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INTERPOLATION AND FUNDAMENTAL PRINCIPLE

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Abstract. In this work we study the spaces of functions analytic in convex domains in the complex plane. We consider subspaces of such spaces, which are invariant with respect to the differentiation operator. We study the fundamental principle problem for an invariant subspace, that is, the problem on representing all its elements by a series of eigenfunctions and generalized eigenfunctions of the differentiation operator in this subspace, which are the exponentials and exponential monomials. We provide a complete description of the space of sequences of the coefficients of the series, by which we represent the functions from the invariant subspace. We also study the multiple interpolation problem in the spaces of entire functions of exponential type. We consider the duality of interpolation problem and fundamental principle. This duality problem is completely solved. We established the duality of the fundamental principle problem and interpolation problem for an arbitrary convex domain with no restrictions.

Keywords: exponential monomial, convex domain, fundamental principle, interpolation, duality.

Mathematics Subject Classification: 30D10

1. Introduction

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and their multiplicities n_k. We suppose that $|\lambda_k|$ does not decrease and $|\lambda_k| \to \infty$, $k \to \infty$. Let $D \subset \mathbb{C}$ be a convex domain and $H(D)$ be the space of functions analytic in the domain D with the topology of uniform convergence on compact subsets in D. We note that $H(D)$ is a Fréchet — Schwarz space [1, Ch. I, Thm. 4.6]. The symbol $W(\Lambda, D)$ stands for the closure of the linear span of system

$$\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$$

in the space $H(D)$.

If the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$, then $W(\Lambda, D)$ is a non–trivial ($\neq H(D)$, $\{0\}$) closed subspace in $H(D)$. It follows from the definition of $W(\Lambda, D)$ that it is invariant with respect to the differentiation operator. At the same time the system $\mathcal{E}(\Lambda)$ is the set of eigenfunctions and generalized eigenfunctions of the differentiation operator in $W(\Lambda, D)$ and Λ is its multiple spectrum.

Let $W \subset H(D)$ be a nontrivial closed subspace invariant with respect to the differentiation operator and $\Lambda = \{\lambda_k, n_k\}$ is its multiple spectrum. This is an at most countable set with the only accumulation point ∞ [2, Ch. II, Sect. 7]. In the case, when the spectrum W is finite, it coincides with the space of solutions to homogeneous linear differential equation of finite order with constant coefficients. As a more general example of the invariant space, the set of solutions to the convolution equation $\mu(g(z + w)) \equiv 0$ (or system of such equations) serves, where $\mu \in H^*(D)$ and $H^*(D)$ is the space of linear continuous functionals on the

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space $H(D)$. Particular cases of the convolution equation are linear differential, difference, differential–difference equations with constant coefficients of finite and infinite orders, as well as some types of integral equations.

The main problem in the theory of invariant subspace is the representation of all its functions by means of eigenfunctions and generalized eigenfunctions $z^n e^{\lambda_k z}$ of the differentiation operator. If W is the space of solutions to a linear differential equations with constant coefficients of finite order, then it coincides with the linear span of the system $\mathcal{E}(\Lambda)$. This result is known as the Euler fundamental principle. In this connection the problem of representation of functions $g \in W$ by means of series over the elements in the system $\mathcal{E}(\Lambda)$, that is, by the series

$$\sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n e^{\lambda_k z}, \tag{1.1}$$

(which converge in the topology of the space $H(D)$) is called the fundamental principle problem for an invariant subspace. A first step to the representation (1.1) is to solve the spectral synthesis problem, that is, to find out the conditions, under which the system $\mathcal{E}(\Lambda)$ is complete in the subspace W (in other words, when $W = W(\Lambda, D)$). It is natural to consider the fundamental principle problem only for invariant subspaces admitting the spectral synthesis, that is, for the subspaces of form $W(\Lambda, D)$.

By the end of 40s in the last century a close relation was observed between the fundamental principle problem and the interpolation problem in the spaces of entire functions of exponential type. They turned out to be dual. A.F. Leontiev seems to be first who employed the solvability of the interpolation problem for expansions of solutions of convolution equations into the exponential series. After him this relation was used systematically. The interpolation problem in the space of entire functions is of an independent interest and has a rich history. The studies of the mentioned dual problems conducted first independently have a rich history. Its main milestones are reflected in the works [3] and [4]. In the latter work there was obtained the most general present result on the duality of fundamental principle problem and interpolation problem for an arbitrary convex domain $D \subset \mathbb{C}$ under a single restriction for the relative multiplicity Λ :

$$n_{k(p)}/|\lambda_{k(p)}| \to 0$$

for each sequence $\{\lambda_{k(p)}\}$ such that

$$\lambda_{k(p)}/|\lambda_{k(p)}| \to e^{-i\varphi} \quad \text{and} \quad H(\varphi, D) < +\infty,$$

where

$$H(\varphi, D) = \sup_{z \in D} \operatorname{Re}(z e^{-i\varphi})$$

is the support function of the domain D.

In the present work the mentioned duality problem is completely solved. We establish the duality of fundamental principle problem and interpolation problem for an arbitrary convex domain $D \subset \mathbb{C}$ with not restrictions.

2. Fundamental principle

Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain and $W(\Lambda, D)$ be a nontrivial subspace in the space $H(D)$. By the Hahn — Banach theorem the latter is equivalent to the existence of a nonzero functional $\mu \in H^*(D)$, which vanishes on $W(\Lambda, D)$.

By $\widehat{\mu}$ we denote the Laplace transform of the functional $\mu \in H^*(D)$: $\widehat{\mu}(\lambda) = \mu(e^{\lambda z})$. The function $\widehat{\mu}$ is entire and has an exponential type, that is, for some A, $B > 0$ the inequality $|\widehat{\mu}(\lambda)| \leqslant A \exp(B|\lambda|)$, $\lambda \in \mathbb{C}$, holds. It is known [1, Ch. III, Sect. 12, Thm. 12.3] that the

Laplace transform is an algebraic and topological isomorphism between the spaces $H^*(D)$ and P_D, where P_D is the inductive limit of Banach spaces

$$P_s = \left\{ f \in H(\mathbb{C}) : \|f\|_s = \sup_{re^{i\varphi}\in\mathbb{C}} |f(re^{i\varphi})| \exp(-rH(-\varphi, K_s)) < \infty \right\},$$

$\mathcal{K}(D) = \{K_s\}_{s=1}^{\infty}$ is the sequence of compact sets exhausting the domain D, that is, $K_s \subset \operatorname{int} K_{s+1}$, $s \geqslant 1$, (int denotes the interior of a set), and $D = \cup_{s=1}^{\infty} K_s$. The definition of $\mathcal{K}(D)$ implies that there exist numbers $\alpha_s > 0$, $s \geqslant 1$ such that

$$H(\varphi, K_s) + \alpha_s \leqslant H(\varphi, K_{s+1}), \quad \varphi \in [0, 2\pi]. \tag{2.1}$$

We note that

$$P_1 \subset P_2 \subset \cdots \subset P_s \subset \tag{2.2}$$

and the set P_D consists of the union of sets P_s, $s \geqslant 1$. The space P_D is a so–called LN^* space, that is, it is the union of the sequence of Banach spaces, for which the embeddings (2.2) are true and completely continuous (this is impled by the estimate (2.1)). Therefore, P_D is separable and complete [1, Ch. I, Sect. 2, Thm. 2.4].

Since $W(\Lambda, D)$ is nontrivial, there exists a nonzero functional $\mu \in H^*(D)$, which vanishes on all functions in the system $\mathcal{E}(\Lambda)$. Then its Laplace transform $\widehat{\mu} = f \in P_D$ vanishes at the points λ_k with the multiplicity at least n_k.

Let K be the adjoint diagram of a function f [2, Ch. I, Sect. 5]. By Pólya theorem [2, Ch. I, Sect. 5, Thm. 5.4] the identity

$$h_f(\varphi) = H(-\varphi, K), \quad \varphi \in [0, 2\pi],$$

is true, where h_f is the (upper) indicator of the function f

$$h_f(\varphi) = \varlimsup_{r\to+\infty} \frac{\ln|f(re^{i\varphi})|}{r}.$$

The inclusion $f \in P_D$ means that

$$H(-\varphi, K) = h_f(\varphi) \leqslant H(-\varphi, K_s) < H(-\varphi, D), \quad \varphi \in [0, 2\pi], \tag{2.3}$$

for some $s \geqslant 1$. This implies the inclusion $K \subset D$.

The existence of the function f with the mentioned properties implies [5, Ch. IV, Sect. 1, Subsect. 2] the existence of a biorthogonal to $\mathcal{E}(\Lambda)$ system of functionals

$$\Xi(\Lambda, D) = \{\mu_{k,n}\}_{k=1,n=0}^{\infty,n_k-1} \subset H^*(D)$$

such that

$$\mu_{k,n}(z^l e^{\lambda_j z}) = 1 \quad \text{for} \quad j = k, \quad l = n, \qquad \mu_{k,n}(z^l e^{\lambda_j z}) = 0 \quad \text{otherwise}.$$

It is constructed by means of the function f. Since there are infinitely many functions f with the mentioned properties, the system $\Xi(\Lambda, D)$ is not uniquely determined.

Suppose that the series (1.1) converges uniformly to a function g on compact subsets of the domain D. Using the continuity and linearity of the functionals $\mu_{k,n}$, we then obtain $d_{k,n} = \mu_{k,n}(g)$, $k \geqslant 1$, $n = \overline{0, n_k - 1}$. Hence, the following statement is true.

Lemma 2.1. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain $W(\Lambda, D)$ be a nontrivial subspace in the space $H(D)$. If a function $g \in W(\Lambda, D)$ is represented by the series (1.1) converging in the topology of $H(D)$, then the representation is unique and its coefficients can be calculated by the formula $d_{k,n} = \mu_{k,n}(g)$, where $\Xi(\Lambda, D) = \{\mu_{k,n}\}$ is an arbitrary biorthogonal sequence to the system $\mathcal{E}(\Lambda)$.*

Let $\Lambda=\{\lambda_k,n_k\}$, $\lambda_k=r_ke^{i\varphi_k}$, $k\geqslant 1$, and D be a convex domain. We are going to describe the sequence of the coefficients $\{d_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$, for which the series (1.1) converges in the domain D. For each $s\geqslant 1$ we introduce the Banach space

$$Q_s(\Lambda)=\{d=\{d_{k,n}\}:\|d\|_s=\sup_{k,n}|d_{k,n}|s^n\exp(r_kH(-\varphi_k,K_s))<\infty\}.$$

By (2.1) the inequalities

$$\|d\|_1\leqslant\|d\|_2\leqslant\cdots\leqslant\|d\|_s\leqslant\cdots,\qquad d\in Q(D,\Lambda), \tag{2.4}$$

hold. Therefore, the chain of embeddings

$$Q_1(\Lambda)\supset Q_2(\Lambda)\supset\cdots\supset Q_s(\Lambda)\supset\cdots \tag{2.5}$$

holds. We let

$$Q(D,\Lambda)=\bigcap_{s=1}^{\infty}Q_s(\Lambda).$$

In the space $Q(D,\Lambda)$ we introduce the metric

$$\rho(d,b)=\sum_{s=1}^{\infty}2^{-s}\frac{\|d-b\|_s}{1+\|d-b\|_s}.$$

With this metric, $Q(D,\Lambda)$ becomes the Fréchet space. It is easy to observe that for a convex domain $D_1\supset D$ the embedding $Q(D_1,\Lambda)\subset Q(D,\Lambda)$ is true. We let $\Lambda=\{\lambda_k,n_k\}$ and

$$m(\Lambda)=\overline{\lim_{k\to\infty}}\frac{n_k}{|\lambda_k|},\quad n(\Lambda)=\overline{\lim_{j\to\infty}}\frac{j}{|\xi_j|},\quad \sigma(\Lambda)=\overline{\lim_{j\to\infty}}\frac{\ln j}{|\xi_j|},$$

where $\{\xi_j\}$ is a non–decreasing in the modulus sequence formed by the points λ_k, and each λ_k appears in this sequence exactly n_k times.

Lemma 2.2. *Let D be a convex domain, $\Lambda=\{\lambda_k,n_k\}$ and $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Then for each $s\geqslant 1$ there exist $C_s>0$ and an index $m(s)$ such that*

$$\sum_{k=1,n=0}^{\infty,m_k-1}|d_{k,n}|\sup_{z\in K_s}|z^ne^{z\lambda_k}|\leqslant C_s\|d\|_{m(s)},\quad d=\{d_{k,n}\}\in Q(D,\Lambda). \tag{2.6}$$

Proof. Since the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$, there exists an entire function vanishing at the points λ_k with multiplicities at least n_k. Then by the Lindelöf theorem [6, Ch. I, Thm. 15] we have $n(\Lambda)<\infty$. This implies that $\sigma(\Lambda)=0$. Thus, all assumptions of Lemma 2.6 from the work [7] are satisfied. The statement of the cited lemma coincides with the statement of this lemma. The proof is complete. □

We are going to show that under some natural conditions the space $Q(D,\Lambda)$ coincides with the space of the coefficients converging in the domain D series of the form (1.1).

Let $\overline{\lambda}$ be the complex conjugate of λ. By $\Theta(\Lambda)$ we denote the set of the limits of all converging sequences of form $\{\overline{\lambda}_{k_j}/|\lambda_{k_j}|\}_{j=1}^{\infty}$. It is obvious that $\Theta(\Lambda)$ is a closed subset of the circumference $S(0,1)=\{z\in\mathbb{C}:|z|=1\}$. We let

$$m(\Lambda,\mu)=\sup\overline{\lim_{j\to\infty}}\frac{n_{k_j}}{\lambda_{k_j}},$$

where the supremum is taken over the subsequences $\{\lambda_{k_j}\}$ such that $\overline{\lambda}_{k_j}/|\lambda_{k_j}|\to\mu$. If $\mu\notin\Theta(\Lambda)$, then the identity $m(\Lambda,\mu)=0$ is obviously true.

Let D be a convex domain. We let

$$J(D)=\{e^{i\varphi}\in S(0,1):H(\varphi,D)=+\infty\}.$$

By $\overline{J(D)}$ we denote the closure of the set $J(D)$.

Lemma 2.3. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$ and*

$$m(\Lambda, \mu) = 0, \quad \mu \in \Theta(\Lambda) \setminus \overline{J(D)}. \tag{2.7}$$

Then the following statements are equivalent.

1) *The series (1.1) converges in the domain D.*

2) *The inclusion $d = \{d_{k,n}\} \in Q(D, \Lambda)$ holds.*

Proof. As in Lemma 2.2, the inequality $n(\Lambda) < \infty$ holds. This implies that $m(\Lambda) < \infty$ and $\sigma(\Lambda) = 0$. Hence, all assumptions of Theorem 2.1 in the work [7] are satisfied. The statement of this theorem coincides with the statement of this theorem. The proof is complete. □

Remark 2.1. *According to Lemmas 2.2 and 2.3, under the assumptions of Lemma 2.3 the pointwise convergence of the series (1.1) in the domain D is equivalent to its absolute and uniform convergence on the compact subsets in this domain.*

The condition (2.7) is a natural restriction for the sequence Λ. It was proved in [8, Thm. 4.2] that this condition is necessary for the validity of the fundamental principle in the invariant subspace $W(\Lambda, D)$. We formulate this result in a convenient for us form.

Lemma 2.4. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Assume that each function $g \in W(\Lambda, D)$ is expanded into the series (1.1), which converges uniformly on compact subsets in the domain D. Then (2.7) holds.*

Let $\mathbb{E}$ be an operator, which maps the element $d = \{d_{k,n}\} \in Q(D, \Lambda)$ into the sum of series (1.1).

Lemma 2.5. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Then the operator $\mathbb{E}$ is defined on the entire space $Q(D, \Lambda)$, $\mathbb{E}(Q(D, \Lambda)) \subseteq W(\Lambda, D)$, the operator $\mathbb{E}$ is injective and continuous. If, in addition, the operator $\mathbb{E}$ is surjective, then it is an isomorphism of linear topological spaces $Q(D, \Lambda)$ and $W(\Lambda, D)$.*

Proof. By (2.6), for each element $d = \{d_{k,n}\} \in Q(D, \Lambda)$, the series (1.1) converges uniformly on compact subsets in the domain D. Therefore, the operator $\mathbb{E}$ is defined on the entire space $Q(D, \Lambda)$ and $\mathbb{E}(d) = g \in H(D)$. Moreover, since the function g is represented by the series (1.1), we have $g \in W(\Lambda, D)$. By (2.6) for each $s \geqslant 1$ we have

$$\sup_{z \in K_s} |\mathbb{E}(d)| \leqslant C_s \|d\|_{m(s)}, \quad d \in Q(D, \Lambda).$$

This means that $\mathbb{E} : Q(D, \Lambda) \to W(\Lambda, D)$ is a continuous operator. According to Lemma 2.1, the operator $\mathbb{E}$ is injective. Assume that $\mathbb{E}$ is surjective. Then by the theorem on open mapping for Fréchet spaces [9, Ch. VI, Sect.3, Thm. 8] the operator $\mathbb{E}$ is an isomorphism of linear topological spaces $Q(D, \Lambda)$ and $W(\Lambda, D)$. The proof is complete. □

Theorem 2.1. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Then the following statements are equivalent.*

1) *The fundamental principle holds in the space $W(\Lambda, D)$.*
2) *The operator $\mathbb{E} : Q(D, \Lambda) \to W(\Lambda, D)$ is an isomorphism.*

Proof. Suppose that Assertion 1) holds true, that is, each function $g \in W(\Lambda, D)$ is expanded into the series (1.1), which converges uniformly on compact subsets in the domain D. Then by Lemma 2.4 the identity (2.7) holds. This is why in accordance with Lemma 2.3 each function

$g \in W(\Lambda, D)$ is expanded into the series (1.1) with coefficients $\{d_{k,n}\} \in Q(D, \Lambda)$. This means that $\mathbb{E}$ is surjective. Therefore, by Lemma 2.5 Assertion 2) is true.

Suppose that Assertion 2) is true. Then $\mathbb{E}$ is surjective, that is, each function $g \in W(\Lambda, D)$ is expanded into the series (1.1) with the coefficients $\{d_{k,n}\} \in Q(D, \Lambda)$. By (2.6) this series converges uniformly on the compact sets in the domain D. Then Assertion 1) is true. The proof is complete. □

3. Interpolation

By means of the Laplace transform the fundamental principle problem is reduced to the multiple interpolation problem in the space of entire functions of exponential type. We are going to study this problem.

Let $f \in P_D$. Then there exists an index s such that the indicator h_f satisfies the inequality (2.3). We note one more property of the indicator [6, Ch. I, Sect. 18, Thm. 28]: for each $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that

$$\ln |f(re^{i\varphi})| \leqslant (h_f(\varphi) + \varepsilon)r, \quad \varphi \in [0, 2\pi], \quad r \geqslant R(\varepsilon). \tag{3.1}$$

By (2.1) this implies that the inequality (2.3) yields the inclusion $f \in P_D$. Thus, $f \in P_D$ if and only if (2.3) is satisfied. In other words, $f \in P_D$ if and only if its adjoint diagram K lies in the domain D.

We also introduce the spaces of complex sequences

$$\mathcal{R}_s(\Lambda) = \{b = \{b_{k,n}\} : \|b\|^s = \sup_{k,n} |b_{k,n}| s^{-n} \exp(-r_k H(-\varphi_k, K_s)) < \infty\}, \quad s \geqslant 1,$$

where $r_k e^{i\varphi_k} = \lambda_k$ and $\{K_s\} = \mathcal{K}(D)$. Let $\mathcal{R}(D, \Lambda)$ be the inductive limit of the Banach spaces $\mathcal{R}_s(\Lambda)$. Then

$$\mathcal{R}(D, \Lambda) = \bigcup_{s=1}^{\infty} \mathcal{R}_s(\Lambda).$$

The space $\mathcal{R}(D, \Lambda)$ is LN^* space.

We consider the multiple interpolation problem

$$f^{(n)}(\lambda_k) = b_{k,n}, \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1. \tag{3.2}$$

First of all, let us find natural estimates for the complex sequence $\{b_{k,n}\}$ under the assumption $f \in P_D$.

On the space P_D we define a linear operator Σ so that it maps each function f into the sequence $b = \{b_{k,n}\} = \{f^{(n)}(\lambda_k)\}$.

Lemma 3.1. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Then for each function $f \in P_D$ the sequence $b = \Sigma(f)$ belongs to the space $\mathcal{R}(D, \Lambda)$. The operator $\Sigma : P_D \to R(D, \Lambda)$ is continuous.*

Proof. Let $f \in P_s \subset P_D$. The definitions of the indicator h_f and space P_s imply (2.3). This means that the adjoint diagram K of the function f lies in the compact set K_s. Then the contour ∂K_{s+1} envelops, by (2.1), the compact set K_s, as well as the adjoint diagram K. This is why we have the integral representation for an entire function of exponential type [2, Ch. I, Sect. 5, Thm. 5.2]

$$f(\lambda) = \frac{1}{2\pi i} \int\limits_{\partial K_{s+1}} e^{z\lambda} \gamma(z)dz,$$

where γ is the Borel associated function for f. We recall that the adjoint diagram is a convex hull of the set of singular points of the function γ. Differentiating under the integral sign, for all $k \geqslant 1$ and $n = \overline{0, n_k - 1}$ we get

$$|f^{(n)}(\lambda_k)| \leqslant \frac{1}{2\pi i} \sup_{z \in K_{s+1}} |z^n e^{z\lambda_k}| \sup_{z \in K_{s+1}} |\gamma(z)| l_{s+1}, \qquad (3.3)$$

where l_{s+1} is the length of the contour ∂K_{s+1}. We choose an index $m(s) \geqslant s+1$ such that

$$\max_{z \in K_{s+1}} |z| \leqslant m(s). \qquad (3.4)$$

Then by (3.3) we get the inclusion $\Sigma(f) \in \mathcal{R}_{m(s)}(\Lambda) \subset \mathcal{R}(D, \Lambda)$.

Let us prove the continuity of Σ. In order to do this, we need to estimate γ at all points of the contour ∂K_{s+1}. Let $z_0 \in \partial K_{s+1}$. For each boundary point of the compact set K_{s+1}, there is at least one support line, which passes this point. In other words, there exists $\varphi \in [0, 2\pi]$ such that $H(\varphi, K_{s+1}) = \mathrm{Re}(z_0 e^{-i\varphi})$. In the half–plane

$$\{z : \mathrm{Re}(ze^{-i\varphi}) > H(\varphi, K)\}$$

we have the integral representation for the function γ [2, Ch.I, Sect. 5, Thm. 5.3]:

$$\gamma(z) = \int_0^{+\infty} \mu f(t\mu) e^{-zt\mu} dt, \quad \mu = e^{-i\varphi}.$$

Since by (2.1) and (2.3)

$$\mathrm{Re}(z_0 e^{-i\varphi}) = H(\varphi, K_{s+1}) > H(\varphi, K_s) \geqslant H(\varphi, K) = h_f(-\varphi),$$

the representation holds at the point z_0. Then in view of the inclusion $f \in P_s$, the choice of φ and (2.1) we find

$$|\gamma(z_0)| \leqslant \int_0^{+\infty} |f(t\mu)| e^{-\mathrm{Re}(z_0 t\mu)} dt \leqslant \|f\|_s \int_0^{+\infty} \exp\left(tH(\varphi, K_s) - tH(\varphi, K_{s+1})\right) dt$$
$$\leqslant \|f\|_s \int_0^{+\infty} \exp(-t\alpha_s) dt = \frac{\|f\|_s}{\alpha_s}.$$

By (3.3) and (3.4) this implies

$$\left|f^{(n)}(\lambda_k)\right| \leqslant \frac{l_{s+1}\|f\|_s}{2\pi\alpha_s} (m(s))^n \exp\left(r_k H(-\varphi_k, K_{s+1})\right), \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1.$$

Since $m(s) \geqslant s+1$, by (2.1)

$$H(-\varphi_k, K_{s+1}) \leqslant H(-\varphi_k, K_{m(s)}), \quad k \geqslant 1.$$

This is why in accordance with the previous inequality we have

$$\|\Sigma(f)\|^{m(s)} \leqslant \frac{l_{s+1}\|f\|_s}{2\pi\alpha_s}.$$

The operator Σ is continuous on the inductive limit P_D if its restriction on each P_s is continuous [9, Ch. V, Sect. 2, Prop. 5]. Since Σ maps the space P_s into $\mathcal{R}_{m(s)}(\Lambda)$, the continuity of $\Sigma : P_D \to \mathcal{R}(D, \Lambda)$ follows from the continuity of the mapping $\Sigma : P_s \to \mathcal{R}_{m(s)}(\Lambda)$, which holds by the latter inequality. The proof is complete. $\square$

Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$ and $I(\Lambda, D)$ be the kernel of the operator $\Sigma : P_D \to \mathcal{R}(D, \Lambda)$. This is a closed subspace in P_D. Since $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$, this subspace is nontrivial. The subspace $I(\Lambda, D) \subset P_D$ consists exactly of the functions, which vanish (at least) at the points λ_k with the multiplicities at least n_k.

The quotient space $P_D/I(\Lambda, D)$, as P_D, is LN^* space and is the union of an increasing sequence of Banach spaces $P_{s,0}$. An element $[f] \in P_D/I(\Lambda, D)$ belongs to $P_{s,0}$ if and only if some representative $g \in P_D$ of the equivalency class $[f]$ belongs to P_s. At the same time, the norm $\|[f]\|_s$ is equal to the infimum of the norms $\|g\|_s$ over all representatives $g \in P_s$ in the class $[f]$. In the usual way the operator Σ generates the operator Σ_0 acting from $P_D/I(\Lambda, D)$ into $\mathcal{R}(D, \Lambda)$. The mapping

$$\Sigma_0 : P_D/I(\Lambda, D) \to \mathcal{R}(D, \Lambda) \tag{3.5}$$

is injective and also continuous by Lemma 3.1 and the definition of the quotient topology. Thus, the following statement holds.

Lemma 3.2. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. The operator (3.5) is continuous and injective.*

Lemma 3.3. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Suppse that the operator (3.5) is surjective. Then Σ_0 is an isomorphism between the linear topological spaces $P_D/I(\Lambda, D)$ and $\mathcal{R}(D, \Lambda)$.*

Proof. By Lemma 3.2, the operator (3.5) is continuous and injective. If it is also surjective, then by the theorem on open mapping [9, App. 1, Thm. 2] for separable spaces covered by a countable family of its Fréchet subspaces (LN^* spaces are obviously among these subspaces) the operator Σ_0 is an isomorphism of linear topological space. The proof is complete. □

Remark 3.1. *The surjectivity of the operator Σ_0 (or equivalently of the operator Σ) means that the interpolation problem (3.2) is solvable in the space P_D for each right hand side $b = \{b_{k,n}\} \in \mathcal{R}(D, \Lambda)$.*

Remark 3.2. *Let $\operatorname{ort} W(\Lambda, D)$ be the family of functionals $\upsilon \in H^*(D)$ annihilating the subspace $W(\Lambda, D)$. The system $\mathcal{E}(\Lambda)$ is complete in $W(\Lambda, D)$ and hence, $\upsilon \in \operatorname{ort} W(\Lambda, D)$ if and only if*

$$\upsilon(z^n e^{\lambda_k z}) = 0, \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1.$$

Thus, in view of the definition of Laplace transform we find that the subspace $I(\Lambda, D)$ is the set of Laplace transforms of the functionals $\upsilon \in \operatorname{ort} W(\Lambda, D)$.

4. Duality

Theorem 4.1. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. The following statements are equivalent.*

1) *The fundamental principle holds in the subspace $W(\Lambda, D)$.*
2) *The interpolation problem (3.2) is solvable in the space P_D for each right hand side $b = \{b_{k,n}\} \in \mathcal{R}(D, \Lambda)$.*

Proof. Assume that Assertion 1) holds and $b = \{b_{k,n}\} \in \mathcal{R}(D, \Lambda)$. We choose an index $s \geqslant 1$ such that $b \in \mathcal{R}_s(\Lambda)$. Then

$$|b_{k,n}| \leqslant \|b\|^s s^n \exp(r_k H(-\varphi_k, K_s)), \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1. \tag{4.1}$$

By Theorem 2.1 and Lemma 2.1 each function $g \in W(\Lambda, D)$ is represented by the series (1.1), where $d = \{d_{k,n}\} \in Q(D, \Lambda)$. In particular, $d \in Q_{s+1}(\Lambda)$, that is,

$$|d_{k,n}| \leqslant \|d\|_{s+1}(s+1)^{-n} \exp\left(-r_k H(-\varphi_k, K_{s+1})\right), \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1.$$

By (4.1) and (2.1) this implies

$$\sum_{k=1,n=0}^{\infty,n_k-1} |d_{k,n}||b_{k,n}| \leqslant \|d\|_{s+1}\|b\|^s \sum_{k=1}^{\infty} n_k \exp\left(r_k(H(-\varphi_k, K_s) - H(-\varphi_k, K_{s+1}))\right)$$
$$\leqslant \|d\|_{s+1}\|b\|^s \sum_{k=1,n=0}^{\infty,n_k-1} n_k e^{-r_k \alpha_s}.$$

As in Lemma 2.3, the identity $\sigma(\Lambda) = 0$ holds. This is why by [10, Lm. 2.1] the latter series converges. Therefore,

$$\sum_{k=1,n=0}^{\infty,n_k-1} |d_{k,n}||b_{k,n}| \leqslant C_s \|d\|_{s+1}\|b\|^s, \quad b \in \mathcal{R}_s(\Lambda), \quad d \in Q(D, \Lambda).$$

Thus, in view of Theorem 2.1, the linear continuous functional

$$\upsilon(g) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} b_{k,n}$$

is well–defined on the subspace $W(\Lambda, D)$. By the Hahn — Banach theorem, it can be continued to a linear continuous functional on the entire space $H(D)$. Let $f \in P_D$ be the Laplace transform of the functional $\upsilon \in H^*(D)$. It follows from the definitions of the Laplace transform and functional υ that

$$f^{(n)}(\lambda_k) = \upsilon(z^n e^{\lambda_k z}) = b_{k,n}, \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1.$$

This means that Assertion 2) is true.

Let Assertion 2) be true. Since $H(D)$ is a Fréchet — Schwarz space, in view of Remark 3.2 on Lemma 3.3 the isomorphisms hold [11]:

$$W(\Lambda, D) \cong (H^*(D)/\operatorname{ort} W(\Lambda, D))^* \cong (P_D/I(\Lambda, D))^*.$$

Let g be an arbitrary function in the subspace $W(\Lambda, D)$ and functionals $\theta \in (H^*(D)/\operatorname{ort} W(\Lambda, D))^*$ and $\omega \in (P_D/I(\Lambda, D))^*$ are associated with this function under the mentioned isomorphisms. We fix $z \in D$. If δ_z is the Dirac functional concentrated at the point z, then

$$g(z) = \delta_z(g) = [\delta_z](g) = \theta([\delta_z]) = \omega([f_z]),$$

where f_z is the Laplace transform of the functional δ_z and $[\delta_z]$, $[f_z]$ are the equivalence classes respectively from the spaces $H^*(D)/\operatorname{ort} W(\Lambda, D)$, $P_D/I(\Lambda, D)$. It is easy to see that the function $f_z(\lambda)$ coincides with $e^{z\lambda}$. Thus, the identity

$$g(z) = \omega([e^{z\lambda}])$$

holds. According to Assertion 2), Lemma 3.3 and Remark 3.1, the space $P_D/I(\Lambda, D)$ is isomorphic to $\mathcal{R}(D, \Lambda)$. This is why the exists a functional $h \in (\mathcal{R}(D, \Lambda))^*$ such that

$$g(z) = \omega([e^{z\lambda}]) = h(\Sigma(e^{z\lambda})) = h(\{z^n e^{\lambda_k z}\}). \tag{4.2}$$

We choose a number s such that $b = \{b_{k,n}\} = \{z^n e^{\lambda_k z}\} \in \mathcal{R}_s(\Lambda)$. The functional h is continuous on $\mathcal{R}(D,\Lambda)$. This is why its restriction on the Banach space $\mathcal{R}_{s+1}(\Lambda)$ (as well as on each other $\mathcal{R}_m(\Lambda)$) is continuous [9, Ch. V, Sect. 2, Prop. 5], that is,

$$|h(\{a_{k,n}\})| \leqslant c_s \|a\|^{s+1}, \quad a = \{a_{k,n}\} \in \mathcal{R}_{s+1}(\Lambda). \tag{4.3}$$

We consider the elements $e^{k,n} = \{a_{l,j}^{k,n}\} \in \mathcal{R}_{s+1}(\Lambda)$, where $a_{l,j}^{k,n} = 1$ if $l = k$, $j = n$, and $a_{l,j}^{k,n} = 0$ otherwise. We let

$$d_{k,n} = h(e^{k,n}), \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1, \tag{4.4}$$

$$b(m,p) = \sum_{k=1,n=0}^{m-1,n_k-1} b_{k,n} e^{k,n} + \sum_{n=1}^{p} b_{m,n} e^{m,n}, \quad p = \overline{0, n_m - 1}.$$

Then by (4.2), (4.3), (2.1) we obtain

$$\begin{aligned}
&\left| g(z) - \sum_{k=1,n=0}^{m-1,n_k-1} d_{k,n} z^n e^{\lambda_k z} - \sum_{n=1}^{p} d_{m,n} z^n e^{\lambda_m z} \right| = |h(b - b(m,p))| \\
&\leqslant c \sup_{k \geqslant m, n = \overline{1, n_k}} |b_{k,n}| (s+1)^{-n} \exp\left(-r_k H(-\varphi_k, K_{s+1})\right) \\
&\leqslant c_s \sup_{k \geqslant m, n = \overline{1, n_k}} |b_{k,n}| s^{-n} \exp\left(-r_k H(-\varphi_k, K_s)\right) \exp\left(-r_k (H(-\varphi_k, K_{s+1}) - H(-\varphi_k, K_s))\right) \\
&\leqslant c_s \|b\|^s \sup_{k \geqslant m, n = \overline{1, n_k}} e^{-r_k \alpha_s} \to 0, \quad m \to \infty.
\end{aligned}$$

Thus, at each point $z \in D$ the representation

$$g(z) = \sum_{k=1,n=0}^{\infty, n_k - 1} d_{k,n} z^n e^{\lambda_k z}$$

holds. By (4.3) and (4.4) we have

$$|d_{k,n}| \leqslant c_s \|e^{k,n}\|^{s+1} = c_s (s+1)^{-n} \exp\left(-r_k H(-\varphi_k, K_{s+1})\right), \quad n = \overline{0, n_k - 1}, \quad k \geqslant 1.$$

Since the inclusion $e^{k,n} \in \mathcal{R}_{s+1}(\Lambda)$ and inequality (4.3) are true for all $s \geqslant 0$, this implies $d = \{d_{k,n}\} \in Q(D,\Lambda)$. Then by Lemma 2.2 the latter series converges uniformly on the compact subsets in the domain D and Assertion 1) holds. The proof is complete. □

By Theorems 2.1, 4.1 and Lemma 3.3 we obtain the following result.

Theorem 4.2. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. The following statements are equivalent.*

1) *The fundamental principle holds in the space $W(\Lambda, D)$.*
2) *The operator $\mathbb{E} : Q(D,\Lambda) \to W(\Lambda, D)$ is isomorphism.*
3) *The operator $\Sigma_0 : P_D / I(\Lambda, D) \to \mathcal{R}(D,\Lambda)$ is isomorphism.*
4) *The interpolation problem (3.2) is solvable in the space P_D for each right hand side $b = \{b_{k,n}\} \in \mathcal{R}(D,\Lambda)$.*

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