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INTERPOLATION SETS IN SPACES OF FUNCTIONS OF FINITE ORDER IN HALF–PLANE

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Abstract. We consider free interpolation problems, the study of which was initiated by A.F. Leontiev. We obtain new criterions for the interpolation property of sets in the space of analytic in the upper half–plane functions of finite order. We provide examples of interpolation sets in the space of analytic in the upper half–plane functions of finite order. These examples are similar to interpolation sets in the space of analytic and bounded in the upper half–plane functions. In particular, we provide examples of sets satisfying the Newman condition and uniform Frostman condition.

Keywords: free interpolation, half–plane, finite order, interpolation set.

Mathematics Subject Classification: 30E05, 30D15

1. Introduction

1.1. Notation and terminology. If an inequality (identity) holds for all sufficiently large values of a variable, then it is called *asymptotic inequality (identity)*. By $K, M, \dots, \varepsilon, \delta, \dots$ we denote positive constants, which can change in arguing. For instance, we can use the phrase «if $f(r) < 3M$, then $f(r) < M$».

By $\mathbb{N} := \{1, 2, \dots\}$ we denote the set of (*natural*) numbers, $\mathbb{C}$ is the *complex plane with the real axis* $\mathbb{R}$ and *positive semi–axis* $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geqslant 0\}$, $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the *upper half–plane*. One–point sets are written without braces if this causes no ambiguity. For instance, $\overline{\mathbb{R}} := \mathbb{R} \cup \pm\infty$ and $\overline{\mathbb{R}}_+ := \mathbb{R} \cup +\infty$ are respectively *extended* real axis and positive semi–axis with usual *modulus* $|\cdot|$ as for $\mathbb{C}$, and $|\pm\infty| := +\infty$, ∞ is the infinity in the complex half–plane $\mathbb{C}_+$, that is, the sequence of points $z_n \to \infty$ as $n \to \infty$ if $\lim\limits_{n\to\infty} |z_n| = +\infty$, $\overline{n_1, n_2}$ is the set of integer numbers $n : n_1 \leqslant n \leqslant n_2$. The *open circle of radius* r *centered at a point* a is denoted by $C(a, r)$, by $B(a, r) = \overline{C(a, r)}$ we denote a *closed circle*, G_+ denotes the *intersection of the set* G *with the half–plane* $\mathbb{C}_+$, that is, $G_+ := G \cap \mathbb{C}_+$, $\overline{G}$ is the *closure of the set* G.

By a^+ we denote $(|a| + a)/2$, in particular, $\ln^+ 0 := 0$. By $[\cdot]$ we denote the integer part of a number, $A = \{a_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$ is a sequence of points without repetitions with limiting points only on the real axis and at ∞. Hereinafter, unless otherwise stated, we suppose $r_n = |a_n|$, $\theta_n = \arg a_n$, $r = |z|$, $\theta = \arg z$, where $0 \leqslant \arg z \leqslant \pi$ for $z \in \overline{\mathbb{C}}_+$, $n_A^+(G) := n^+(G) = \sum\limits_{a_n \in G} \sin\theta_n$, in particular, $n_A^+(R) := n^+(C(0, R))$.

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1.2. Definition of order of analytic in upper half–plane function. Let ρ ($r \in \mathbb{R}_+$) be the proximate order in the Valiron sense, $\lim\limits_{r\to\infty} \rho(r) = \varrho > 0$. We denote $V(r) = r^{\rho(r)}$. Let f be a holomorphic function in $\mathbb{C}_+$. The proximate order ρ is called semi–formal order of a function f if there exists a constant $M > 0$ (depending on f and independent of z) such that for all $z \in \mathbb{C}_+$ the inequality

$$\ln|f(z)| < MV(|z|)$$

holds and Levin condition is satisfied: there exist numbers $q \in (0, 1)$ and $\delta \in (0, \pi/2)$ such that in each domain

$$D(R, q, \delta) = \{z : \, qR < |z| < R/q, \, \delta < \arg z < \pi - \delta\}$$

there exist a point z, at which the inequality $\ln|f(z)| > -MV(|z|)$ holds.

The definition of semi–formal order of function is due to A.F. Grishin (see, for instance, [1]). We denote by $[\rho, \infty)^+$ the space of functions, for which ρ is the semi–formal order.

1.3. Interpolation problem in space $[\rho, \infty)^+$. The problems we consider in this work are free interpolation problems, the study of which was initiated by Leontiev [2]–[4]. A multiple interpolation problem in the space $[\rho, \infty)^+$, $\varrho > 0$, was solved in works [5], [6]. The formulation of problem and theorem given below are its particular case, when the multiplicities of the interpolation nodes are equal to one. The case $\varrho = 0$ (zero order) was considered in [7].

We give some notion and notation. We introduce the *Nevanlinna canonical factor*

$$B_q(u, v) = \begin{cases} \dfrac{\bar{v}(u - v)}{v(u - \bar{v})} & \text{for} \quad q = 0, \\ B_0(u, v) \exp\left(\displaystyle\sum_{j=1}^{q} \frac{u^j}{j} \left(\frac{1}{v^j} - \frac{1}{\bar{v}^j} \right) \right) & \text{for} \quad q \in \mathbb{N}. \end{cases}$$

Let ρ be the proximate order in the Valiron sense, $\lim\limits_{r\to\infty} \rho(r) = \varrho > 0$, $q = [\varrho]$. If a sequence A such that

$$n_A^+(R) \leqslant KV(R) \tag{1.1}$$

for some $K > 0$ (that is, it has a finite upper argument density) and ϱ is a non–integer number, then the infinite product

$$E(z) := E_A(z) = \prod_{r_n \leqslant 1} B_0(z, a_n) \prod_{r_n > 1} B_q(z, a_n)$$

converges uniformly on compact sets in $\mathbb{C}_+$.

The function $E(z)$ is called the *canonical function* (*canonical product*) *of sequence* A.

The case of integer $\varrho \geqslant 1$ is more complicated. In this case for the uniform convergence of the function $E(z)$ on compact sets in $\mathbb{C}_+$ a finite upper argument density is not enough, we need some argument symmetry of the points a_n. For constructing the canonical product we add a factor without zero, the total measure of which is concentrated on the real axis (see [6]). Such function is called the *adjoint function* of the sequence A.

By a given sequence A we define the families of functions

$$\Phi_D^+(z, \alpha) = \frac{n_D^+(C(z, \alpha|z|) \setminus \{a_n\})}{V(|z|)},$$

where a_n is the point in the support of the sequence A closest to the point z (if there are several such points, we choose an arbitrary of them). We let

$$I_A^+(z, \delta) = \sin\theta \int_0^{\delta} \frac{\Phi_D^+(C(z, \alpha)\, d\alpha}{\alpha(\alpha + \sin\theta)^2}, \quad \theta = \arg z.$$

Definition 1.1. *The sequence* $A = \{a_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$, *all limiting points of which belong to* $\mathbb{R} \cup \infty$, *is called interpolating for the space* $[\rho, \infty)^+$ *if for each sequence of complex numbers* b_n, $n \in \mathbb{N}$, *obeying the condition*

$$\sup_n \frac{\ln^+ |b_n|}{V(|a_n|)} < \infty,$$

there exists a function $F \in [\rho, \infty)^+$ *such that*

$$F(a_n) = b_n, \quad n \in \mathbb{N}. \tag{1.2}$$

Hereafter we suppose that the condition $|a_n| \geqslant 1$ holds; this is a technical condition and it can be easily omitted. Let us formulate the versions of the theorem from [5], [6] for the case of simple interpolation.

Theorem 1.1. *The following three statements are equivalent.*

1) *The sequence* A *is interpolating for the space* $[\rho, \infty)^+$.
2) *If* $\varrho = \lim\limits_{r\to\infty} \rho(r)$ *is non–integer, then the canonical product of the sequence* A *satisfies the condition*

$$\sup_n \frac{1}{V(|a_n|)} \ln \frac{1}{\operatorname{Im} a_n |E'(a_n)|} < \infty. \tag{1.3}$$

2') *If* $\varrho \geqslant 1$ *is integer, then it follows from 1) that the condition (1.3) is satisfied for each adjoint function* $E(z)$ *of the sequence* A. *And vice versa, if (1.3) holds for at least one adjoint function* $E(z)$ *of the sequence* A, *then 1) holds.*
3) *Condition* (1.1) *holds and for each* $\delta > 0$

$$\sup_{z\in\mathbb{C}_+} I_A^+(z, \delta) < \infty.$$

Theorem 1.2. *The following two statements are equivalent.*

1) *The sequence* A *is an interpolating sequence for the space* $[\rho, \infty)^+$.
2) *The condition* (1.1) *holds and for each* $\delta > 0$

$$\ln \left| \frac{a_n - \bar{a}_k}{a_n - a_k} \right| \leqslant V(r_n), \quad n \neq k, \tag{1.4}$$

$$\Phi_z^+(\alpha) \leqslant \alpha, \quad \frac{\sin\theta}{2} \leqslant \alpha \leqslant \delta \quad (\theta = \arg z), \tag{1.5}$$

$$\Phi_z^+(\alpha) \leqslant \frac{\sin\theta}{\ln \frac{\sin\theta}{\alpha}}, \quad 0 \leqslant \alpha \leqslant \frac{\sin\theta}{2}. \tag{1.6}$$

1.4. Interpolation condition in space $[\rho, \infty)^+$**.** Let a sequence $A = \{a_n = r_n e^{i\theta_n}\}_{n=1}^{\infty}$ belong to the upper half–plane $A \in \mathbb{C}_+$ and there exist $K > 0$ such that conditions (1.1) are satisfied and

$$\prod_{\substack{r_n/2 < r_k < 3r_n/2 \\ k \neq n}} \left| \frac{a_k - a_n}{a_k - \bar{a}_n} \right| \geqslant \exp\left[-KV(r_n)\right]. \tag{1.7}$$

In this case we say that the sequence A satisfies the interpolation condition for the space $[\rho, \infty)^+$ (or, following [10], $\mathcal{I}_+(\rho)$–condition). The meaning of the condition (1.7) is that each point in the sequence A is located far enough from other points of this sequence. By Theorem 1.1, if the sequence A satisfies the interpolation condition (1.7), then it is interpolating for the space $[\rho, \infty)^+$.

Interpreting the half–plane $\mathbb{C}_+$ as a model of the plane in the Lobachevskii geometry, by $\sigma(z_1,z_2)$ we denote the non–Euclidean distance between aribtrary points z_1 and z_2 in the half–plane $\mathbb{C}_+$:

$$\sigma(z_1,z_2)=\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|,\qquad u=\frac{z_1-z_2}{z_1-\overline{z_2}}.$$

We can write the condition (1.7) as

$$\prod_{\substack{r_n/2<r_k<3r_n/2\\ k\neq n}}\tan\left(2\sigma(a_k,a_n)\right)\geqslant\exp\left[-KV(r_n)\right].$$

Here is another writing of the condition (1.7):

$$\sum_{\substack{r_n/2<r_k<3r_n/2\\ k\neq n}}G(a_k,a_n)\leqslant KV(r_n),$$

where

$$G(z,\zeta)=\ln\left|\frac{z-\bar{\zeta}_n}{z-\zeta}\right|$$

is the Green function of the half–plane $\mathbb{C}_+$.

2. Other conditions

In this section $A=\{a_n=r_ne^{i\theta_n}\}$ still denotes a sequence in the upper half–plane, $A_n\overset{\text{def}}{=}A\setminus\{a_n\}$, $a_n\in A$.

1) (*Sparse conditions.*) The sequence A is called *sparse* (or ρ_+–sparse) if there exists $K>0$ such that the inequality

$$\left|\frac{a_k-a_n}{a_k-\bar{a}_n}\right|\geqslant\exp\left[-K(r_n)\right]$$

holds.

It is clear that an interpolating for the space $[\rho,\infty)^+$ sequence A is necessarily ρ_+–sparse.

2) (*Newman condition.*) We renumerate the sequence A so that

$$\left|\frac{a_{n+1}-i}{a_{n+1}+i}\right|>\left|\frac{a_n-i}{a_n+i}\right|.$$

Let A satisfy the condition (1.1). Suppose that there exists c, $0<c<1$, such that the sequence A converges exponentially to the boundary, that is, the inequality

$$\frac{\operatorname{Im}a_{n+1}}{\operatorname{Im}a_n}\cdot\frac{r_n^2}{r_{n+1}^2}\leqslant c$$

holds. Following [10], we say that the sequence A satisfies the condition $(\mathrm{N}_+(\rho))$.

3) (*Frostman condition.*) The sequence A is said to satisfy the uniform Frostman condition for order ρ (following [10], condition $\mathcal{F}_+(\rho)$) if it satisfies the condition (1.1) and there exists $K>0$ such that

$$\sup_{t\in\mathbb{R}}\sum_{r_n/2\leqslant r_k\leqslant 3r_n/2}\frac{\operatorname{Im}a_k}{|\bar{a}_k-t|}\leqslant KV(r_n).$$

3. Various reformulations of interpolation condition

Theorem 3.1. *Let the sequence* $A = \{a_n = r_n e^{i\theta_n}\}_{n=1}^{\infty}$ *belong the half–plane* $\mathbb{C}_+$. *The following statements are equivalent*

1. *The sequence* A *is an interpolating sequence for the space* $[\rho, \infty)^+$.

2.1. *The condition* (1.1) *holds.*

2.2. *There exists* $K > 0$ *such that the inequality*

$$\tau_n(A) \stackrel{\text{def}}{=} \sum_{\substack{r_n/2<r_k<3r_n/2 \\ k\neq n}} \ln\left(1 + \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - a_n|^2}\right) \leqslant KV(r_n) \tag{3.1}$$

holds.

3. *The sequence* A *satisfies the condition* $\mathcal{R}_+(\rho)$ *and*

3.1. *The condition* (1.1) *is satisfied;*

3.2. *There exists* $K > 0$ *such that the inequality*

$$S_n(A) \stackrel{\text{def}}{=} \sum_{r_n/2<r_k<3r_n/2} \ln\left(1 + \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - \bar{a}_n|^2}\right) \leqslant KV(r_n)$$

holds.

There exists $K > 0$ *such that the inequality*

$$\widehat{S}_n(A) \stackrel{\text{def}}{=} \sum_{r_n/2<r_k<3r_n/2} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - \bar{a}_n|^2} \leqslant KV(r_n) \tag{3.2}$$

holds.

Proof. We first of all mention two important simple identities

$$\left|\frac{a-b}{\bar{a}-b}\right|^2 = 1 - \frac{4\operatorname{Im} a \operatorname{Im} b}{|\bar{a}-b|^2}, \quad a, b \in \mathbb{C}_+, \tag{3.3}$$

$$\left|\frac{\bar{a}-b}{a-b}\right|^2 = 1 + \frac{4\operatorname{Im} a \operatorname{Im} b}{|a-b|^2}, \quad a, b \in \mathbb{C}_+, \quad a \neq b. \tag{3.4}$$

Indeed,

$$\begin{aligned}\left|\frac{a-b}{\bar{a}-b}\right|^2 &= 1 - \frac{|\bar{a}-b|^2 - |a-b|^2}{|\bar{a}-b|^2} 1 - \frac{(\bar{a}-b)(a-\bar{b}) - (a-b)(\bar{a}-\bar{b})}{|\bar{a}-b|^2} \\ &= 1 - \frac{|a|^2 - ba - \bar{a}\bar{b} + |b|^2 - |a|^2 + b\bar{a} + a\bar{b} - |b|^2}{|\bar{a}-b|^2} \\ &= 1 + \frac{(a-\bar{a})(b-\bar{b})}{|\bar{a}-b|^2} = 1 - \frac{4\operatorname{Im} a \operatorname{Im} b}{|\bar{a}-b|^2}.\end{aligned}$$

Similarly,

$$\left|\frac{\bar{a}-b}{a-b}\right|^2 = 1 - \frac{|a-b|^2 - |\bar{a}-b|^2}{|a-b|^2} = 1 - \frac{(a-\bar{a})(b-\bar{b})}{|\bar{a}-b|^2} = 1 + \frac{4\operatorname{Im} a \operatorname{Im} b}{|\bar{a}-b|^2}.$$

Let us prove the implication 1) $\Longrightarrow$ 2). It follows from the identity (3.4) and condition (1.4) that

$$\prod_{\substack{r_n/2<r_k<3r_n/2 \\ k\neq n}} \left(1 + \frac{4\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - a_n|^2}\right) \leqslant \exp[KV(r_n)].$$

Taking the logarithm, by this inequality we get (3.1).

We proceed to proving the implication 2) $\Longrightarrow$ 3). It follows from (3.1) and (3.4) that

$$2\ln\left|\frac{\bar a_n - a_k}{a_n - a_k}\right| \leqslant \tau_n(A) \leqslant KV(r_n)$$

for $n \neq k$. This implies the inequality

$$\left|\frac{a_n - a_k}{\bar a_n - a_k}\right| \geqslant \exp\left[-\frac{K}{2}V(r_n)\right],$$

and this is why the sequence A satisfies the condition $\mathcal{R}_+(\rho)$. Moreover, $S_n(A) \leqslant \tau_n(A)$, since for $a, b \in \mathbb{C}_+$ the inequality

$$\left|\frac{a-b}{\bar a - b}\right| \leqslant 1$$

is true.

In order to prove the inequality (3.2), we observe that if the sequence A obeys the interpolation condition, then by the identity (3.3) and the elementary inequality $-\ln(1-x) \geqslant x$ $(0 \leqslant x < 1)$ we have

$$\begin{aligned} KV(r_n) &\geqslant -\ln \prod_{\substack{r_n/2<r_k<3r_n/2\\ k\neq n}} \left|\frac{a_k - a_n}{a_k - \bar a_n}\right| \\ &= -\sum_{r_n/2<r_k<3r_n/2} \ln\left(1 - \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - \bar a_n|^2}\right) \geqslant \sum_{\substack{r_n/2<r_k<3r_n/2\\ k\neq n}} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - \bar a_n|^2}\end{aligned}$$

for each point $a_n \in A$. This implies (3.2).

We are going to prove the implication 3) $\Longrightarrow$ 1). We note that the conditions (1.1) and (3.2) imply the inequality

$$\sup_n \sum_{k=1}^{\infty} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_k - \bar a_n|^2 r_k^{\varrho+1}} < \infty. \tag{3.5}$$

Indeed, the condition (1.1) implies the convergence of the series

$$K_1 = \sum_{k=1}^{\infty} \frac{\sin\theta_k}{r_k^{\varrho+1}} < \infty. \tag{3.6}$$

By (3.2) and (3.6) we obtain

$$\sum_{k\neq n} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|\bar a_k - a_n|^2 r_k^{\varrho+1}} = \sum_{\substack{r_n/2<r_k<3r_n/2\\ k\neq n}} + \sum_{|a_k - a_n|>r_n/2} \leqslant 2^{\varrho+2}\widehat{S}_n(A) + 4K_1 \sin\theta_n.$$

Since for $k = n$ the corresponding summand in (3.5) is of the form $1/(4r_n^{\varrho+1})$, we obtain (3.5).

We note that the condition (3.5) played a main role in works [5], [6] (see also [7]–[9]) for constructing a series, which solved the interpolation problem. The proof is complete. □

4. Union of interpolating sequences

It is easy to understand that the union of two interpolating sequences not necessarily possesses the interpolation property since the points of one set can closely approach the points of the other set. However, the following lemma holds.

Lemma 4.1. *The sequence A obeying the condition $\mathcal{R}_+(\rho)$, which is equal to the union of several interpolating sequences, is an interpolating sequence for the space $[\rho,\infty)^+$.*

Proof. Let A_1, A_1, ..., A_q be interpolating sequences. Then the measures μ_{A_j} $(j = 1, 2, \dots, q)$ satisfy the conditions (1.5) and (1.6). The sum of these measures obviously satisfies these conditions and hence, the measure μ_A satisfies them as well, where $A = \bigcup\limits_{j=1}^{q} A_j$. If A is a sequence satisfying the condition $\mathcal{R}_+(\rho)$, then it also satisfies the condition (1.4) of Theorem 1.2. Therefore, by Theorem 1.2 the sequence A is an interpolating sequence in the space. □

5. Relations between conditions $\mathcal{I}_+(\rho)$ and $N_+(\rho)$

Lemma 5.1. *A sequence A satisfying the condition $N_+(\rho)$ is an interpolating sequence for the space $[\rho, \infty)^+$.*

Proof. The condition $N_+(\rho)$ and the elementary inequality

$$\frac{(|a+i| - |a-i|)|b+i|}{|a+i|(|b+i| - |b-i|)} \leqslant 8\,\frac{\operatorname{Im} a}{\operatorname{Im} b} \cdot \frac{|b|^2}{|a|^2}$$

imply

$$\frac{|a_{n+1}+i| - |a_{n+1}-i|}{|a_{n+1}+i|} \leqslant c\frac{|a_n+i| - |a_n-i|}{|a_n+i|}.$$

Since for all points a, b in the upper half–plane $\mathbb{C}_+$

$$\left|\frac{a-b}{a-\bar{b}}\right| \geqslant \frac{|a-i||b+i| - |a+i||b-i|}{|a+i||b+i| - |a-i||b-i|},$$

we have

$$\prod_{\substack{r_n/2<r_j<3r_n/2\\ j\neq n}} \left|\frac{a_j-a_n}{a_j-\bar{a}_n}\right| \geqslant \prod_{\substack{r_n/2<r_j<3r_n/2\\ j>n}} \frac{|a_j-i||a_n+i| - |a_j+i||a_n-i|}{|a_j+i||a_n+i| - |a_j-i||a_n-i|}$$
$$\cdot \prod_{\substack{r_n/2<r_j<3r_n/2\\ j<n}} \frac{|a_n-i||a_j+i| - |a_n+i||a_j-i|}{|a_j+i||a_n+i| - |a_j-i||a_n-i|} := \prod_{j>n} \cdot \prod_{j<n}. \tag{5.1}$$

If $j > n$, then

$$\frac{|a_j+i| - |a_j-i|}{|a_j+i|} \leqslant c^{j-n}\frac{|a_n+i| - |a_n-i|}{|a_n+i|},$$

and thus,

$$|a_j-i||a_n+i| - |a_j+i||a_n-i| \geqslant (1-c^{j-n})|a_j+i|(|a_n+i| - |a_n-i|).$$

On the other hand,

$$|a_j+i||a_n+i| - |a_j-i||a_n-i| \leqslant (1+c^{j-n})|a_j+i|(|a_n+i| - |a_n-i|).$$

Thus,

$$\prod_{j>n} \geqslant \prod_{j=1}^{\infty} \frac{1-c^j}{1+c^j}.$$

We then find

$$-\ln\prod_{j>n} = -\sum_{j=1}^{\infty} \ln\frac{1-c^j}{1+c^j} \leqslant 2\sum_{j=1}^{\infty} \frac{c^j}{1+c^j}.$$

By elementary calculations we obtain

$$\ln\prod_{j>n} \geqslant 2\sum_{j=1}^{\infty} \frac{c^j}{1+c^j} = \frac{2}{\ln c \cdot \ln(1+c)}. \tag{5.2}$$

If $j < n$, then
$$\frac{|a_n+i|-|a_n-i|}{|a_n+i|} \leqslant c^{n-j}\frac{|a_j+i|-|a_j-i|}{|a_j+i|}.$$
Thus,
$$|a_n-i||a_j+i|-|a_n+i||a_j-i| \geqslant (1-c^{n-j})|a_n+i|(|a_j+i|-|a_j-i|),$$
$$|a_n+i||a_j+i|-|a_j-i||a_n-i| \leqslant (1+c^{n-j})|a_n+i|(|a_j+i|-|a_j-i|).$$
Hence,
$$\prod_{j<n} \geqslant \prod_{j=1}^{\infty}\frac{1-c^{n-j}}{1+c^{n-j}},$$
and
$$\ln\prod_{j<n} \geqslant \frac{2}{\ln c\cdot\ln(1+c)}. \tag{5.3}$$
It follows from (5.2) and (5.3) that the sequence A obeys the condition $(\mathcal{I}_+(\rho))$. □

Corollary 5.1. *Each sequence, the limiting points of which are located on the real axis and at infinity, contains an interpolating for the space $[\rho,\infty)^+$ subsequence.*

Corollary 5.2. *Let a sequence A satisfy the condition $(\mathcal{I}_+(\rho))$ and all its points are on the imaginary axis. Then the necessary condition for A being interpolating for the space $[\rho,\infty)^+$ is*
$$\frac{a_n}{a_{n+1}} \leqslant \exp[KV(r_n)]$$
for some $K > 0$, and the sufficient condition is
$$\frac{a_n}{a_{n+1}} \leqslant c < 1.$$

Proof. We have already shown that the interpolation is possible if a_n tends to the boundary (in the present case to infinity) with an exponential rate. And vice versa, if the interpolation is possible, then there exists $K > 0$ such that
$$\exp[-KV(r_n)] \leqslant \frac{a_{n+1}-a_n}{a_{n+1}-\bar{a}_n} = \frac{a_{n+1}-a_n}{a_{n+1}+a_n} \leqslant \frac{a_{n+1}}{a_n}.$$
□

6. Interpolation of sparse sequence obeying uniform Frostman condition

In this section $A = \{a_n = r_n e^{i\theta_n}\}$ still denotes a sequence in the upper half–plane, $A_n \stackrel{\text{def}}{=} A\setminus\{a_n\}$, $a_n \in A$.

Lemma 6.1. *A ρ_+–sparse sequence A satisfying the condition $(\mathcal{F}_+(\rho))$ is an interpolating sequence.*

Proof. Let A_0 be a finite subsequence of the sequence A. The function
$$f(z) = \sum_{\substack{r_n/2\leqslant r_k\leqslant 3r_n/2\\ a_k\in A_0}} \frac{\operatorname{Im} a_k}{|\bar{a}_k - z|}$$
is subharmonic in $\overline{\mathbb{C}_+}$ and this is why
$$\sum_{\substack{r_n/2\leqslant r_k\leqslant 3r_n/2\\ a_k\in A_0}} \frac{\operatorname{Im} a_k}{|\bar{a}_k - z|} \leqslant \max_{t\in\mathbb{R}} f(t) \leqslant \sup_{t\in\mathbb{R}} \sum_{r_n/2\leqslant r_k\leqslant 3r_n/2} \frac{\operatorname{Im} a_k}{|\bar{a}_k - t|}.$$

Hence,

$$\sum_{r_n/2\leqslant r_k\leqslant 3r_n/2}\frac{\operatorname{Im} a_k}{|\bar a_k-a_n|}\leqslant KV(r_n)$$

for each point a_n in the set A. At the same time

$$\frac{\operatorname{Im} a_n}{|\bar a_k-a_n|}<1,\quad a_k,a_n\in\mathbb{C}_+.$$

Therefore,

$$\sum_{r_n/2<r_k<3r_n/2}\frac{\operatorname{Im} a_k\operatorname{Im} a_n}{|\bar a_k-a_n|^2}\leqslant KV(r_n).$$

Together with the condition of ρ_+–sparseness this inequality implies the interpolation property of the sequence A (Theorem 3.1). □

7. Sets close to interpolating ones

In what follows we are interesting in sets close to interpolating ones in the space $[\rho,\infty)^+$, that is, to the sets obeying the $\mathcal{I}_+(\rho)$–condition.

By $\Omega(a,r)$ we denote the circle

$$\Omega(a,r)\overset{\text{def}}{=}\left\{z\in\mathbb{C}:\left|\frac{z-a}{z-\bar a}\right|\leqslant r\right\}.$$

The set E' is *K–translated with respect to the set E ($E\subset\mathbb{C}_+$) for proximate order ρ* if there exists a mapping ω of the set E onto the set E' such that $\omega(\xi)\in\Omega(\xi,\exp[-KV(|\xi|)])$ for all $\xi\in E$. In this case we say that the mapping ω is *K–translation of the set E for proximate order ρ*.

We being with proving some useful inequalities.

Lemma 7.1. *Let $a,b,c,d\in\mathbb{C}_+$,*

$$u=\frac{a-b}{\bar a-b},\qquad v=\frac{c-a}{\bar c-a},\qquad w=\frac{c-d}{\bar c-d}.$$

1) *If*

$$\frac{b-c}{b-\bar c}\leqslant\frac{\delta}{4},\qquad u\geqslant\delta,$$

then $v\geqslant\delta^\alpha$, where

$$\alpha=\frac{\ln(\delta/2)}{\ln\delta}.$$

2) *If*

$$\frac{b-c}{b-\bar c}\leqslant\frac{\delta}{4},\qquad \frac{a-d}{\bar a-d}\leqslant\frac{\delta}{4},\qquad u\geqslant\delta,$$

then $w\geqslant\delta^{\alpha\beta}$, where

$$\beta=\frac{\ln(\delta/4)}{\ln\delta/2}.$$

Proof. We apply the conformal mapping

$$w=\frac{z-i}{z+i}$$

of the half–plane $\mathbb{C}_+$ onto the unit circle $C(0,1)$ and similar lemma for the circle [10, Lm. 8] and this completes the proof. □

Lemma 7.2. *Let $a, b \in \mathbb{C}_+$, $|a| \geqslant 1$. If*

$$\frac{a-b}{b-\bar{a}} \leqslant \varepsilon, \qquad 0<\varepsilon<1,$$

then

1) *the inequality holds*

$$|a-b| \leqslant \frac{2\varepsilon \operatorname{Im} a}{1-\varepsilon};$$

2) *the inequality holds*

$$\left|\arg \frac{a}{b}\right| \leqslant \frac{2\varepsilon \operatorname{Im} a}{1-\varepsilon}.$$

Proof. 1) We have

$$\frac{b-\bar{a}}{a-b} \geqslant \frac{1}{\varepsilon}.$$

This yields

$$\left|1-\frac{2\varepsilon \operatorname{Im} a}{a-b}\right| \geqslant \frac{1}{\varepsilon}.$$

This inequality implies

$$|a-b| + 2\operatorname{Im} a \geqslant \frac{|a-b|}{\varepsilon}$$

and hence, we obtain the inequality in Assertion 1).

2) The needed inequality follows from that in Assertion 1) since for $|a|>1$ and $|a-b| \leqslant r<1$, we have $|\arg a - \arg b| \leqslant r$.

The proof is complete. □

Lemma 7.3. *Let a sequence $A=\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$, $a_n=r_n e^{i\theta_n}$, obeys the relation*

$$\sum_{a_n \in C(0,r)} \sin\theta_n \leqslant C_1 V(r) \tag{7.1}$$

for some $C_1>0$. Then for a given $K>0$ there exists $C_2=C_2(K)>0$ such that each K–translated sequence $A'=\{a'_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$, $a'_n=r'_n e^{i\theta'_n}$, for proximate order $\rho(r)$ with respect to the sequence A satisfies the condition (7.1) with the constant C_2.

Proof. Using Assertion 2 of Lemma 7.2, we obtain

$$\begin{aligned}\sum_{a'_n \in C(0,r)} \sin\theta'_n &\leqslant \sum_{a_n \in C(0,2r)} \sin\theta_n + \sum_{a'_n \in C(0,r)} |\sin\theta'_n - \sin\theta_n| \\ &\leqslant \sum_{a_n \in C(0,2r)} \sin\theta_n + \sum_{a_n \in C(0,2r)} \sin\theta_n \exp(-C_2 V(r_n)), \quad C_2>0.\end{aligned} \tag{7.2}$$

The series $\sum\limits_{a_n \in C(0,2r)} \sin\theta_n \exp(-C_2 V(r_n))$ converges. Now inequalities (7.1) and (7.2) imply the statement of the lemma. □

Theorem 7.1. *Let a sequence A obeys the $\mathcal{I}_+(\rho)$–condition. Then there exists a number $K>0$ such that the following statements hold:*

1) *If $a_n \in A$, $\zeta \in \Omega\left(a_n, \exp\left[-KV(r_n)\right]\right)$, then*

$$|E_n^A(\zeta)| := \prod_{\substack{r_n/2<r_k<3r_n/2 \\ a_k \in A_n}} \left|\frac{a_k-\zeta}{\bar{a}_k-\zeta}\right| \geqslant \exp\left[-KV(r_n)\right].$$

2) *Each sequence A', which is K–translated for proximate order ρ with respect to A satisfies $\mathcal{I}_+(\rho)$–condition.*

3) *Each K–translation for proximate order ρ of the sequence A is a one–to–one correspondence.*

4) *If $a_k, a_n \in A$ and $a_k \neq a_n$, then*

$$\Omega\left(a_k, \exp\left[-KV(r_k)\right]\right) \cap \Omega\left(a_n, \exp\left[-KV(r_n)\right]\right) = \varnothing.$$

Proof. 1) Condition (5.1) implies the existence of a number $K > 0$ such that for each $a_n \in A$

$$\prod_{r_n/2<r_k<3r_n/2//\omega\in A_n} \left|\frac{a_k - a_n}{a_k - \bar{a}_n}\right| \geqslant \exp\left[-KV(r_n)\right].$$

This inequality yields

$$\left|\frac{a_k - a_n}{a_k - \bar{a}_n}\right| \geqslant \exp\left[-KV(r_n)\right].$$

We apply Assertion 1) in Lemma 7.1 with $\delta = \exp\left[-KV(r_n)\right]$. We let

$$a = a_k, \qquad b = a_n, \qquad \xi, \eta \in E, \qquad \zeta \in \Omega\left(\eta, \frac{1}{2}\exp\left[-KV(|\eta|\right]\right),$$

and we get

$$\prod_{|\xi|/2<|\eta|<3|\xi|/2//\eta\in E_\xi} \left|\frac{\xi - \zeta}{\bar{\xi} - \zeta}\right| \geqslant (\delta)^{\ln(\delta/2)/\ln(\delta)} = \frac{1}{2}\exp\left[-KV(|\xi|\right].$$

2) Arguing as in the proof of Assertion 1), we get the existence of numbers C_1, $K > 0$ and a proximate order $\rho(r)$, $\lim\limits_{r\to\infty} \rho(r) = \rho$, such that the condition (1.7) holds for each $\zeta \in E$

$$\prod_{\substack{|\zeta|/2<|\gamma|<3|\zeta|/2 \\ \gamma\in E_\zeta}} \left|\frac{\gamma - \zeta}{\gamma - \bar{\zeta}}\right| \geqslant \exp\left[-C_1V(|\zeta|\right].$$

This inequality yields

$$\left|\frac{\gamma - \zeta}{\gamma - \bar{\zeta}}\right| \geqslant -C_1V(|\zeta|.$$

We let

$$K = C_1, \qquad \xi, \eta \in E', \qquad \eta \in \frac{1}{4}\Omega\left(\gamma, \exp\left[-KV(|\gamma|\right]\right), \qquad \xi \in \frac{1}{4}\Omega\left(\zeta, \exp\left[-KV(|\zeta|)\right]\right).$$

Applying Assertion 2) of Lemma 7.1 with $\delta = \exp\left[-KV(|\xi|\right]$, we get

$$\prod_{\substack{|\xi|/2<|\eta|<3|\xi|/2 \\ \eta\in E_\xi}} \left|\frac{\xi - \eta}{\bar{\xi} - \eta}\right| \geqslant (\delta)^{\ln(\delta/2)/\ln(\delta)\cdot\ln(\delta/4)/\ln(\delta/2)} = \frac{1}{4}\exp\left[-KV(|\xi|\right].$$

Assertions 3) and 4) follow from Assertion 2). Indeed, if

$$\xi, \eta \in E', \qquad \eta \in \frac{1}{4}\Omega\left(\gamma, \exp\left[-KV(|\gamma|\right]\right), \qquad \xi \in \frac{1}{4}\Omega\left(\zeta, \exp\left[-KV(|\zeta|\right]\right),$$

then

$$\left|\frac{\xi - \eta}{\bar{\xi} - \eta}\right| \geqslant -C_1V(|\xi| > 0,$$

that is, $\xi \neq \eta$. □

8. Weakly regular sets

In [11], weakly regular sets in the upper half–plane $[\rho(r),\infty)_+$ were studied. Their definition is as follows.

Definition 8.1. *A sequence of points $A=\{a_n\}_{n=1}^{\infty}$ in the upper half–plane $\mathbb{C}_+$ is called weakly regular sequence with respect to the proximate order ρ (or $WR^+(\rho)$–set) if one of the following conditions* (C) *or* (C′) *is satisfied:*

(C$_+$)*–condition*

1) *there are no multiple points or points with the same absolute value in A;*
2) $A\cap C(0,2)=\varnothing$;
3) *the condition holds*

$$n^+(C(0,r))\leqslant KV(r),\quad K>0\,;$$

4) *there exists a number $d>0$ such that for all points a_n, $a_k\in A$ such that $|a_n|\geqslant|a_k|$ we have*

$$|a_n|\geqslant|a_k|+d\,\mathrm{Im}\,a_k/V(|a_k|).$$

(C$'_+$)*–condition*

1) *there are no multiple points or points with the same absolute value in A;*
2) $A\cap C(0,2)=\varnothing$;
3) *the condition holds*

$$n^+(C(0,r))\leqslant KV(r),\quad K>0\,;$$

4) *there exists a number $d>0$ such that the circles of radii*

$$r_n=d(\sin(\arg a_n))^{1/2}|a_n|^{1-\frac{\rho(|a_n|)}{2}}$$

centered a_n are disjoint.

The sets obeying (C$_+$)–condition played an important role in [12], [13].

By using the geometric criterion of the interpolation property of sequence, the following theorem was proved.

Theorem 8.1. *Let a sequence $A=\{a_n\}_{n=1}^{\infty}$, $A\in\mathbb{C}_+$, be a $WR^+(\rho)$–set. Then A is an interpolating sequence in the space $[\rho,\infty)_+$.*

9. Interpolation problem for compactly supported sequences

We recall that a sequence of complex numbers $\{b_n\}_{n=1}^{\infty}$ is called *compactly supported* if all terms in this sequence vanish starting from some one, that is, $b_n=0$ for $n\geqslant n_0\geqslant 1$.

Interpolation problems in spaces of entire functions take special place for compactly supported sequences and are related with the distribution of zeroes of entire functions. In particular, Bratishchev and Korobejnik [14] considered a problem on multiple interpolation in the space $[\rho,\infty)$ of entire functions of finite order (include zero order) and normal type for compactly supported sequences. The obtained result can be formulated as follows.

Theorem 9.1 (Bratishchev, Korobejnik)**.** *The following three statements are equivalent.*

1) *The interpolation problem 1.2 in the space $[\rho,\infty)$ is solvable for each compactly supported sequence.*

2) *The set of interpolation nodes* $\{a_n\}_{n=n_0}\infty$ *obeys the condition*

$$\limsup_{r\to\infty} \frac{n(r)}{rV'(r)} < \infty,$$

where $n(r)$ *is the number of points in the sequence* $\{a_n\}_{n=n_0}^{\infty}$ *located in the circle* $|z| < r$.

3) *The interpolation problem (1.2) in the space* $[\rho, \infty)$ *is solvable for at least one non–zero compactly supported sequence.*

We consider problem on simple interpolation for compactly supported sequences in the space $[\rho, \infty)_+$.

Theorem 9.2. *The following three statements are equivalent.*

1) *The interpolation problem (1.2) in the space* $[\rho, \infty)_+$ *is solvable for each compactly supported sequence.*
2) *The condition* (1.1) *is satisfied.*
3) *The interpolation problem (1.2) in the space* $[\rho, \infty)_+$ *is solvable for at least one non–zero compactly supported sequence.*

Proof. The implication 1) $\Rightarrow$ 3) is trivial. Let us prove the implication 3) $\Rightarrow$ 2). Suppose that the interpolation problem (1.2) is solvable in the space $[\rho, \infty)_+$ for a non–zero compactly supported sequence $\{b_n\}_{n=1}^{\infty}$, where $b_n = 0$ for all $n > n_0$ and $b_{n_0} \neq 0$. Let a function $F \in [\rho, \infty]_+$ be such that

$$F(a_n) = b_n, \quad n = 1, 2, \dots, n_0, \qquad F(a_n) = 0, \quad n = n_0 + 1, \dots.$$

The zero set $\{z_n\}$ of the function contains the sequence $\{a_n\}_{n=n_0}^{\infty}$ and by Theorem 1.2 it obeys the condition (1.1). Since just finitely many terms in the sequence $\{a_n\}_{n=1}^{\infty}$ does not belong to the set $\{z_n\}$, the sequence $\{a_n\}_{n=1}^{\infty}$ also satisfies the condition (1.1). The proof of the implication 3) $\Rightarrow$ 2) is complete.

We proceed to proving the implication 2) $\Rightarrow$ 1). Suppose that the condition (1.1) is satisfied. Then the canonical product $E(z)$ of the sequence $\{a_n\}_{n=1}^{\infty}$ belongs to the space $[\rho, \infty)_+$. Let $\{b_n\}_{n=1}^{\infty}$ be a compactly supported sequence. Since the series

$$F(z) = \sum_{n=1}^{\infty} \frac{E(z) b_n}{(z - a_n) E'(a_n)}$$

contains just finitely many non–zero terms, the function $F(z)$ belongs to the space $[\rho, \infty)_+$. The function $F(z)$ solves the interpolation problem (1.2). The proof of the implication 2) $\Rightarrow$ 1) is complete. □

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