf{p}^t\mathbf{q})^2\mathbf{q} - 2s((\mathbf{p}^t\mathbf{q})\mathbf{q} + i\mathbf{q}_x) - (s^2 + is_x)\mathbf{q},\\
\mathcal{F}_2 =& (\mathbf{p}_x^t\mathbf{q}_x - \mathbf{p}^t\mathbf{q}_{xx} - \mathbf{q}^t\mathbf{p}_{xx}) - 2(\mathbf{p}^t\mathbf{q})^3 + 3s^2(\mathbf{p}^t\mathbf{q}) + 3is(\mathbf{p}^t\mathbf{q}_x - \mathbf{q}^t\mathbf{p}_x),\\
H_3 =& i\mathbf{p}_{xxx} + 3(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{p} + 3(\mathbf{p}^t\mathbf{q}_x)\mathbf{p}_x + 3i(\mathbf{p}^t\mathbf{q})(\mathbf{p}_x^t\mathbf{q})\mathbf{p} + 3i(\mathbf{p}^t\mathbf{q})^2\mathbf{p}_x\\
&- 3s\mathbf{p}_{xx} - 3(s_x + is^2 + is(\mathbf{p}^t\mathbf{q}))\mathbf{p}_x - (s_{xx} - s^3 - 6s^2(\mathbf{p}^t\mathbf{q}) + 6s(\mathbf{p}^t\mathbf{q})^2)\mathbf{p}\\
&- 3is(s_x + (\mathbf{q}^t\mathbf{p}_x) - 2(\mathbf{p}^t\mathbf{q}_x))\mathbf{p},\\
G_3 =& i\mathbf{q}_{xxx} - 3(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{q} - 3(\mathbf{q}^t\mathbf{p}_x)\mathbf{q}_x + 3i(\mathbf{p}^t\mathbf{q})(\mathbf{q}_x^t\mathbf{p})\mathbf{q} + 3i(\mathbf{p}^t\mathbf{q})^2\mathbf{q}_x\\
&+ 3s\mathbf{q}_{xx} + 3(s_x - is^2 - is(\mathbf{p}^t\mathbf{q}))\mathbf{q}_x + (s_{xx} - s^3 - 6s^2(\mathbf{p}^t\mathbf{q}) + 6s(\mathbf{p}^t\mathbf{q})^2)\mathbf{q}\\
&- 3is(s_x + (\mathbf{p}^t\mathbf{q}_x) - 2(\mathbf{q}^t\mathbf{p}_x))\mathbf{q}.
\end{aligned}
$$

3. Integrable evolutionary nonlinear equations

We define the second equation in the Lax pair as

$$i\Psi_{t_k} = W_k\Psi, \tag{3.1}$$

where $W_k = V_{2k} + s_kJ$, $\partial_{t_k}s = \partial_x s_k$. Then the compatibility conditions of the Lax pair imply the following integrable nonlinear evolutionary equations

$$
\begin{aligned}
\mathbf{p}_{t_k} &= i\mathbf{H}_{k+1} - is_k\mathbf{p},\\
\mathbf{q}_{t_k} &= i\mathbf{G}_{k+1} + is_k\mathbf{q}.
\end{aligned}
\tag{3.2}
$$

Equations (3.2), (2.4) yield

$$\partial_{t_k}(\mathbf{p}^t\mathbf{q}) = \partial_x(\mathcal{F}_k).$$

Hence, in Equations (3.2) we can let

$$s = \alpha(\mathbf{p}^t\mathbf{q}), \quad s_k = \alpha\mathcal{F}_k, \tag{3.3}$$

where α is some real number. The equations (3.2) are of simplest form in three cases, for $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$.

Letting $\alpha = 0$, we have

$$
\begin{aligned}
&i\mathbf{p}_{t_1} - \mathbf{p}_{xx} + 2i(\mathbf{p}^t\mathbf{q}_x)\mathbf{p} - 2(\mathbf{p}^t\mathbf{q})^2\mathbf{p} = 0,\\
&i\mathbf{q}_{t_1} + \mathbf{q}_{xx} + 2i(\mathbf{q}^t\mathbf{p}_x)\mathbf{q} + 2(\mathbf{p}^t\mathbf{q})^2\mathbf{q} = 0
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&\mathbf{p}_{t_2} + \mathbf{p}_{xxx} - 3i(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{p} - 3i(\mathbf{p}^t\mathbf{q}_x)\mathbf{p}_x + 3(\mathbf{p}^t\mathbf{q})(\mathbf{p}_x^t\mathbf{q})\mathbf{p} + 3(\mathbf{p}^t\mathbf{q})^2\mathbf{p}_x = 0,\\
&\mathbf{q}_{t_2} + \mathbf{q}_{xxx} + 3i(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{q} + 3i(\mathbf{q}^t\mathbf{p}_x)\mathbf{q}_x + 3(\mathbf{p}^t\mathbf{q})(\mathbf{q}_x^t\mathbf{p})\mathbf{q} + 3(\mathbf{p}^t\mathbf{q})^2\mathbf{q}_x = 0.
\end{aligned}
$$

In this case Equation (3.4) is a vector form of the Gerdjikov — Ivanov equation.

For $\alpha = 1$ the evolutionary equations (3.2) cast into the form

$$
\begin{aligned}
&i\mathbf{p}_{t_1} - \mathbf{p}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{p}_x = 0,\\
&i\mathbf{q}_{t_1} + \mathbf{q}_{xx} - 2i(\mathbf{p}^t\mathbf{q})\mathbf{q}_x = 0
\end{aligned}
\tag{3.5}
$$

and

$$\begin{aligned}&\mathbf{p}_{t_2}+\mathbf{p}_{xxx}+3i(\mathbf{p}^t\mathbf{q})\mathbf{p}_{xx}+3i(\mathbf{p}_x\mathbf{q}^t)\mathbf{p}_x-3(\mathbf{p}^t\mathbf{q})^2\mathbf{p}=0,\\&\mathbf{q}_{t_2}+\mathbf{q}_{xxx}-3i(\mathbf{p}^t\mathbf{q})\mathbf{q}_{xx}-3i(\mathbf{q}_x\mathbf{p}^t)\mathbf{q}_x-3(\mathbf{p}^t\mathbf{q})^2\mathbf{q}=0.\end{aligned}$$

It is easy to see that Equation (3.5) is the vector form of Chen — Lee — Liu equation.

Equation (3.2) with $\alpha=2$ read as

$$\begin{aligned}&i\mathbf{p}_{t_1}-\mathbf{p}_{xx}-2i(\mathbf{p}^t\mathbf{q})\mathbf{p}_x-2i(\mathbf{p}\mathbf{q}^t)_x\mathbf{p}=0,\\&i\mathbf{q}_{t_1}+\mathbf{q}_{xx}-2i(\mathbf{p}^t\mathbf{q})\mathbf{q}_x-2i(\mathbf{p}\mathbf{q}^t)_x\mathbf{q}=0\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\mathbf{p}_{t_2}&+\mathbf{p}_{xxx}+6i(\mathbf{p}^t\mathbf{q})\mathbf{p}_{xx}+3i(\mathbf{p}^t\mathbf{q}_x)\mathbf{p}_x+6i(\mathbf{p}_x\mathbf{q}^t)\mathbf{p}_x+3i(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{p}\\&-15(\mathbf{p}^t\mathbf{q})^2\mathbf{p}_x-12(\mathbf{p}^t\mathbf{q})(\mathbf{p}^t\mathbf{q}_x)\mathbf{p}-3(\mathbf{p}^t\mathbf{q})(\mathbf{p}_x^t\mathbf{q})\mathbf{p}=0,\\\mathbf{q}_{t_2}&+\mathbf{q}_{xxx}-6i(\mathbf{p}^t\mathbf{q})\mathbf{q}_{xx}-3i(\mathbf{q}^t\mathbf{p}_x)\mathbf{q}_x-6i(\mathbf{q}_x\mathbf{p}^t)\mathbf{q}_x-3i(\mathbf{p}_x^t\mathbf{q}_x)\mathbf{q}\\&-15(\mathbf{p}^t\mathbf{q})^2\mathbf{q}_x-12(\mathbf{p}^t\mathbf{q})(\mathbf{q}^t\mathbf{p}_x)\mathbf{q}-3(\mathbf{p}^t\mathbf{q})(\mathbf{p}^t\mathbf{q}_x)\mathbf{q}=0.\end{aligned}$$

Equation (3.6) is a new version of vector Kaup — Newell equation.

4. Stationary equations

Apart of recurrent relations, Equation (1.1) also yields stationary equations satisfied by multi–phase solutions

$$\begin{aligned}\big(i\partial_x V_n^0+[V_n^0,R+sJ]\big)&+\sum_{k=1}^{n-1}c_k\left(i\partial_x V_{n-k}^0+[V_{n-k}^0,R+sJ]\right)\\&+ic_n\partial_x R+c_{n+1}\left(i\partial_x Q+[Q,R+sJ]\right)+[J_n,R]=\mathbf{0}\end{aligned}$$

and

$$\begin{aligned}\big(i\partial_x V_{n-1}^0+[V_{n-1}^0,R+sJ]+[V_n^0,Q]\big)&+\sum_{k=1}^{n-2}c_k\big(i\partial_x V_{n-1-k}^0+[V_{n-1-k}^0,R+sJ]+[V_{n-k}^0,Q]\big)\\&+c_{n-1}\big(i\partial_x R+[V_1^0,Q]\big)+c_n\left(i\partial_x Q+s[Q,J]\right)+[J_n,Q]=\mathbf{0}.\end{aligned}$$

Since the structures of the matrices V_k^0 depend on the parity, the scalar forms of the stationary equations also depend on the parity. The compatibility conditions of these two matrix equations produce restrictions for the constants c_k. In particular, the recurrent equations (2.4) imply the following realness conditions. If $\mathbf{q}=\sigma\mathbf{p}^*$, where $\sigma=\pm1$, then

$$\mathbf{G}_k=-\sigma\mathbf{H}_k^*,\quad F_k^*=F_k^t,\quad \mathcal{F}_k^*=\mathcal{F}_k.$$

These conditions lead to the following symmetry of the matrices $V_k^0(\lambda)$:

$$\big(V_{2k}^0(\lambda)\big)^\dagger=V_{2k}^0(\lambda^*),\quad \big(V_{2k-1}^0(\lambda)\big)^\dagger=-\sigma V_{2k-1}^0(\lambda^*).$$

Here † stands for the Hermitian conjugation. We note that similar relations hold for the matrices J, Q, R

$$J^\dagger=J,\qquad Q^\dagger=-\sigma Q,\qquad R^\dagger=R.$$

It follows from these symmetry relations that each stationary equations splits into two parts, each of which is transformed according its own rule. Therefore, one of these parts vanishes identically. It is easy to understand that this implies that all coefficients with odd indices are zero. This is why $J_{2k-1}=0$, and the matrices $M_n(\lambda)$ satisfy the following conditions

$$(M_{2k}(\lambda))^\dagger=M_{2k}(\lambda^*),\qquad (M_{2k-1}(\lambda))^\dagger=-\sigma M_{2k-1}(\lambda^*).\tag{4.1}$$

The second set of stationary equations is implied by the constancy condition of the coefficients in the equation of the spectral curve. We recall that the spectral curve is the characteristic equation of the matrix $M_n(\lambda)$:

$$\Gamma: \quad \mathcal{R}(\mu,\lambda)=\det(\mu I - M_n(\lambda))=0$$

or

$$\mathcal{R}(\mu,\lambda)=\mu^3+\mathcal{A}(\lambda)\mu+\mathcal{B}(\lambda)=0, \tag{4.2}$$

where

$$\begin{aligned}\mathcal{A}(\lambda)&=-\frac{1}{3}\lambda^{2n+4}-\frac{2c_2}{3}\lambda^{2n+2}+\sum_{k=2}^{n+2}\mathcal{A}_k\lambda^{2n+4-2k},\\ \mathcal{B}(\lambda)&=\frac{2}{27}\lambda^{3n+6}+\frac{2c_2}{9}\lambda^{3n+4}+\sum_{k\geqslant 2}\mathcal{B}_k\lambda^{3n+6-2k}.\end{aligned}$$

We also recall that the coefficients $\mathcal{A}_k$ and $\mathcal{B}_k$ are additional integrals of the multi–phase solutions. At the same time the highest coefficients $\mathcal{A}_k$ and $\mathcal{B}_k$ for $n\geqslant 1$ are related by the identities

$$\mathcal{B}_2+\frac{1}{3}\mathcal{A}_2=\frac{1}{9}c_2^2, \qquad \mathcal{B}_3+\frac{1}{3}\mathcal{A}_3+\frac{1}{3}c_2\mathcal{A}_2=-\frac{1}{27}c_2^3. \tag{4.3}$$

It follows from Conditions (4.3) that the discriminant of the polynomial $\mathcal{R}(\mu)$ is a polynomial on λ of the degree $6n+4$. Since in the general case the curve (4.2) contains three infinite points $\mathcal{P}_\infty^{1,2,3}$

$$\begin{aligned}\mu(\mathcal{P})&=\frac{\lambda^n}{3}\left(-2\lambda^2-2c_2+(3A_2+c_2^2)\lambda^{-2}+O(\lambda^{-4})\right), \quad \mathcal{P}\to\mathcal{P}_\infty^1,\\ \mu(\mathcal{P})&=\frac{\lambda^n}{3}\left(\lambda^2+c_2\pm\sqrt{\frac{26c_2^3}{3}\lambda^{-1}+O(\lambda^{-2})}\right), \qquad \mathcal{P}\to\mathcal{P}_\infty^{2,3},\end{aligned}$$

and respectively Γ contains no infinite branching points, in the general case the curve (4.2) contains $6n+4$ branching points. Using Riemann — Hurwitz formula, we obtain that in the general case the genus of the spectral curve is equal to $g=3n$.

For even n the curve (4.2) possesses the holomorphic involution $\tau_h:(\mu,\lambda)\to(\mu,-\lambda)$. For odd n the holomorphic involution of the curve (4.2) has the form $\tau_h:(\mu,\lambda)\to(-\mu,-\lambda)$. It follows from the conditions (4.1) that the spectral curve (4.2) possesses the antiholomorphic involution

- $\tau_a:(\mu,\lambda)\to(\mu^*,\lambda^*)$ for even n,
- $\tau_a:(\mu,\lambda)\to(-\sigma\mu^*,\lambda^*)$ for odd n.

5. Case $n=1$

For $n=1$ the matrix M reads as ($c_1=0$ and $J_1=\mathbf{0}$)

$$M=V_1+c_2V_{-1}.$$

In this case the stationary equations are of the form

$$\begin{aligned}
&\partial_x^2 p_1 + i(c_2 - 2p_1q_1 - p_2q_2 + 2s)\partial_x p_1 - ip_1q_2\partial_x p_2\\
&\qquad + ((c_2+s)(2p_1q_1 + 2p_2q_2 - s) + is_x)p_1 = 0,\\
&\partial_x^2 p_2 + i(c_2 - p_1q_1 - 2p_2q_2 + 2s)\partial_x p_2 - ip_2q_1\partial_x p_1\\
&\qquad + ((c_2+s)(2p_1q_1 + 2p_2q_2 - s) + is_x)p_2 = 0,\\
&\partial_x^2 q_1 - i(c_2 - 2p_1q_1 - p_2q_2 + 2s)\partial_x q_1 + ip_2q_1\partial_x q_2\\
&\qquad + ((c_2+s)(2p_1q_1 + 2p_2q_2 - s) - is_x)q_1 = 0,\\
&\partial_x^2 q_2 - i(c_2 - p_1q_1 - 2p_2q_2 + 2s)\partial_x q_2 + ip_1q_2\partial_x q_1\\
&\qquad + ((c_2+s)(2p_1q_1 + 2p_2q_2 - s) - is_x)q_2 = 0.
\end{aligned} \tag{5.1}$$

Following [23], [26], [31], in Equations (5.1) we make the change

$$p_j = \sqrt{u_j}\exp\left\{-\int \frac{w_j}{2u_j}dx\right\}, \quad q_j = \sqrt{u_j}\exp\left\{\int \frac{w_j}{2u_j}dx\right\}, \tag{5.2}$$

where $u_j = p_jq_j$, $w_j = p_j\partial_x q_j - q_j\partial_x p_j$. After simplifications we obtain

$$\begin{aligned}
w_1 &= ic_5 + i(c_2 - u_1 - u_2 + 2s)u_1,\\
w_2 &= ic_6 + i(c_2 - u_1 - u_2 + 2s)u_2
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&2u_1\partial_x^2 u_1 - (\partial_x u_1)^2 + (c_2^2 - 2(c_5+c_6) + 4c_2u_2 + 3u_2^2)u_1^2\\
&\qquad + (4c_2 + 6u_2)u_1^3 + 3u_1^4 - c_5^2,\\
&2u_2\partial_x^2 u_2 - (\partial_x u_2)^2 + (c_2^2 - 2(c_5+c_6) + 4c_2u_1 + 3u_1^2)u_2^2\\
&\qquad + (4c_2 + 6u_1)u_2^3 + 3u_2^4 - c_6^2.
\end{aligned} \tag{5.4}$$

Here c_5 and c_6 are the integration constants. We observe that Equations (5.4) do not involve the function s and coincide completely with Equations (23) in the work [23].

In this case the coefficients in Equation (4.2) of the spectral curve read as

$$\begin{aligned}
\mathcal{A}(\lambda) &= -\frac{1}{3}\lambda^6 - \frac{2c_2}{3}\lambda^4 + \mathcal{A}_2\lambda^2 + \mathcal{A}_3,\\
\mathcal{B}(\lambda) &= \frac{2}{27}\lambda^9 + \frac{2c_2}{9}\lambda^7 + \mathcal{B}_2\lambda^5 + \mathcal{B}_3\lambda^3 + \mathcal{B}_4\lambda,
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
\mathcal{A}_2 =& -\frac{c_2^2 + 3c_5 + 3c_6}{3},\\
\mathcal{A}_3 =& \frac{(\partial_x u_1)^2}{4u_1} + \frac{(\partial_x u_2)^2}{4u_2} + \frac{1}{4}\left(u_1^3 + u_2^3\right) + \frac{1}{4}(2c_2 + 3u_2)u_1^2 + \frac{1}{4}(2c_2 + 3u_1)u_2^2\\
& + c_2u_1u_2 + \frac{c_2^2 - 2c_5 - 2c_6}{4}(u_1 + u_2) + \frac{c_5^2}{4u_1} + \frac{c_6^2}{4u_2} - \frac{c_2(c_5+c_6)}{2},\\
\mathcal{B}_2 =& \frac{2c_2^2 + 3c_5 + 3c_6}{9},\\
\mathcal{B}_3 =& -\frac{1}{3}\mathcal{A}_3 + \frac{2c_2^3}{27} + \frac{c_2(c_5+c_6)}{3},\\
\mathcal{B}_4 =& -\frac{c_2}{3}\mathcal{A}_3 - \frac{u_2(\partial_x u_1)^2}{4u_1} - \frac{u_1(\partial_x u_2)^2}{4u_2} + \frac{1}{2}(\partial_x u_1)(\partial_x u_2)\\
& - \frac{c_5^2u_2}{4u_1} - \frac{c_6^2u_1}{4u_2} + \frac{1}{2}c_5c_6.
\end{aligned}$$

It is easy to see that the coefficients (5.5) in Equation (4.2) of the spectral curve are also independent of the parameter s. Therefore, the amplitudes of the solutions of vector form of derivative nonlinear Schrödinger equation are independent of the particular form of the equation. Hence, the coefficients in the equations of spectral curves of the multi–phase solutions of Equations (3.6), (3.5) and (3.4) are also independent of the type of equation. We recall that solutions of scalar form of derivative nonlinear Schrödinger equation possess a similar property [28].

Since the stationary equations (5.4) pass one into the other under the change $u_1 \leftrightarrow u_2$, and the coefficients (5.5) are invariant under the change $(u_1, c_5) \leftrightarrow (u_2, c_6)$, from the functions u_1, u_2 we proceed to the functions u, v:

$$u = u_1 + u_2, \qquad v = u_1 - u_2. \tag{5.6}$$

In the new notation Equations (5.4) and the coefficients (5.5) are of form [23]:

$$\begin{aligned} &u\partial_x^2 v + v\partial_x^2 u - (\partial_x u)(\partial_x v) + (c_2^2 - 2(c_5 + c_6) + 4c_2 u + 3u^2)uv + c_6^2 - c_5^2 = 0, \\ &2u\partial_x^2 u + 2v\partial_x^2 v - (\partial_x u)^2 - (\partial_x v)^2 \\ &\qquad + (c_2^2 - 2(c_5 + c_6) + 4c_2 u + 3u^2)(v^2 + u^2) - 2(c_5^2 + c_6^2) = 0 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \mathcal{A}_3 =& \frac{1}{4}(u + c_2)(u^2 + c_2 u - 2(c_5 + c_6)) \\ &+ \frac{(2(c_5^2 + c_6^2) + (\partial_x u)^2 + (\partial_x v)^2)u - 2(c_5^2 - c_6^2 + (\partial_x u)(\partial_x v))v}{4(u^2 - v^2)}, \\ \mathcal{B}_4 =& -\frac{c_2}{3}\mathcal{A}_3 - \frac{((c_5 + c_6)^2 + (\partial_x u)^2)v^2 + ((c_5 - c_6)^2 + (\partial_x v)^2)u^2}{4(u^2 - v^2)} \\ &- \frac{(c_5^2 - c_6^2 + (\partial_x u)(\partial_x v))uv}{2(u^2 - v^2)}. \end{aligned} \tag{5.8}$$

Using relations (5.8), we can pass from Equations (5.7) to the identities

$$\partial_x^2 u = -2u^3 - 3c_2 u^2 - (c_2^2 - 2(c_5 + c_6))u + 2\mathcal{A}_3 + c_2(c_5 + c_6) \tag{5.9}$$

and

$$\begin{aligned} 6v\partial_x^2 v - 3(\partial_x v)^2 + 3(c_2^2 - 2(c_5 + c_6) + 4c_2 u + 3u^2)v^2 \\ - 12\mathcal{B}_4 - 4c_2\mathcal{A}_3 - 3(c_5 - c_6)^2 = 0. \end{aligned} \tag{5.10}$$

Integrating (5.9), we find

$$(\partial_x u)^2 = -u^4 - 2c_2 u^3 - (c_2^2 - 2(c_5 + c_6))u^2 + (4\mathcal{A}_3 + 2c_2(c_5 + c_6))u + c_7, \tag{5.11}$$

where $c_7 \in \mathbb{R}$ is the integration constant. Therefore, $u(x)$ is an elliptic function with simple poles or its degeneration. Since Equation (5.11) is an autonomous differential equation, the dependence of the function u on the variables t_k is of the form

$$u(x, t_k) = u(x + \phi_1(t_k)).$$

The constant $\mathcal{B}_4$ is related with the constant c_7 by means of the relation

$$\mathcal{B}_4 = \frac{1}{4}c_7 + \frac{1}{4}(c_5 + c_6)^2 - \frac{1}{3}c_2\mathcal{A}_3.$$

Following [23], we make the change $v = \widehat{v}u$ in Equation (5.10). After simplification with using the relations (5.7), (5.9), (5.10) and (5.11) we get the following differential equation for the function $\widehat{v}$

$$(\partial_x \widehat{v})^2 = \frac{c_7\widehat{v}^2 + 2(c_5^2 - c_6^2)\widehat{v} - c_7 - 2(c_5^2 + c_6^2)}{u^2}. \tag{5.12}$$

Integrating (5.12) with $c_7 \neq 0$, we obtain

$$\widehat{v} = \frac{\sqrt{((c_5+c_6)^2-\kappa^2)\,((c_5-c_6)^2-\kappa^2)}}{\kappa^2}\sin(\kappa\theta) + \frac{c_5^2-c_6^2}{\kappa^2}, \tag{5.13}$$

where $\partial_x\theta = \pm u^{-1}$, $\kappa^2 = -c_7$.

For $c_7 = 0$ Equation (5.12) reads as

$$(\partial_x\widehat{v})^2 = \frac{2(c_5^2-c_6^2)\widehat{v} - 2(c_5^2+c_6^2)}{u^2}. \tag{5.14}$$

Integrating (5.14) with $c_5^2 \neq c_6^2$, we find

$$\widehat{v} = \frac{c_5^2+c_6^2}{c_5^2-c_6^2} + (c_5^2-c_6^2)\frac{\theta^2}{2}. \tag{5.15}$$

It follows from the formulas (5.13) and (5.15) that the dependence of the function $\widehat{v}$ on the variables t_k is of the form $\widehat{v}(x,t_k) = \widehat{v}(\theta(x,t_k))$, where $\theta(x,t_k) = \theta(x+\phi_1(t_k)) + \phi_2(t_k)$.

Since for $n = 1$ the discriminant of the polynomial $\mathcal{R}(\mu)$ equals

$$-c_7\lambda^{10} - (2\mathcal{A}_3(c_5+c_6) + c_2(c_5+c_6)^2 + 3c_2c_7)\lambda^8 + \ldots,$$

for $c_7 \neq 0$ the genus of the spectral curve is $g = 3$. This statement is true once the curve is connected and non–degenerate. Hence, for $c_7 \neq 0$ the spectral curve has the genus $g = 3$ (or is the degeneration of the curve of genus $g = 3$) and at the same time, the corresponding solution is two–phase. The first phase contains $\phi_1(t_k)$, while the other does $\phi_2(t_k)$.

Since for $c_7 = 0$ the discriminant equals

$$-(2\mathcal{A}_3 + c_2(c_5+c_6))(c_5+c_6)\lambda^8 + \ldots,$$

for $\mathcal{A}_3 \neq -c_2(c_5+c_6)/2$ the spectral curve has the genus $g = 2$ (or a degeneration of the curve of genus $g = 2$). We note that as $c_7 = 0$, the condition $c_6^2 \neq c_5^2$ holds since otherwise Equation (5.14) has no real solutions. We can also say that as $c_6^2 = c_5^2$, the condition $c_7 \neq 0$ is satisfied.

It follows from the formula (5.15) that as $c_7 = 0$, a nonlinear resonance arises, when the periodic oscillations of the function $\widehat{v}$ transform into its quadratic growth.

As $c_7 = 0$ and $\mathcal{A}_3 = -c_2(c_5+c_6)/2$, Equation (5.11) reads as

$$(\partial_x u)^2 = u^2(2(c_5+c_6) - (u+c_2)^2).$$

In this case the function u is expressed in terms of the elementary functions

$$u = \frac{a}{\cosh(ax+\phi_1(t))}, \quad a = \sqrt{2(c_5+c_6-c_2) - c_2^2}.$$

At the same time, the genus of the corresponding spectral curve is $g = 1$.

Thus, the procedure of constructing simplest nontrivial solutions of vector derivative forms of nonlinear Schrödinger equation consists of the following steps.

- Choose a function $u(x)$ satisfying Equation (5.11) and the constants c_2, c_5, c_6, c_7 and $\mathcal{A}_3$.
- Using the formula (5.13), by the function $u(x)$ and constants find the functions $\theta(x)$ and $\widehat{v}(x)$.
- If $c_7 = 0$, find the function $\widehat{v}$ by equation (5.15).
- Knowing the functions $u(x)$ and $\widehat{v}(x)$, find the function $v(x) = \widehat{v}(x)u(x)$.
- Using formula (5.6), obtain the functions

$$u_1 = \frac{1}{2}(u+v) = \frac{1}{2}(1+\widehat{v})u, \quad u_2 = \frac{1}{2}(u-v) = \frac{1}{2}(1-\widehat{v})u.$$

- By the formula (5.3) find $w_1(x)$ and $w_2(x)$. Only at this step there arise the differences in solutions of various versions of vector derivative forms of the nonlinear Schrödinger equation.

- By the formulas (5.2) find $p_j(x)$ and $q_j(x)$. Here, after calculating the integrals, in the variables of the exponentials of the functions p_j and q_j there arise additional summands depending on the variables t_k.
- Find specific values of the functions $\phi_m(t_k)$ by Equations (3.2).

Examples of simplest nontrivial solutions of the vector derivative nonlinear Schrödinger equation for $s = 0$ were provided in the work [23]. At the same time the solutions corresponding to $n = 1$ possess joint properties independent of the form of solution. In particular, as we have obtained above, the spectral curve is defined by Equations (4.2), (5.5), where

$$\begin{aligned}
\mathcal{A}_2 &= -\frac{c_2^2 + 3(c_5 + c_6)}{3},\\
\mathcal{B}_2 &= \frac{2c_2^2 + 3(c_5 + c_6)}{9},\\
\mathcal{B}_3 &= -\frac{1}{3}\mathcal{A}_3 + \frac{2c_2^3}{27} + \frac{c_2(c_5 + c_6)}{3},\\
\mathcal{B}_4 &= \frac{1}{4}c_7 + \frac{1}{4}(c_5 + c_6)^2 - \frac{1}{3}c_2\mathcal{A}_3.
\end{aligned}$$

As $= 1$, the equation of the spectral curve depends on the constants $\mathcal{A}_3$, c_2, c_7 and $(c_5 + c_6)$. It follows from Equation (5.11) that the equation for the function $u(x)$ depends on the same constants. We observe that these constants are determined uniquely by Equation (5.11). We also note that the five coefficients in the equation of spectral curve are determined by these four constants and, as one can easily verify, they are related by the equation

$$\mathcal{A}_3 + 3\mathcal{B}_3 + (4\mathcal{A}_2 + 3\mathcal{B}_2)\sqrt{\frac{1}{3}\mathcal{A}_2 + \mathcal{B}_2} = 0. \tag{5.16}$$

Hence, with the solutions of the stationary equations (5.1) not all curves of form (4.2), (5.5) are associated but only ones, the coefficients of which obey the condition (5.16).

We recall that in the work [23] we pointed out the possibility of the geometric approach to the analysis of simplest nontrivial solutions of vectors equations. The geometric interpretation is as follows. Let

$$p_1 = |\mathbf{p}|\, e^{i\alpha_1}\cos(\phi), \quad p_2 = |\mathbf{p}|\, e^{i\alpha_2}\sin(\phi)$$

and $\mathbf{q} = \sigma\mathbf{p}^*$, where $\sigma = \pm 1$. We then have

$$u_1 = p_1 q_1 = \sigma\,|\mathbf{p}|^2\cos^2(\phi), \qquad u_2 = p_2 q_2 = \sigma\,|\mathbf{p}|^2\sin^2(\phi)$$

and

$$\begin{aligned}
u &= u_1 + u_2 = \sigma\,|\mathbf{p}|^2,\\
v &= u_1 - u_2 = \sigma\,|\mathbf{p}|^2\cos(2\phi),\\
\widehat{v} &= v/u = \cos(2\phi) \leqslant 1.
\end{aligned}$$

If the reduction is of form

$$q_1 = \sigma p_1^*, \quad q_2 = -\sigma p_2^*, \tag{5.17}$$

then the angle ϕ becomes pure imaginary $\phi = i\widehat{\phi}$, $\widehat{\phi} \in \mathbb{R}$. Then the orientation of the vector $\mathbf{p}$ is determined by the function $\widehat{v} = \cosh(2\widehat{\phi}) \geqslant 1$. Thus, if $\widehat{v} < 1$, the solution satisfies the reduction $\mathbf{q} = \sigma\mathbf{p}^*$. If $\widehat{v} > 1$, the reduction satisfies the relation (5.17). As $\widehat{v} = 1$, the second component of the vector $\mathbf{p}$ is absent ($u_2 = 0$). The sign of the reduction σ is determined by the sign of u:

$$\sigma = \mathrm{sign}(u).$$

Thus, for $n = 1$, the equations for the spectral curve and length of the vector are determined by the same constants. Hence, by the equation of spectral curve we can uniquely determine the equation for the length of the solution vector. At the same time, as it follows from Equation (5.13), the direction of the vector depends on the length and the constants c_7, $(c_5 + c_6)$, $(c_5 - c_6)$. Therefore, in the case $n = 1$ the direction of the solution vector depends on the additional parameter $(c_5 - c_6)$, which can not be determined by the equation of the spectral curve. It follows from Equation (5.13) that as $c_7 \neq 0$ the solution vector oscillates around the direction defined by the identity

$$\cos(2\phi_0) = \widehat{v}_0 = -\frac{(c_5 - c_6)(c_5 + c_6)}{c_7}.$$

Thus, the parameter $c_5 - c_6$ determines the direction, around which the solution vector oscillates.

Sometimes the solutions to vector equations are constructed via those to scalar equations by the rule $p_2 = mp_1$, $q_2 = \pm m^* q_1$. The sign '$-$' corresponds to the reduction (5.17). In this case

$$u_2 = \pm |m|^2 u_1 \quad \text{and} \quad \widehat{v} = \frac{1 \mp |m^2|}{1 \pm |m^2|} = const.$$

It follows from Equation (5.13) that for $c_7 \neq 0$ the function $\widehat{v}$ is constant in one of the following two cases

$$c_7 = -(c_5 + c_6)^2 \quad \text{and} \quad c_7 = -(c_5 - c_6)^2.$$

If $c_7 = -(c_5 + c_6)^2$, then the equation of the spectral curve reads as

$$\left(\mu - \frac{1}{3}\lambda^3 - \frac{1}{3}c_2\lambda\right)\left(\mu^2 + \left(\frac{1}{3}\lambda^3 + \frac{1}{3}c_2\lambda\right)\mu \right.$$
$$\left. -\frac{2}{9}\lambda^6 - \frac{4}{9}c_2\lambda^4 - \frac{1}{9}(c_2^2 + 9(c_5 + c_6))\lambda^2 + \mathcal{A}_3\right) = 0.$$

In this case the spectral curves splits into two components, one of which is a rational curve. The genus of the second component is equal to 2. In this case the direction of the solution vector is determined by the equation

$$\widehat{v} = \widehat{v}_0 = \frac{c_5 - c_6}{c_5 + c_6} \quad \text{or} \quad \frac{c_6}{c_5} = \pm |m|^2 .$$

In this case the length of the solution vector can be both an elliptic function and its degeneration.

If $c_7 = -(c_5 - c_6)^2$, then the spectral curve for $c_5 \neq c_6$ can be either a curve of genus 3 or its degeneration. This seems to be related with the fact that the equation of the spectral curve is independent of the parameter $(c_5 - c_6)$. In this case the length of the solution vector can be both the elliptic function and its degeneration, while the direction is determined by the identity

$$\widehat{v} = \widehat{v}_0 = \frac{c_5 + c_6}{c_5 - c_6} \quad \text{or} \quad \frac{c_6}{c_5} = \mp |m|^2 .$$

In contrast to the Manakov system [1] and Kundu — Eckhaus equation [26], the construction of solutions to vector equations by those of their scalar analogues not always leads to the splitting of the spectral curve into separate components.

In conclusion of this section we note that as $c_7 = 0$, the solution vector loses oscillations of its direction depending on $\phi_2(t_k)$ since in this case the orientation of the solution vector is determined by the formula (5.15).

6. Dynamics of multi–phase solutions

Now a natural question arises how general the geometric approach is. Can it be applied to more complicated solutions? In this section we try to answer this question.

Instead of the relations (5.2) we consider the following identities

$$p_j(x,t) = \sqrt{u_j(x,t)}e^{i\alpha_j(x,t)}, \qquad q_j(x,t) = \sqrt{u_j(x,t)}e^{-i\alpha_j(x,t)}. \tag{6.1}$$

Substituting the expressions (6.1) into Equations (3.2) with $t = t_1$ and simplifying (without using stationary equations), we find

$$\begin{aligned} \partial_{t_1} u &= \partial_x((2s-u)u + iw), \\ \partial_{t_1}\widehat{v} &= 2s\partial_x\widehat{v} - \frac{i\partial_x w}{u}\widehat{v} + i\frac{\partial_x\widehat{w}}{u}. \end{aligned} \tag{6.2}$$

Here

$$w = w_1 + w_2, \qquad \widehat{w} = w_1 - w_2.$$

It follows from the relations (5.2) that $\partial_x\alpha_j = iw_j/(2u_j)$. The expressions for $\partial_{t_1}\alpha_j$ are very bulky and this is why we omit them.

In the case $s = \alpha(\mathbf{p}^t\mathbf{q}) = \alpha u$ (vector derivative equations of nonlinear Schrödinger equations) Equations (6.2) cast into the form

$$\begin{aligned} \partial_{t_1} u &= \partial_x((2\alpha-1)u^2 + iw), \\ \partial_{t_1}\widehat{v} &= 2\alpha u\partial_x\widehat{v} - \frac{i\partial_x w}{u}\widehat{v} + \frac{\partial_x\widehat{w}}{u}. \end{aligned} \tag{6.3}$$

Here the dynamics of the length of vector is determined by the length itself and by an additional function w, the complexity of which increases with the index n. However, the direction of the vector depends on the direction itself, the length and additional functions w and $\widehat{w}$.

As $n = 1$, equations (5.3) yield the identities

$$\begin{aligned} w &= i(c_5 + c_6) + i(c_2 + (2\alpha-1)u)u, \\ \widehat{w} &= i(c_5 - c_6) + i(c_2 + (2\alpha-1)u)\widehat{v}u. \end{aligned} \tag{6.4}$$

Therefore, for $n = 1$ equations (6.3) are of the form

$$\begin{aligned} \partial_{t_1} u &= -c_2\partial_x u, \\ \partial_{t_1}\widehat{v} &= (u - c_2)\partial_x\widehat{v}. \end{aligned}$$

Hence, as $n = 1$, the length of each solution vector to each vector derivative nonlinear Schrödinger equation depends on $(x - c_2t_1)$. At the same time, its orientation varies in a more complicated way.

In the case $t = t_2$ the dynamics of the length of solution is described by the following formula

$$\begin{aligned} \partial_{t_2} u =& \partial_x\Bigg((3s^2 - 2u^2)u + 3isw + \frac{3(w^2 - 2w\widehat{w}\widehat{v} + \widehat{w}^2)}{4u(\widehat{v}^2 - 1)} \\ &+ \frac{3(\partial_x u)^2}{4u} + \frac{3u(\partial_x\widehat{v})^2}{4(1-\widehat{v}^2)} - \partial_x^2 u\Bigg). \end{aligned} \tag{6.5}$$

If the evolution of the vector is determined by the second equation in the hierarchy, then the evolution of the length depends also on its direction. However, since for $n = 1$ the direction of vector depends on its length, the evolution of solution vector in the case $n = 1$ is rather simple

$$\begin{aligned} \partial_{t_2} u &= (c_2^2 - 2(c_5 + c_6))\partial_x u, \\ \partial_{t_2}\widehat{v} &= (c_2^2 - 2(c_5 + c_6) - 2c_2 u)\partial_x\widehat{v}. \end{aligned}$$

Therefore, for $n = 1$ and $t = t_2$, the length of solution vector also depends on the linear combination of the variables $(x+(c_2^2-2(c_5+c_6))t_2)$. And again the evolution of the orientation is determined in a more complicated way. While simplifying the evolution equations, we employed the relations (5.9), (5.11), (5.12) and (6.4).

7. Concluding remarks

Simplification of Equations (6.2) and (6.5) for $n = 1$ indicates that if for other values n, as in the case $n = 1$, the functions w, $\widehat{v}$ and $\widehat{v}$ are expressed only via the function u, then the evolution of the length of solution vector will depend also only on the length itself and will be independent of the orientation of the solution. Then the geometric interpretation of the solution will make sense not only in the case $n = 1$. The consideration of the case $n = 2$ allowed us to answer partially this question. We note that the case $n = 2$ is also interesting by the fact that the monodromy matrix, stationary equations and equation of spectral curve possess other symmetries. The increasing of the genus of spectral curve to $g = 6$ for $n = 2$ and the appearance of the entries of matrix J_2 in the stationary equations can produce additional variants of the behavior of solution vector.

We also note that in the present case we do not study in detail the dependence of the components of solution vector on the function s. This is related with the fact that it follows from Equations (5.2) and (5.3) that this dependence is of rather simple form

$$p_j(s) = p_j(0)\exp\left\{-i\int s(x)dx\right\}, \quad q_j(s) = q_j(0)\exp\left\{i\int s(x)dx\right\}. \tag{7.1}$$

Formula (7.1) points to a proper interpretation of Equations (3.4)–(3.6) as vector derivative forms of nonlinear Schrödinger equation since as in the scalar case (see, for instance, [32]– [35]) there exists a gauge transformation of form (7.1) transforming one equation into the other with preserving the amplitude of solution.

We note that the real and imaginary parts of the components of solutions depends rather essentially on the function s. Since the information is transmitted via optical channels by means of complex codes (see, for instance, [36]), the choice of equation corresponding to a particular waveguide is very important.

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