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ESTIMATES FOR TORSIONAL RIGIDITY OF CONVEX DOMAIN VIA NEW GEOMETRIC CHARACTERISTICS

L.I. GAFIYATULLINA, R.G. SALAKHUDINOV

Abstract. We introduce new geometric characteristics of a convex domain with finite boundary length and provide an algorithm for calculating them. A series of isoperimetric inequalities between new functionals and known integral characteristics of the domain are proved. Some of the inequalities have a wide class of extremal domains. We consider applications of new characteristics to the problem on estimating the torsional rigidity of a convex domain.

Keywords: convex domain, function of distance to the boundary, torsional rigidity, isoperimetric inequality, extremal domain.

Mathematics Subject Classification: 26A99, 26D99, 30C99

1. Introduction

Let G be a simply connected domain in the plane. One of its important characteristics is the functional

$$\mathbf{P}(G) := 2\int\limits_G \mathrm{u}(x,G)\mathrm{dA}, \tag{1.1}$$

where $\mathrm{u}(x,G)$ is the stress function, which is the solution of problem

$$\begin{cases} \triangle u = -2, & x \in G, \\ u = 0, & x \in \partial G, \end{cases}$$

while dA denotes the differential area element. It is well–known that the stress function is well and uniquely defined (see, for instance, [1], [2]).

The first experimental results of calculations with a torsion balance were made in 1784 by Coulomb. He discovered that the force required to twist a homogeneous rod is directly proportional to its length l, the angle θ by which it should be twisted, some physical constant κ depending on the material from which the rod is made, and some characteristic P depending only on the shape of the cross–section of the homogeneous rod

$$F = \kappa l \theta P.$$

The quantity P was later called the torsional rigidity. We note that P is proportional to the functional (1.1). The functional $\mathbf{P}(G)$ is called the torsional rigidity of the domain G. Although the definition (1.1) was not known to Coulomb, he proposed the formula

$$P = \frac{\pi r^4}{2}$$

for calculating the torsional rigidity with a circular cross–section, where r is the cross-section radius. It is well–known that the constant 2 in the definition (1.1) appeared due to the Coulomb's

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formula. It turned out that the functional (1.1) is an important physical characteristic of a domain not only in torsion theory, but also in hydrodynamics.

One of the classical problems of mathematical physics is the calculation of $\mathbf{P}(G)$ for specific cross–sections and the study of its properties. Exact formulas for calculating torsional rigidity turned out to be a difficult problem [2], so the natural problem is to estimate torsional rigidity through simpler characteristics of the domain. This line of research turned out to be closely related to the isoperimetric inequalities in mathematical physics. Hundreds of papers were written in this direction [3], among which are the works of O. Cauchy, V. Saint–Venant, G. Polya, G. Szegö, E. Makai, L.E. Payne, F.G. Avkhadiev.

In 1951, Pólya and Szegö [1] showed that for each convex domain the inequality

$$\frac{1}{2}\mathbf{A}(G)\boldsymbol{\rho}(G)^2 \leqslant \mathbf{P}(G), \tag{1.2}$$

holds, where $\boldsymbol{\rho}(G)$ is the radius of the maximal circle contained in G and $\mathbf{A}(G)$ is the area of the domain G. The identity in (1.2) is attained for the circle.

Later Makai showed [4] that for each convex domain the inequality

$$\mathbf{P}(G) < 4\int\limits_G \rho(x,G)^2 \mathrm{dA}, \tag{1.3}$$

holds, where $\rho(x,G)$ is the function of distance from the point x to the boundary G. The constant 4 in the inequality (1.3) is the best possible and is attained at the limit on the sequence of rectangles $\mathrm{Q}_n = [0,1]\times[0,1/n]$ as $n \to +\infty$. As a corollary of this inequality he obtained the estimate

$$\mathbf{P}(G) < \frac{4}{3}\mathbf{A}(G)\boldsymbol{\rho}(G)^2, \tag{1.4}$$

The constant 4/3 is best possible and is attained as the domain degenerates.

In the second half of the 90s of the 20th century Avkhadiev defined the integral geometric functional

$$\mathbf{I}_p(G) = \int\limits_G \rho(x,G)^p \mathrm{dA}, \tag{1.5}$$

which is called the Euclidean moment of the domain with respect to the boundary of order p. As $p=2$, the functional is naturally called the Euclidean intertia moment of domain [5], while as $p=1$, the stationary Euclidean moment of domain. Avkhadiev demonstrated [5] an important role of the Euclidean moment of inertia in the torsion theory of homogeneous rod with a simply connected cross–section. Namely, Avkhadiev established that $\mathbf{P}(G)$ and $\mathbf{I}_2(G)$ are equivalent quantities in the sense of Pólya and Szegö [1].

In this paper, we introduce new easily computable geometric functionals of the domain and provide new upper bounds for the torsional rigidity of a convex domain will be given, as well as new lower bounds for $\mathbf{P}(G)$.

The main research tool is the evaluation of the functionals of the domain on the level sets of the domain function.

2. Functionals $\mathbf{K}(G)$ and $d(\boldsymbol{\rho}(G))$ and their properties

By

$$G(\mu) := \{z \in G \mid \rho(z,G) \geqslant \mu\}, \quad \mathbf{a}(\mu) := \mathbf{A}(G(\mu)) := \int\limits_{G(\mu)} \mathrm{dA}$$

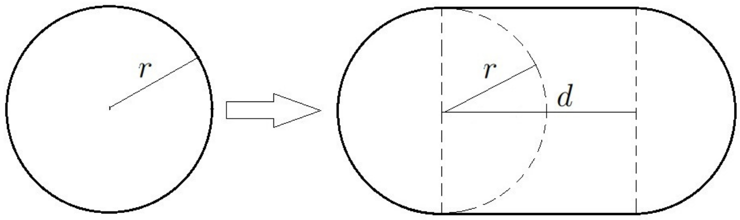

FIGURE 1. Circle and its stretching.

we denote the level set of the distance function $\rho(x, G)$ and the measure of the level set $G(\mu)$, respectively. By $\mathbf{L}(G)$ we denote the length of the boundary of domain G. Let

$$\boldsymbol{l}(\mu, G) := \mathbf{L}(G(\mu)), \qquad \boldsymbol{l}(\boldsymbol{\rho}(G)) := \lim_{\mu \to \boldsymbol{\rho}(G)} \boldsymbol{l}(\mu, G). \tag{2.1}$$

If just a single domain is considered, we briefly denote the functional $\boldsymbol{l}(\mu, G)$ by $\boldsymbol{l}(\mu)$.

A convex domain G is called a stretching of a convex domain G_0 if G_0 can be obtained from G by cutting out a rectangular fragment and connecting the remaining parts by parallel translation so that $\boldsymbol{\rho}(G_0) = \boldsymbol{\rho}(G)$. On the other hand, it is natural to call G_0 a contraction of G. Note that not all domains can be stretched. Indeed, it is easy to see that a triangle and a regular polygon with an odd number of sides are examples of non–stretchable domains. If G is non–stretchable, then we let $G_0 \equiv G$. On the other hand, if $\boldsymbol{l}(\boldsymbol{\rho}(G)) \not\equiv 0$, then the convex domain G is both stretchable and contractible [6]. For example, a stretching of a circle is a Bonnesen–type domain consisting of two semicircles of radius r and a rectangle with sides d and $2r$, see Figure 1. Such domains form a two–parameter family of convex domains depending on the parameters d and r.

As in [7], we denote by Γ the subset of convex domains containing polygons circumscribed about some circle, as well as circular polygons obtained from the circumscribed polygons by replacing some sides or parts of them with arcs of a circle inscribed in the polygon. We complete the formation of the set Γ by adding domains, which are stretchings of elements from Γ. Despite the specific construction of the domains in the class Γ, in what follows we shortly call the elements from the class Γ the polygons, although they form a subclass of convex circular polygons.

For the domains D in Γ we introduce the functional

$$\mathbf{K}(D) := \sup_{\mu} \left(-\boldsymbol{l}'(\mu)\right), \tag{2.2}$$

where $\boldsymbol{l}'(\mu)$ is the derivative of the function $\boldsymbol{l}(\mu)$. It is known (see, for instance, [8]) that for convex polygonal domains the function $\boldsymbol{l}(\mu)$ is piece–wise linear, decreasing and concave, while for the class Γ the function $\boldsymbol{l}(\mu)$ is linear and

$$\mathbf{K}(D) = -\lim_{\mu \to \boldsymbol{\rho}(D)} \boldsymbol{l}'(\mu).$$

The domain D in the set Γ is characterized by the set of parameters α_i, β_j, γ_l. At the same time $\gamma_l = \pi$ for all l and $\sum\limits_{i=1}^{n} \alpha_i - \sum\limits_{j=1}^{m} \beta_j = \pi(n - m - 2)$. On Figure 2 we demonstrate the role of the parameters α_i, β_j, γ_l for a domain in the class Γ.

It is easy to see that

$$\mathbf{L}(D)\boldsymbol{\rho}(G) = 2\boldsymbol{\rho}(D) \sum_{i=1}^{n} \cot(\alpha_i/2) + \boldsymbol{\rho}(D) \sum_{j=1}^{m} \beta_j,$$

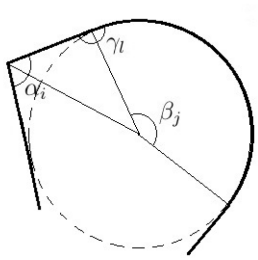

FIGURE 2. Example of domain in the class Γ.

and hence

$$\mathbf{K}(D)=2\sum_{i=1}^{n}\cot(\alpha_i/2)+\sum_{j=1}^{m}\beta_j. \tag{2.3}$$

Since $\alpha_i\in(0,\pi)$, it follows from (2.3) that the functional $\mathbf{K}(G)$ takes finite values and can grows unboundedly if at least one of its angles $\alpha_i\to 0$. For instance, this is seen at the example of the triangle. We note that the value of the functional $\mathbf{K}(D)$ is independent of the angles γ_l.

Thus, for a non–stretchable domain $D\in\Gamma$ we have

$$\mathbf{K}(D)=\frac{\mathbf{L}(D)}{\boldsymbol{\rho}(D)}.$$

If D is a non–stretchable domain in the class Γ, we then obviously obtain

$$\mathbf{K}(D)=\frac{\mathbf{L}(D)-\boldsymbol{l}(\boldsymbol{\rho}(D))}{\boldsymbol{\rho}(D)}. \tag{2.4}$$

Our next aim is to construct an analog of the functional $\mathbf{K}(D)$ for an arbitrary convex bounded domain. To this end, we give an example of a domain not belonging to the class Γ, which is, in a sense, disappointing. Let G be a semicircle of radius r (see Figure 3). For the domain G we have $\boldsymbol{\rho}(G)=r/2$ and

$$\boldsymbol{l}(\mu)=2\sqrt{r^2-2r\mu}+(r-\mu)\left(\pi-2\mathrm{arcsin}\frac{\mu}{r-\mu}\right).$$

We then obtain easily

$$\sup_{\mu}\,(-\boldsymbol{l}'(\mu))=\lim_{\mu\to r/2}\,(-l'(\mu))=+\infty.$$

This simple example shows that extending the definition of (2.2) (or (2.3)) to a wider subclass of convex domains is difficult and ineffective, since for a “good” domain, a semicircle, this functional is not finite. In fact, this example is not the only one. Despite this, below we shall provide a class of domains for which this is possible. In fact, the semicircle example is the key one, and for a correct generalization of the definition of $\mathbf{K}(D)$ to arbitrary convex domains, a criterion for the finiteness of the limit

$$\lim_{\mu\to\boldsymbol{\rho}(G)}\,(-l'(\mu)) \tag{2.5}$$

is needed.

Let G be an arbitrary convex domain. With G we associate a domain $D\in\Gamma$, which contains G, has the same radius of the maximal circle, and the smallest length of the domain boundary.

In the following statements we provide an algorithm for constructing the domain D.

Lemma 2.1. *For each domain $G\in\Gamma$ and sector* $\mathrm{Sec}(\beta)$ *of an opening β with the vertex at the center of the maximal inscribed in the domain G circle the inequalities hold*

$$\inf_{G\in\Gamma}\boldsymbol{L}(\partial G\cap\mathrm{Sec}(\beta))\geqslant\beta\boldsymbol{\rho}(G),\qquad \inf_{G\in\Gamma}\mathbf{A}(G\cap\mathrm{Sec}(\beta))\geqslant\frac{\beta\boldsymbol{\rho}(G)^2}{2},$$

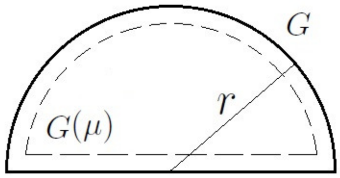

FIGURE 3. Example of domain not belonging to the class Γ.

where ∂G is the boundary of the domain G.

This statement is implied by the definition of the class Γ and it states that the smallest length and area of a domain $G \in \Gamma$ are possessed by a domain, the boundary of which contains an arc of sector of opening β.

Lemma 2.2. *Let G be a convex domain of finite area and $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$. Then there exists a domain $D \in \Gamma$ such that*

$$\boldsymbol{L}(D) := \min\{\boldsymbol{L}(Q) : Q \supset G, \boldsymbol{\rho}(Q) = \boldsymbol{\rho}(G), Q \in \Gamma\}.$$

At the same time,

1) *if a part of boundary of G coincides with an arc of the sector of maximal inscribed circle, then this circle belongs completely to D;*
2) *if the boundary of G strictly contains the arc of sector of maximal inscribed circle, then a part of the boundary D is formed by tangent lines to the domain G. Moreover, a part of the domain D contains a part of the domain G and the considered sector (see Figure 4).*

Proof. Let G be a convex domain. We consider the maximal circle in G and the set of tangent points of the inscribed circle and the boundary of G. We draw tangent lines to G at all such points, excluding tangent lines drawn at interior points of arcs of the inscribed circle that are boundary points of G if such an arc or arcs exist. Note that there are tangent lines to G at the ends of these arcs, since otherwise the arc is not an arc of the inscribed circle in G, or the convexity condition is not satisfied. This is why the drawn tangent lines and arcs of the inscribed circle form a polygon from the class Γ. We shall show that the constructed domain can be taken as the domain D.

We denote by $S(OA_iA_{i+1})$ the circular sector of opening $\beta < \pi$, while $G(OA_iA_{i+1})$ is the curvilinear sector of the opening $\beta < \pi$, where A_i, A_{i+1} are adjacent points of tangency of the inscribed circle and boundary of domain G. We first consider the case when $S(OA_iA_{i+1}) = G(OA_iA_{i+1})$. According to Lemma 2.1, the arc of the circle A_iA_{i+1} is optimal, that is, this arc is a part of the boundary of sought polygon D (see Figure 4).

Let $S(OA_jA_{j+1}) \subset G(OA_jA_{j+1})$, but these sectors not coincide. Under the introduced restrictions, it is impossible to reduce the length of the constructed polygon D: the broken line A_jAA_{j+1} (see Figure 4) cannot be replaced by an arc A_jA_{i+1} of the inscribed circle (since then $G \not\subset D$), or by an arbitrary segment connecting the sides of the broken line A_jAA_{j+1} (since then the constructed domain is a polygon from Γ).

Thus, the boundary of domain D consists of angles of type A_iAA_{i+1} and arcs of the inscribed circle, which are optimal. This completes the construction of the domain D and the proof. $\square$

In the next lemma we consider remaining variants of constructing the domain D.

Lemma 2.3. *Let G be a convex domain of finite area. Then there exists a polygon $D \in \Gamma$ such that*

$$\boldsymbol{L}(D) := \min\{\boldsymbol{L}(Q) : Q \supset G, \boldsymbol{\rho}(Q) = \boldsymbol{\rho}(G), Q \in \Gamma\}.$$

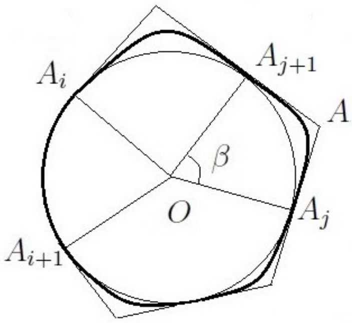

FIGURE 4. Convex domain with $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$.

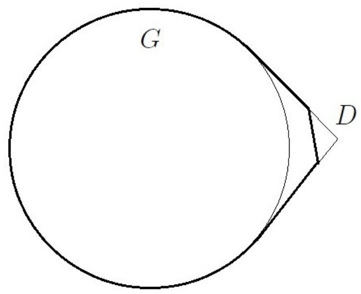

FIGURE 5. Convex domain with a single circumference

Proof. We present an algorithm for constructing a domainn D with a minimal boundary length for a convex domain G based on the location of various tangent lines drawn at the touching points of the maximal inscribed circle and the boundary of domain G.

Let G be a convex domain. In G we inscribe the maximal circle or arcs of a circle. We consider tangent lines only at the ends of arcs common to the circle and the boundary of G if such arcs are present, and also at other touching points of the inscribed circle and the boundary of G that are not connected with arcs.

The construction of the domain D splits into two cases:

1) the set of tangent lines contains no parallel ones or there exist more than one pair of parallel tangent lines and $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$;
2) among all tangent lines, there is one pair of parallel ones and $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$ or there are more than one pair of such tangent lines and $\boldsymbol{l}(\boldsymbol{\rho}(G)) \neq 0$.

In the first case it is obvious that the inscribed circle is unique (see Figures 5, 6). Then the tangent lines and arcs of a circle from G (D) form a domain from the class Γ. According to Lemmas 2.1 and 2.2, the constructed domain is the domain D.

The second case is more interesting. Let there be parallel tangent lines among the drawn ones. Then the inscribed circle may be unique (a semicircle, an ellipse), or there may be infinitely many of them (any contractible domain).

Assume that G is contractible (see Figure 7), i.e. $\boldsymbol{l}(\boldsymbol{\rho}(G)) \neq 0$. We draw tangential lines to the domain G parallel to the set $G(\boldsymbol{\rho}(G))$, which is a segment, and obtain a strip containing G. Then we construct a rectangle P containing G so that all its sides touch the boundary of domain G, or the rectangle has common points with G. Moreover, the two constructed sides of the rectangle may be either tangent lines to the domain G or not (see Figure 7). We inscribe a Bonnesen–type domain B in the resulting rectangle P (see [9]).

We draw tangent lines to G through the common points of the boundary of G and the boundary of B, and if the boundary of G contains an arc of the boundary of B, then through

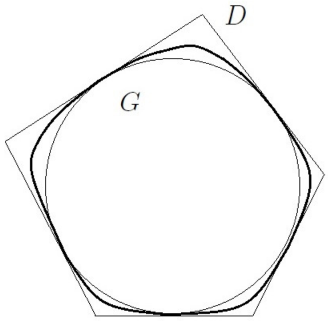

FIGURE 6. Convex domain with a single circumference

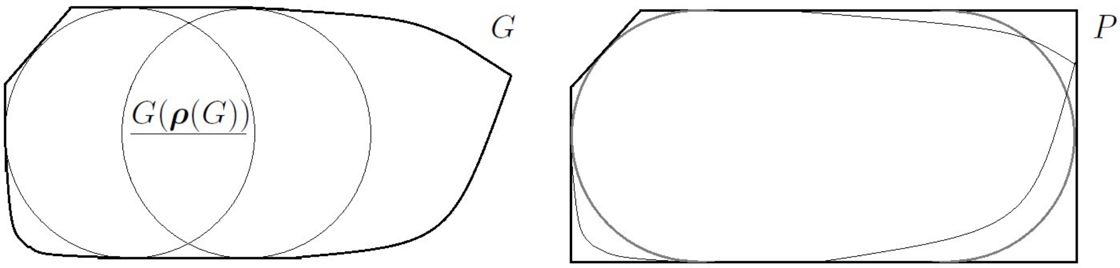

FIGURE 7. Construction of the domain D from the class Γ

the points at the ends of the arcs. The boundary of P contains extreme points of the convex domain G (see [13]), then through these points we also draw tangent lines to B (on Figure 8 this situation corresponds to the tangent passing through the points A_i, A_{i-1}). Note that this tangent line is not tangent to the boundary of G. The drawn tangent lines, the sides of P and the arcs of the boundary of B form a certain polygon $D' \in \Gamma$ containing G (see Figure 7), since B is a stretching of a circle.

Since we need to find a polygon with the smallest boundary length, D' should be optimized. First of all we observe that the rectilinear sides of B are part of D. Indeed, these sides or parts of them cannot be replaced by arcs of the inscribed circle since otherwise the convexity of the polygon is violated. Therefore we obtain a domain that is not from the class Γ (see Figure 8). Similarly, by replacing part of the sides with segments of shorter length, for example, tangent lines to G, we are also led to a domain that is not from the set Γ. Next, consider the set of common points of the boundary of B and D' and denote these points by A_i. If the boundaries of B and D' have common straight line segments or common arcs, then among these points we consider only ones coinciding with the ends of the segments or arcs. For each pair of adjacent points, we apply Lemmas 2.1 and 2.2 and replace parts of the boundaries of the polygon D' with arcs or sides of shorter length (Figure 8). This completes the construction of the domain $D \in \Gamma$ for the case of a contractible domain.

In the case when among the considered tangent lines drawn to the non–contractible domain G, there is one pair of parallel ones, the construction of the domain D is made similarly to the previous case.

This completes the construction of the domain $D \in \Gamma$ and the proof. □

Let G be an arbitrary convex domain and D be the domain corresponding to G. We define a new functional,

$$\mathbf{K}(G) := \mathbf{K}(D). \tag{2.6}$$

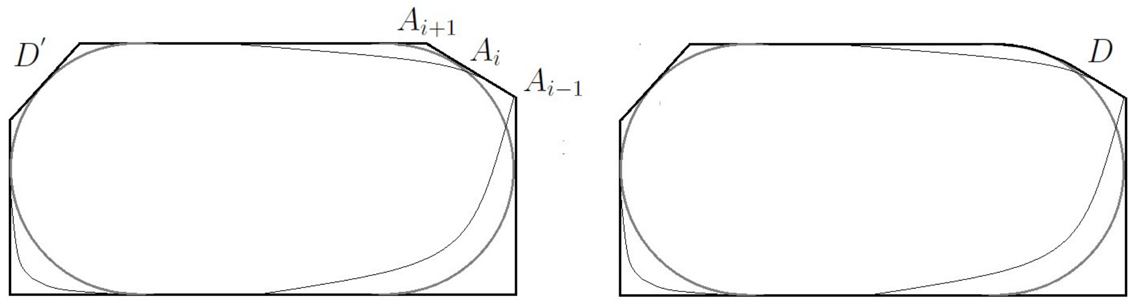

FIGURE 8. Construction of the domain D from the class Γ

As we shall see below, the functional $\mathbf{K}(G)$ is not enough for obtaining the estimates for the torsional rigidity. We define one more functional

$$\boldsymbol{d}(\boldsymbol{\rho}(G)) := \boldsymbol{l}(\boldsymbol{\rho}(D)). \tag{2.7}$$

The definition implies that

$$\boldsymbol{d}(\boldsymbol{\rho}(G)) \geqslant \boldsymbol{l}(\boldsymbol{\rho}(G)) \tag{2.8}$$

for each convex domain G. In contrast to $\boldsymbol{l}(\boldsymbol{\rho}(G))$, this functional describes more precisely how the domain G is stretched. For instance, for the ellipse with the semi–axes a and b we have $\boldsymbol{d}(\boldsymbol{\rho}(G)) = 4(a-b)$ and $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$. This corresponds to our perception of the ellipse as a stretched domain.

Let us provide one more example of applying Lemmas 2.1, 2.2, 2.3 for finding the domain D with the minimal length of the boundary for a convex domain G. We consider a semicircle G of radius r (see Figure 9). In order to construct the domain $D \in \Gamma$, we describe a rectangle around the semicircle. Using Lemma 2.3, we replace parts of the sides of the rectangle with an arc of a circle inscribed in G. As the domain D, we obtain a rectangle with two corners cut off by the arc of the maximal inscribed circle. For the semicircle, $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$, $\boldsymbol{d}(\boldsymbol{\rho}(G)) = \boldsymbol{l}(\boldsymbol{\rho}(D)) = 2r$, $\mathbf{K}(G) = \mathbf{K}(D) = 4 + \pi$.

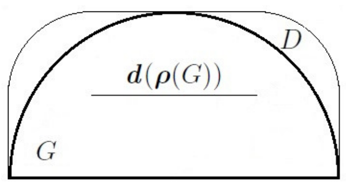

FIGURE 9. Example of the domain D.

Theorem 2.1. *Let G be a bounded convex domain. Then*

$$\lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu)) = +\infty$$

if and only if $G \notin \Gamma$ and $\boldsymbol{d}(\boldsymbol{\rho}(G)) > 0$.

Proof. Sufficiency. Let $G \notin \Gamma$ and $\boldsymbol{d}(\boldsymbol{\rho}(G)) > 0$. Then there are two parallel to the segment $D(\boldsymbol{\rho}(G))$ (the set $G(\boldsymbol{\rho}(G))$ is possibly a point) tangent lines to the domain G at the touching points of the domain G and the maximal inscribed circle. Let H be a circular lune. Due to the convexity of the domain G, there are either exactly two or infinitely many touching points of the inscribed circle and the boundary of domain G. In the first case, we can take a symmetric circular lune, while in the second case a lune bounded by a chord. Since for μ close to $\boldsymbol{\rho}(G)$ the set $G(\mu)$ contains the circular lune $H(\mu)$ (see Figure 10), we obtain

$$\boldsymbol{l}(\mu, G) \geqslant \boldsymbol{l}(\mu, H).$$

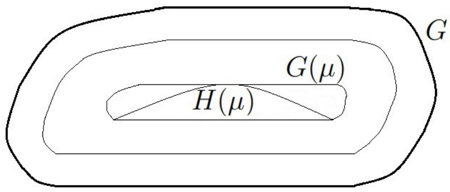

FIGURE 10. Convex domain with the circular lune

At the same, the inequality

$$\boldsymbol{l}(\mu, G) - \boldsymbol{l}(\boldsymbol{\rho}(G)) \geqslant \boldsymbol{l}(\mu, H) - \boldsymbol{l}(\boldsymbol{\rho}(G))$$

holds. We divide the latter inequality by $\mu - \boldsymbol{\rho}(G)$ for $0 \leqslant \mu \leqslant \boldsymbol{\rho}(G)$. Multiplying by (-1), we get

$$-\frac{\boldsymbol{l}(\mu, G) - \boldsymbol{l}(\boldsymbol{\rho}(G))}{\mu - \boldsymbol{\rho}(G)} \geqslant -\frac{\boldsymbol{l}(\mu, H) - \boldsymbol{l}(\boldsymbol{\rho}(G))}{\mu - \boldsymbol{\rho}(G)}.$$

Passing to the limit as $\mu \to \boldsymbol{\rho}(G)$, it is easy to see that

$$\lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu, G)) \geqslant \lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu, H)).$$

The above considered example implies

$$\lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu, H)) = +\infty.$$

This proves the sufficiency.

Necessity. Let G be a convex domain and $G \notin \Gamma$ be such that

$$\lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu)) = +\infty.$$

If $\boldsymbol{d}(\boldsymbol{\rho}(G)) = 0$, then $\boldsymbol{l}(\boldsymbol{\rho}(G)) = 0$. Moreover, by Lemma 2.3 there exists a domain $D \in \Gamma$ such that $D \supset G$, $\boldsymbol{\rho}(D) = \boldsymbol{\rho}(G)$, $\boldsymbol{l}(\mu, D) \geqslant \boldsymbol{l}(\mu, G)$. Therefore,

$$\lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu, D)) \geqslant \lim_{\mu \to \boldsymbol{\rho}(G)} (-\boldsymbol{l}'(\mu, G)).$$

This estimate yields $\mathbf{K}(D) = +\infty$, while by the definition the functional $\mathbf{K}(D)$ has finite values on each convex domain from the class Γ. This contradiction completes the proof. □

A corollary of Theorem 2.1 and Definition (2.7) is a partition of convex domains into two subclasses. The first subclass consists of domains G, for which $\boldsymbol{d}(\boldsymbol{\rho}(G)) > 0$, while for the second class we have $\boldsymbol{d}(\boldsymbol{\rho}(G)) = 0$. The second subclass contains domains close to a circle. We note that the class of domains close to a circle was selected and studied in detail in the monograph by Pólya and Szegö [1, Ch. 6]. Thus, on the class of domains with $\boldsymbol{d}(\boldsymbol{\rho}(G)) = 0$ we can define the functional $\mathbf{K}(G)$ by the formula (2.2), but the study of this case is beyond the scope of this paper. We also note that Lemma 2.2 concerns the class of convex domains with $\boldsymbol{d}(\boldsymbol{\rho}(G)) = 0$, while Lemma 2.3 does the case $\boldsymbol{d}(\boldsymbol{\rho}(G)) > 0$.

Let us mention some main properties of the introduced functional $\mathbf{K}(G)$.

1. Let G_1 and G_2 be similar convex domains, then $\mathbf{K}(G_1) = \mathbf{K}(G_2)$.

For domains from the class Γ, the area and the length of the boundary depend only on the angles, and the similarity transformation preserves the angles between the curves. This immediately implies the stated property. In particular, it implies that the functional $\mathbf{K}(G)$ is not monotone as a function of the domain. For example, for the domains shown on Figure 11 we have $D_1 \subset D_2 \subset D_3$, but

$$\mathbf{K}(D_1) = \mathbf{K}(D_3) = 2\pi \leqslant \mathbf{K}(D_2) = 8.$$

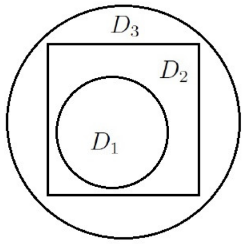

Figure 11. Examples of domains from the class Γ

2. Let G be a contractible convex domain and G_0 be the contraction of G, then $\mathbf{K}(G) = \mathbf{K}(G_0)$.

This statement follows from the definition of $\mathbf{K}(G)$.

3. For each domain $D \in \Gamma$ the identity

$$\boldsymbol{l}(\mu) = \mathbf{K}(D)(\boldsymbol{\rho}(D) - \mu) + \boldsymbol{l}(\boldsymbol{\rho}(D)), \qquad 0 \leqslant \mu \leqslant \boldsymbol{\rho}(D)$$

holds.

We consider the identity

$$\boldsymbol{l}(\mu) = -\int\limits_{\mu}^{\boldsymbol{\rho}(D)} \boldsymbol{l}'(t)\mathrm{d}t + \boldsymbol{l}(\boldsymbol{\rho}(D)). \tag{2.9}$$

The needed property is obtained by applying the identity

$$-\boldsymbol{l}'(t) = 2\sum_{i=1}^{n} \cot(\alpha_i/2) + \sum_{j=1}^{m} \beta_j,$$

which is valid for $D \in \Gamma$, and by integrating (2.9).

4. If G is a convex domain and D is the minimal domain in Γ, $\boldsymbol{\rho}(D) = \boldsymbol{\rho}(G)$ and $G \subset D$, then

$$\boldsymbol{l}(\mu, G) \leqslant \mathbf{K}(G)(\boldsymbol{\rho}(G) - \mu) + \boldsymbol{d}(\boldsymbol{\rho}(G)). \tag{2.10}$$

The identity is attained for domains in the class Γ for each $\mu \in [0, \boldsymbol{\rho}(G)]$.

Since $G(\mu) \subset D(\mu)$, $(0 \leqslant \mu \leqslant \boldsymbol{\rho}(G))$, we have $\boldsymbol{l}(\mu, G) \leqslant \boldsymbol{l}(\mu, D)$ and in view of Property 3 we get

$$\boldsymbol{l}(\mu, G) \leqslant \mathbf{K}(D)(\boldsymbol{\rho}(D) - \mu) + \boldsymbol{l}(\boldsymbol{\rho}(D)) = \mathbf{K}(G)(\boldsymbol{\rho}(G) - \mu) + \boldsymbol{d}(\boldsymbol{\rho}(G)).$$

In particular, the inequality

$$\mathbf{L}(G) \leqslant \mathbf{K}(G)\boldsymbol{\rho}(G) + \boldsymbol{d}(\boldsymbol{\rho}(G)) \tag{2.11}$$

holds.

In Table 1 we provide approximate values of quotient of the left and right hand sides in (2.11) for the ellipse. We see that these values are between numbers, which are close to 1. A corollary of this example is the impossibility of replacing $\boldsymbol{d}(\boldsymbol{\rho}(G))$ by $\boldsymbol{l}(\boldsymbol{\rho}(G))$ in (2.10).

Ellipse with semi–axes a, b	$\dfrac{\mathbf{L}(G)}{\mathbf{K}(G)\boldsymbol{\rho}(G)+\boldsymbol{d}(\boldsymbol{\rho}(G))}$
$a/b=1$	1
$a/b=6/5$	0.977779
$a/b=4/3$	0.967349
$a/b=3/2$	0.95769
$a/b=7/4$	0.948036
$a/b=2$	0.942164
$a/b=3$	0.935708
$a/b=4$	0.938395
$a/b=7$	0.951491
$a/b=12$	0.965792
$a/b=100$	0.994597
$a/b\to\infty$	1

Table 1. Demonstration of the inequality (2.11) for the ellipse.

As a remark to this property, we note that there exist convex domains whose level sets, starting from some μ, are domains from the class Γ. It is easy to provide an example of such domains. We consider a trapezoid, one side of which does not touch the inscribed circle. Let b be the intersection point of the bisectors closest to the boundary of the domain. Then for each $\mu\in[0,\rho(b,G)]$ the level set is a trapezoid $G(\mu)$ similar to G, for $\mu\in[\rho(b,G),\boldsymbol{\rho}(G)]$ the level set of the trapezoid is a triangle, i.e. a domain from the class Γ (see Figure 12).

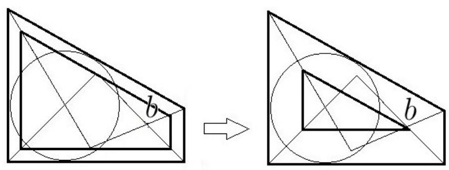

FIGURE 12. Construction of the domain $\Gamma(\mu)$

5. Among all n–polygons D_n circumscribed about a given circle, the smallest value $\mathbf{K}(D_n)$ is attained at the regular n–gon D'_n,

$$\mathbf{K}(D_n)\geqslant\mathbf{K}(D'_n).$$

The statement follows from property 1 and extremal properties of regular polygons [13]

$$\mathbf{K}(D_n)=\frac{\mathbf{L}(D_n)}{\boldsymbol{\rho}(D_n)}\geqslant\frac{\mathbf{L}(D'_n)}{\boldsymbol{\rho}(D'_n)}=\mathbf{K}(D'_n).$$

6. If D_1 and D_2 are non–contractible domains in the class Γ and $\boldsymbol{\rho}(D_1)=\boldsymbol{\rho}(D_2)$ and $D_1\subset D_2$, then $\mathbf{K}(D_1)<\mathbf{K}(D_2)$.

Indeed, since $D_1\in\Gamma$ and $D_2\in\Gamma$, we have

$$\mathbf{K}(D_1)=\frac{\mathbf{L}(D_1)}{\boldsymbol{\rho}(D_1)}=\frac{\mathbf{L}(D_1)}{\boldsymbol{\rho}(D_2)}<\frac{\mathbf{L}(D_2)}{\boldsymbol{\rho}(D_2)}=\mathbf{K}(D_2).$$

Property 6 implies that if D_n and D_{n+1} are circumscribed about a given circle n– and $(n+1)$–gons such that $D_{n+1}\subset D_n$, then the smallest value $\mathbf{K}(D_n)$ is attained at the polygon with the maximal number of sides n:

$$\mathbf{K}(D_{n+1})\leqslant\mathbf{K}(D_n).$$

Since the values of the functional $\mathbf{K}(D)$ for the stretching of a domain from the class Γ and its contraction are equal, it follows that Property 6 is not satisfied for domains, which are stretching the elements from the class Γ. For example, in Figure 13 we demonstrate domains such that $D_1 \subset D_2 \subset D_3$, but $\mathbf{K}(D_1) = \mathbf{K}(D_3) \leqslant \mathbf{K}(D_2)$.

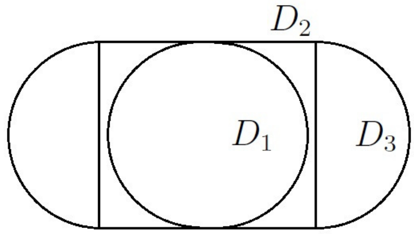

FIGURE 13. Examples of domains from the class Γ

Lemma 2.4. *Let G_1 and G_2 be convex domains such that $G_1 \subset G_2$, $\boldsymbol{\rho}(G_1) = \boldsymbol{\rho}(G_2)$ and $\boldsymbol{l}(\boldsymbol{\rho}(G_1)) = \boldsymbol{l}(\boldsymbol{\rho}(G_2)) = 0$. Let n_1, n_2 be the number of touching points of the inscribed circumference and the boundary of the domains G_1 and G_2, respectively. If $n_1 \geqslant n_2 > 2$, then*

$$\mathbf{K}(G_1) \leqslant \mathbf{K}(G_2).$$

Proof. Let $G_1 \subset G_2$, $\boldsymbol{\rho}(G_1) = \boldsymbol{\rho}(G_2)$ and $n_1 = n_2 > 2$, then the maximal inscribed circumferences in G_1 and G_2 are unique and coincide. The touching points N_1, N_2, N_3 of the maximal inscribed circumference and the boundary of the domains G_1 and G_2 coincide as well. Therefore, while associating the domains from the class Γ with the minimal length of the boundary with the domains G_1 and G_2, we get the same polygon, see Figure 14. This is why $\mathbf{K}(G_1) = \mathbf{K}(G_2)$.

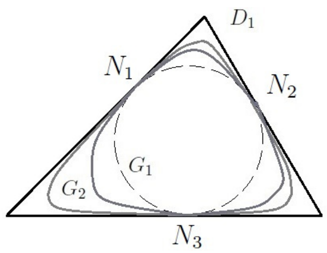

FIGURE 14. Domain from the class Γ

Let $G_1 \subset G_2$, $\boldsymbol{\rho}(G_1) = \boldsymbol{\rho}(G_2)$ and $n_1 > n_2 > 2$, then the maximal inscribed circles in G_1 and G_2 are unique and coincide. To illustrate this case, we present separate figure of the domains for G_1 and G_2, in fact, the general case is obtained by overlaying the figures so that all touching points of the maximal inscribed circle and the boundary of G_2 coincide with the touching points of the maximal inscribed circle and the boundary of G_1, but G_1 has other touching points as well (see Figure 15).

While associating the domains in the class Γ with the minimal length of boundary with the domains G_1 and G_2 we obtain the polygon D_1 associated with G_1 and the polygon D_2 associated with G_2 are circumscribed about the same circle. Since D_1 has more sides than D_2, by Property 6 we obtain that $\mathbf{K}(G_1) < \mathbf{K}(G_2)$. This completes the proof. □

Theorem 2.2. *Let G be a convex domain, then the functional*

$$s_{\boldsymbol{l}}(\mu) = \frac{\boldsymbol{l}(\mu) - \boldsymbol{l}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G) - \mu}$$

increases on the segment $[0; \boldsymbol{\rho}(G)]$.

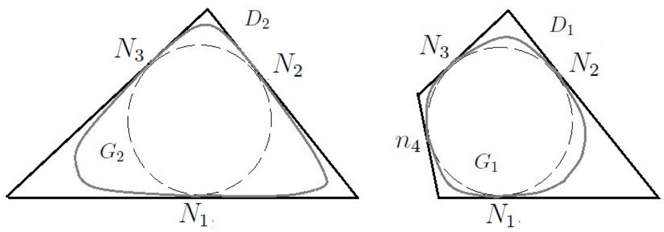

FIGURE 15. Domain from the class Γ

We observe that the function $s_{\boldsymbol{l}}(\mu)$ is almost everywhere differentiable and

$$s'_{\boldsymbol{l}}(\mu)=\frac{\boldsymbol{l}'(\mu)(\boldsymbol{\rho}(G)-\mu)+\boldsymbol{l}(\mu)-\boldsymbol{l}(\boldsymbol{\rho}(G))}{(\boldsymbol{\rho}(G)-\mu)^2}.$$

On the other hand, $\boldsymbol{l}(\mu)$ is a convex upwards function. Therefore, $l'(\mu)$ decreases monotonically almost everywhere. Then, using identity (2.9) valid for all convex domains, we arrive at the statement of the lemma.

Theorem 2.2 implies the inequality

$$s_{\boldsymbol{l}}(\mu)\geqslant 0,$$

which is equivalent to the inequality

$$\boldsymbol{l}(\mu)\geqslant\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)}(\boldsymbol{\rho}(G)-\mu)+\mu\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)},\qquad 0\leqslant\mu\leqslant\boldsymbol{\rho}(G). \tag{2.12}$$

We note that inequality (2.12) can be easily proved without applying Theorem 2.2. As we have mentioned above, $\boldsymbol{l}(\mu)$ is convex upwards, the functional in the right hand side (2.12) is linear and coincides with $\boldsymbol{l}(\mu)$ at the ends of the segment $[0,\boldsymbol{\rho}(G)]$.

In Table 2 we provide examples of calculating the values of the functionals $\mathbf{K}(G)$ and $\boldsymbol{d}(\boldsymbol{\rho}(G))$ for domains considered in the classical monograph by Pólya and Szegö.

Domain	$\mathbf{K}(G)$	$\boldsymbol{d}(\boldsymbol{\rho}(G))$
Circle of radius a	2π	0
Ellipse with semi–axes a and b	2π	$4(a-b)$
Square with side a	8	0
Rectangle with sides a,b, $a\geqslant b$	8	$2(a-b)$
Semicircle of radius a	$4+\pi$	2a
Sector of radius a and opening $\gamma=2\pi\lambda$, $0\leqslant\lambda\leqslant\frac{1}{2}$	$2\left(\cot\frac{\gamma}{2}+2\cot\frac{\pi-\gamma}{2}\right)$	0
Equilateral triangle with side a	$6\sqrt{3}$	0
Triangle with angles 45°, 45°, 90°	11.6569	0
Triangle with angles 30°, 60°, 90°	12.9282	0

Table 2. Values of the functionals $\mathbf{K}(G)$ and $\boldsymbol{d}(\boldsymbol{\rho}(G))$.

3. Formulations of main results

Further estimates for the torsional rigidity are based on the following theorem.

Theorem 3.1. *Let G be a convex domain of a finite area in the plane. Then for $p>1$ the inequality holds*

$$\mathbf{L}(G)(p+2)+\boldsymbol{l}(\boldsymbol{\rho}(G))(p+1)\leqslant\frac{(p+1)(p+2)\mathbf{I}_p(G)}{\boldsymbol{\rho}(G)^{p+1}}\leqslant\mathbf{K}(G)\boldsymbol{\rho}(G)+(p+2)\boldsymbol{d}(\boldsymbol{\rho}(G)).$$

The identity in these inequalities is attained if and only if $G\in\Gamma$.

In construction of new estimates for the torsional rigidity an important role is played by the functions [11]

$$\mathbf{f}_p(\mu) := \mathbf{I}_p(G(\mu)), \tag{3.1}$$

where $0 \leqslant \mu \leqslant \boldsymbol{\rho}(G)$, p is a real parameter and the representation

$$\mathbf{f}_p(\mu) = \int\limits_{\mu}^{\boldsymbol{\rho}(G)} (s-\mu)^p l(s)\mathrm{d}s \tag{3.2}$$

holds for $p > -1$.

Theorem 3.1 is a corollary of the theorem on two–sided estimates for $\mathbf{f}_p(\mu)$.

Theorem 3.2. *Let G be a bounded convex domain. Then for $p \geqslant 0$ the inequalities*

$$\mathbf{f}_p(\mu) \geqslant \frac{\boldsymbol{\rho}(G)^{p+1}}{(p+1)(p+2)} \left(\mathbf{L}(G) + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))((p+1)\boldsymbol{\rho}(G)+\mu)}{(\boldsymbol{\rho}(G)-\mu)} \right) \left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right)^{p+2}, \tag{3.3}$$

$$\mathbf{f}_p(\mu) \leqslant \frac{\boldsymbol{\rho}(G)^{p+2}}{p+1} \left(\frac{\mathbf{K}(G)}{p+2} + \frac{\boldsymbol{d}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)-\mu} \right) \left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right)^{p+2} \tag{3.4}$$

hold. For each $\mu \in [0, \boldsymbol{\rho}(G)]$ the identities in these inequalities is attained at the domains in the class Γ.

We mention two corollaries from Theorem 3.2

Corollary 3.1. *Let G be a convex domain with a finite Euclidean moment of order $p \geqslant 1$. Then*

$$\mathbf{f}'_p(\mu) \leqslant -\frac{1}{p+1} \left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(p\boldsymbol{\rho}(G)+\mu)}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)} \right) (\boldsymbol{\rho}(G)-\mu)^{p+1}, \tag{3.5}$$

$$\mathbf{f}'_p(\mu) \geqslant -\frac{1}{p+1} \left(\mathbf{K}(G) + \frac{(p+1)\boldsymbol{d}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)-\mu} \right) (\boldsymbol{\rho}(G)-\mu)^{p+1}. \tag{3.6}$$

Corollary 3.2. *Let G be a convex domain with a finite Euclidean moment of order $p \geqslant 1$. Then*

$$\mathbf{f}''_p(\mu) \geqslant \left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))((p-1)\boldsymbol{\rho}(G)+\mu)}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)} \right) (\boldsymbol{\rho}(G)-\mu)^{p}, \tag{3.7}$$

$$\mathbf{f}''_p(\mu) \leqslant \left(\mathbf{K}(G) + \frac{p\boldsymbol{d}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)-\mu} \right) (\boldsymbol{\rho}(G)-\mu)^{p}. \tag{3.8}$$

By inequality (3.5) we conclude that the functional $\mathbf{f}_p(\mu)$ decreases monotonically on $[0, \boldsymbol{\rho}(G)]$, while (3.8) implies that $\mathbf{f}_p(\mu)$ is convex downwards on $[0, \boldsymbol{\rho}(G)]$.

On the base of the functional

$$\mathbf{H}(G;p) := \frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p+1}} \left(\mathbf{I}_p(G) - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{p+1}}{p+1} \right), \qquad p > -1,$$

introduced in [7] we consider the functional

$$\mathbf{H}_p(\mu) := \frac{1}{\boldsymbol{\rho}(G(\mu))^{p+2}} \left(\mathbf{f}_p(\mu) - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{p+1} \right), \tag{3.9}$$

where $0 \leqslant \mu \leqslant \boldsymbol{\rho}(G)$.

We are in position to formulate theorems providing lower and upper bounds for the torsional rigidity of a convex domain.

Theorem 3.3. *Let G be a convex domain of finite area in the plane. Then*

$$\mathbf{P}(G) \leqslant \frac{2\boldsymbol{\rho}(G)^3}{3}\left(\mathbf{K}(G)\boldsymbol{\rho}(G) + 2\boldsymbol{d}(\boldsymbol{\rho}(G)) - \pi\boldsymbol{\rho}(G)\right). \tag{3.10}$$

The identity is attained at the limit, for instance, on the sequence of rectangles $\mathrm{Q}_n = [0,1] \times [0,1/n]$ as $n \to +\infty$.

Table 3 demonstrates that the Makai inequality (1.3) provides a sharper estimate for the torsional rigidity in comparison with (3.10). The advantage of the obtained inequality (3.10) for the torsional rigidity is that $\mathbf{P}(G)$ is estimated via easily calculated characteristics of the domain G.

By Pólya — Szegö inequality (1.2) we easily estimate the torsional rigidity via the length of the boundary of domain

$$\mathbf{P}(G) \geqslant \frac{\mathbf{A}(G)\boldsymbol{\rho}(G)^2}{2} \geqslant \frac{\boldsymbol{\rho}(G)^3}{4}\left(\mathbf{L}(G) + \boldsymbol{l}(\boldsymbol{\rho}(G))\right),$$

and the identities are attained, for instance, for the circle.

Theorem 3.4. *Let G be a convex domain of finite area in the plane. Then for $q > 0$ the inequality holds*

$$\mathbf{P}(G) \geqslant \frac{\boldsymbol{\rho}(G)^3}{2(2+q)}\left(\mathbf{L}(G) + \boldsymbol{l}(\boldsymbol{\rho}(G))(q+1) + \pi q\boldsymbol{\rho}(G)\right). \tag{3.11}$$

For $q = 0$, the identity in (3.11) is attained for the circle.

The results obtained in Theorems 3.3 and 3.4 are interesting since the torsional rigidity of a convex domain G is estimated via easier calculable geometric characteristics of the domain G.

4. Proof of main results

Proof of Theorem 3.2. Let us obtain the lower bound for $\mathbf{f}_p(\mu)$. We apply the identity (3.2) and the inequality (2.12) and integrate it. Then we have

$$\begin{aligned}\mathbf{f}_p(\mu) &= \int\limits_{\mu}^{\boldsymbol{\rho}(G)} (s-\mu)^p l(s)\mathrm{d}s \geqslant \int\limits_{\mu}^{\boldsymbol{\rho}(G)} (s-\mu)^p \left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)}(\boldsymbol{\rho}(G)-s) + s\frac{\boldsymbol{l}(\boldsymbol{\rho}(G))}{\boldsymbol{\rho}(G)}\right)\mathrm{d}s \\ &= \frac{\boldsymbol{\rho}(G)^{p+2}}{(p+1)(p+2)}\left(1-\frac{\mu}{\boldsymbol{\rho}(G)}\right)^{p+2}\left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))((p+1)\boldsymbol{\rho}(G)+\mu)}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)}\right).\end{aligned}$$

We proceed to the upper bound. In order to do this, we are going to establish that the functional $\mathbf{H}_p(\mu)$ is monotonically increasing in $p \geqslant 0$ for an arbitrary convex domain with a bounded Euclidean moment of order p.

As it is known [7], the following inequality holds

$$(p+1)(p+2)\mathbf{H}_p(\mu) \geqslant p(p+1)\mathbf{H}_{p-1}(\mu).$$

Applying this inequality on the level sets $G(\mu)$ and taking into consideration the properties of the function $\mathbf{f}_p(\mu)$ (3.1), we obtain the inequality

$$\frac{p+2}{(\boldsymbol{\rho}(G)-\mu)}\left(\mathbf{f}_p(\mu) - \frac{\boldsymbol{l}(\boldsymbol{\rho}(G(\mu)))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)}\right) \geqslant -\mathbf{f}_p'(\mu) - \boldsymbol{l}(\boldsymbol{\rho}(G(\mu)))(\boldsymbol{\rho}(G)-\mu)^p.$$

We multiply this inequality by a positive on the segment $[0,\boldsymbol{\rho}(G)]$ function $1/(\boldsymbol{\rho}(G)-\mu)^{p+2}$:

$$\frac{(p+2)\mathbf{f}_p(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+3}} + \frac{\mathbf{f}_p'(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+2}} - \frac{(p+2)l(\boldsymbol{\rho}(G(\mu)))}{(p+1)(\boldsymbol{\rho}(G)-\mu)^2} + \frac{l(\boldsymbol{\rho}(G(\mu)))}{(\boldsymbol{\rho}(G)-\mu)^2} \geqslant 0.$$

Domain	$\dfrac{3\mathbf{P}(G)}{2\boldsymbol{\rho}(G)^3\left(2\boldsymbol{d}(\boldsymbol{\rho}(G))+\mathbf{K}(G)\boldsymbol{\rho}(G)-\pi\boldsymbol{\rho}(G)\right)}$	$\dfrac{\mathbf{P}(G)}{4\mathrm{I}_2(G)}$
Circle of radius r	$0,75$	$0,75$
Ellipse, $a/b=6/5$	0.703834	0.767925
Ellipse, $a/b=4/3$	0.692329	0.787693
Ellipse, $a/b=3/2$	0.685232	0.812711
Ellipse, $a/b=7/4$	0.68005	0.845916
Ellipse, $a/b=2$	0.676727	0.87273
Ellipse, $a/b=3$	0.664704	0.934614
Ellipse, $a/b=7$	0.6447	0.974535
Ellipse, $a/b=12$	0.61617	0.995411
Ellipse, $a/b=100$	0.592589	0.99993
Ellipse, $a/b\to\infty$	0.58905	1
Square with side a	0.694435	0.843462
Rectangle, $a/b=2$	0.853665	0.914729
Rectangle, $a/b=3$	0.908926	0.947939
Rectangle, $a/b=4$	0.934152	0.962788
Rectangle, $a/b=5$	0.948443	0.971053
Rectangle, $a/b=6$	0.957634	0.976324
Rectangle, $a/b=7$	0.964041	0.97996
Rectangle, $a/b=8$	0.968776	0.982637
Rectangle, $a/b=10$	0.975277	0.986291
Rectangle, $a/b=12$	0.979537	0.98867
Rectangle, $a/b=100$	0.997616	0.998692
Rectangle, $a/b=\infty$	1	1
Semicircle of radius a	0.595121	0.885363
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/12$	0.596293	0.91068
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/10$	0.602724	0.900422
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/8$	0.603784	0.888036
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/6$	0.584973	0.873561
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/4$	0.492653	0.860148
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=1/3$	0.360299	0.859949
Sector of radius r and opening $\gamma=2\pi\lambda$, $\lambda=5/12$	0.200915	0.868803
Narrow sector $r=1$, $\gamma=2\pi\lambda\to 0$	0.5	1
Equilateral triangle, side a	0.644978	0.900001
Triangle with angles 45°, 45°, 90°	0.624461	0.912417
Triangle with angles 30°, 60°, 90°	0.608295	0.920522
Regular hexagon	0.729602	0.797505

Table. Illustration of Theorem 3.3 in comparison with Makai inequality (1.3).

Then

$$\left(\frac{\mathbf{f}_p(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+2}}\right)' \geqslant \frac{l(\boldsymbol{\rho}(G(\mu)))}{(p+1)(\boldsymbol{\rho}(G)-\mu)^2}.$$

This inequality is equivalent to the estimate

$$\frac{d}{d\mu}\left(\frac{(p+1)\mathbf{f}_p(\mu)-\boldsymbol{l}(\boldsymbol{\rho}(G(\mu)))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\right) \geqslant 0.$$

Applying the definition of the functional $\mathbf{H}_p(\mu)$ and the inequality $\mathbf{H}_p(\boldsymbol{\rho}(G)) \geqslant \mathbf{H}_p(\mu)$, $\mu \in [0, \boldsymbol{\rho}(G)]$, we find

$$\lim_{\mu\to\boldsymbol{\rho}(G)}\left(\frac{(p+1)\mathbf{f}_p(\mu)-\boldsymbol{l}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\right) \geqslant \frac{\mathbf{f}_p(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+2}} - \frac{l(\boldsymbol{\rho}(G))}{(p+1)(\boldsymbol{\rho}(G)-\mu)}. \tag{4.1}$$

Since $\mathbf{H}_p(\mu)$ is an increasing function as $p \geqslant 0$, the functional in the left hand side in (4.1) increases monotonically. Then inequality (4.1) holds also for $\boldsymbol{d}(\boldsymbol{\rho}(G))$:

$$\lim_{\mu\to\boldsymbol{\rho}(G)}\left(\frac{(p+1)\mathbf{f}_p(\mu)-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\right) \geqslant \frac{\mathbf{f}_p(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+2}} - \frac{\boldsymbol{d}(\boldsymbol{\rho}(G))}{(p+1)(\boldsymbol{\rho}(G)-\mu)}. \tag{4.2}$$

We consider the quotient in the left hand side of inequality (4.1) and transform it by applying identities (3.2) and Property 4 of the functional $\mathbf{K}(G)$

$$\begin{aligned}
&\frac{(p+1)\mathbf{f}_p(\mu)-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\\
&\quad=\frac{(p+1)\displaystyle\int_{\mu}^{\boldsymbol{\rho}(G)}(s-\mu)^p l(s,G)\mathrm{d}s-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\\
&\quad\leqslant\frac{(p+1)\displaystyle\int_{\mu}^{\boldsymbol{\rho}(G)}(s-\mu)^p l(s,D)\mathrm{d}s-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\\
&\quad\leqslant\frac{(p+1)\left(\displaystyle\int_{\mu}^{\boldsymbol{\rho}(G)}(s-\mu)^p\left(\mathbf{K}(G)(\boldsymbol{\rho}(G)-s)+\boldsymbol{d}(\boldsymbol{\rho}(G))\right)\mathrm{d}s\right)-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\\
&\quad\leqslant\frac{\mathbf{K}(G)}{(p+1)(p+2)}.
\end{aligned}$$

Thus, the left hand side in the inequality (4.1) is bounded from above. Therefore,

$$\lim_{\mu\to\boldsymbol{\rho}(G)}\left(\frac{(p+1)\mathbf{f}_p(\mu)-\boldsymbol{d}(\boldsymbol{\rho}(G))(\boldsymbol{\rho}(G)-\mu)^{p+1}}{(p+1)(\boldsymbol{\rho}(G)-\mu)^{p+2}}\right) \leqslant \frac{\mathbf{K}(G)}{(p+1)(p+2)}.$$

This implies

$$\frac{\mathbf{K}(G)}{(p+1)(p+2)} \geqslant \frac{\mathbf{f}_p(\mu)}{(\boldsymbol{\rho}(G)-\mu)^{p+2}} - \frac{\boldsymbol{d}(\boldsymbol{\rho}(G))}{(p+1)(\boldsymbol{\rho}(G)-\mu)}.$$

This inequality is equivalent to the inequality (3.4). All inequalities in Theorem 3.2 become identities for the domain in the class Γ. This completes the proof. □

Proof of Theorem 3.1. We define the functional considered in [10]; for each simply connected domain G for $p \geqslant p_0 > 0$ we let

$$\mathbf{i}_p(\mu) := p \int_{\mu}^{\boldsymbol{\rho}(G)} t^{p-1}\mathbf{a}(t)\mathrm{d}t. \tag{4.3}$$

For $\mu = 0$ this is the Euclidean moment of the domain G with respect to the boundary, that is, $\mathbf{i}_p(0) = \mathbf{I}_p(G)$. The following identities are known for the derivatives [11]:

$$\mathbf{f}_2'(\mu) = -2\mathbf{i}_1(\mu), \qquad \mathbf{f}_2''(\mu) = 2\mathbf{a}(\mu). \tag{4.4}$$

It is known [12] that

$$\mathbf{I}_p(G) = p(p-1) \int_0^{\boldsymbol{\rho}(G)} \mu^{p-2}\mathbf{i}_1(\mu)\mathrm{d}\mu.$$

We take into consideration that $\mathbf{i}_1(\mu) = \mathbf{f}_1(\mu)$ [12] and applying Theorem 3.2, we obtain

$$\begin{aligned}\mathbf{I}_p(G) &\geqslant p(p-1) \int_0^{\boldsymbol{\rho}(G)} \mu^{p-2}\frac{(\boldsymbol{\rho}(G)-\mu)^3}{6}\left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(2\boldsymbol{\rho}(G)+\mu)}{\boldsymbol{\rho}(G)(\boldsymbol{\rho}(G)-\mu)}\right)\mathrm{d}\mu \\ &= \frac{\boldsymbol{\rho}(G)^{p+2}}{(p+1)(p+2)}\left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(p+1)}{\boldsymbol{\rho}(G)}\right), \quad p > 1.\end{aligned}$$

Applying the inequality (3.4) in Theorem 3.2, we get the upper bound for $\mathbf{I}_p(G)$

$$\begin{aligned}\mathbf{I}_p(G) &= p(p-1) \int_0^{\boldsymbol{\rho}(G)} \mu^{p-2}\mathbf{f}_1(\mu)\mathrm{d}\mu \\ &\leqslant p(p-1) \int_0^{\boldsymbol{\rho}(G)} \frac{\mu^{p-2}}{2}\left(\frac{\boldsymbol{d}(\boldsymbol{\rho}(G))}{(\boldsymbol{\rho}(G)-\mu)} + \frac{\mathbf{K}(G)}{3}\right)(\boldsymbol{\rho}(G)-\mu)^3\mathrm{d}\mu \\ &= \frac{\boldsymbol{\rho}(G)^{p+1}}{(p+1)(p+2)}\left(\mathbf{K}(G)\boldsymbol{\rho}(G) + \boldsymbol{d}(\boldsymbol{\rho}(G))(p+2)\right).\end{aligned}$$

□

Proof of Theorem 3.3. Let $p \geqslant q$ and $0 \leqslant q \leqslant 2$. Then the torsional rigidity satisfies the following inequality [6]:

$$\mathbf{P}(G) \leqslant \frac{4}{3(q+2)}\left(\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\mathbf{I}_p(G) - (p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right) - \frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+2)}.$$

Applying Theorem 3.1 valid for $p > 1$ to the functional $\mathbf{I}_p(G)$, we get

$$\mathbf{P}(G) \leqslant \frac{4\boldsymbol{\rho}(G)^3}{3(q+2)}\left(\boldsymbol{d}(\boldsymbol{\rho}(G)))(q+2) + \mathbf{K}(G)\boldsymbol{\rho}(G) - \frac{\pi(2-q)\boldsymbol{\rho}(G)}{2(q+1)}\right).$$

Letting $q = 0$, we arrive at the statement of the theorem. □

Proof of Theorem 3.4. Let $q > 0$ and $0 \leqslant p \leqslant q$, then the inequality holds [6]:

$$\mathbf{P}(G) \geqslant \frac{1}{2(q+2)}\left[\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\mathbf{I}_p(G) + (q-p)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right] + \frac{\pi q\boldsymbol{\rho}(G)^4}{2(q+2)}.$$

This inequality and Theorem 3.1 with $p > 1$ give

$$\mathbf{P}(G) \geqslant \frac{\boldsymbol{\rho}(G)^4}{2(2+q)}\left(\frac{\mathbf{L}(G)}{\boldsymbol{\rho}(G)} + \frac{\boldsymbol{l}(\boldsymbol{\rho}(G))(q+1)}{\boldsymbol{\rho}(G)} + \pi q\right).$$

The proof is complete. □

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Liliya Ilgizyarovna Gafiyatullina,
Lobachevskii Institute of Mathematics and Mechanics,
Kazan (Volga Region) Federal University,
Kremlevskaya str. 35,
420008, Kazan, Russia
E-mail: gafiyat@gmail.ru

Rustem Gumerovich Salakhudinov,
Lobachevskii Institute of Mathematics and Mechanics,
Kazan (Volga Region) Federal University,
Kremlevskaya str. 35,
420008, Kazan, Russia
E-mail: rsalakhud@gmail.com