WELL-POSEDNESS AND STABILITY RESULT FOR TIMOSHENKO SYSTEM WITH THERMODIFFUSION EFFECTS AND TIME-VARYING DELAY TERM

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Abstract. The main aim of the present paper is to investigate a new Timoshenko beam model with thermal and mass diffusion effects combined with a time-varying delay. Heat and mass exchange with the environment during a thermodiffusion in the Timoshenko beam, where the heat conduction is given by the classical Fourier law and acts on both the rotation angle and the transverse displacements. The heat conduction is given by the Cattaneo law. Under an appropriate assumption on the weights of the delay and the damping, we prove a well-posedness result, more precisely, we prove the existence of the weak solution. Then we proceed to the strong solution using the classical elliptic regularity and we get the result by applying the Lax-Milgram theorem, the Lumer-Phillips corollary and the Hille-Yosida theorem. We show the exponential stability result of the system in the case of nonequal speeds of wave propagation by using a multiplier technique combined with an appropriate Lyapunov functions. Our result is optimal in the sense that the assumptions on the deterministic part of the equation as well as the initial condition are the same as in the classical PDEs theory. To achieve our goals, we employ of the semigroup method and the energy method.

Keywords: Timoshenko beam, diffusion, time varying delay, existence and uniqueness, exponential stability.

Mathematics Subject Classification: 74K10, 37N15, 74F05, 65M60.

1. Introduction

The shear deformation and rotational bending effects on the beam are described mathematically by the Timoshenko system. S. P. Timoshenko first introduced the following system [10]:

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi)_x & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi) & \text{in } (0, L) \times \mathbb{R}_+, \end{cases}$$

where ψ is the rotation angle of the beam's filament and φ is the transverse displacement of the beam. The coefficients ρ, I_ρ, E, I and K represent repsectively for the following quantities: the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus.

In [8] Rivera and Racke examined the following system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) = 0, \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0. \end{cases}$$

Using the energy approach, they established that this system is exponentially stable.

A. Rahmoune, Ou. Khaldi, D. Ouchenane, F. Yazid. Well-posedness and stability result for Timoshenko system with thermodiffusion effects and time-varying delay term.

Submitted October 21, 2023.

Time delays appear in many applications of the majority of phenomena naturally governed by partial differential equations problems relying not only on the present state but also on some past occurrences. The delay can cause an instability. To the best of our knowledge, Said Houari was the first who discussed the Timoshenko system with a time delay [9]. More precisely, in [9] the Timoshenko system

$$\begin{cases}\rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t)+\mu_1\psi_t(x,t)+\mu_2\psi_t(x,t-\tau)=0,\end{cases}$$

and under the assumption $\mu_1 \geqslant \mu_2$ they proved the well-posedness and exponential decay. The work was extended upon by Kirane et al. [5], who also introduced the case of time-varying delay and established several estimates of general decay.

In [2], Apalara considered a one-dimensional Timoshenko system with linear friction damping and a constant delay operating on the displacement equation.

$$\begin{cases}\rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)+\mu_1\varphi_t(x,t)+\mu_2\varphi_t(x,t-\tau)=0,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t)=0.\end{cases}$$

Under appropriate assumptions on the weight of the delay and the wave speeds, the well-posedness and asymptotic stability results of the system were established. The stability results also showed that the dissipation through the frictional damping is strong enough to uniformly stabilize the system even in the presence of delay. This paper is similar to the one by Said Houari [9], but it includes a delay in the first equation, like the one considered in the manuscript.

We may believe that the dissipation cannot be fully explained by the thermal conduction in the Timoshenko beam and the area of diffusion in solids cannot be neglected. A natural question appears on what happens when the thermal effect and diffusion effect are included in Timoshenko beams. The diffusion is a random movement of a group of particles from high concentration areas to low concentration regions. The domains of strain, temperature, and mass diffusion cause the thermodiffusion in an elastic solid. Recently, Aouadi et al., in [1] studied the following problem

$$\begin{cases}\rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0,\\ \rho_2\psi_{tt}-\alpha\psi_{xx}+k(\varphi_x+\psi)-\gamma_1\theta_x-\gamma_2P_x=0,\\ c\theta_t+dP_t-\kappa\theta_{xx}-\gamma_1\psi_{tx}=0,\\ d\theta_t+rP_t-\hbar P_{xx}-\gamma_2\psi_{tx}=0.\end{cases}$$

Under different boundary conditions, they proved that the system is exponentially stable if and only if

$$\frac{k}{\rho_1}\neq\frac{\alpha}{\rho_2}. \tag{1.1}$$

Being inspired by previous works, the main goal of this paper is to demonstrate the well-posedness and establish a general energy decay, which results in the usual exponential decay. Our result is dependent on the kernel of the time varying delay term and the construction of an appropriate Lyapunov functional, which allows us to estimate the energy of the system. However, the study of the asymptotic behavior of the solution for various types of problems, such as the Timoshenko system [7], remains crucial. The study the exponential behavior of solutions to a variety of problems, such as the Timoshenko system [7], is still important.

In this article, we are intersted with the following problem

$$\begin{cases}\rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)+\mu_1\varphi_t(x,t)+\mu_2\varphi(x,t-\tau(t))=0,\\ \rho_2\psi_{tt}-\alpha\psi_{xx}+k(\varphi_x+\psi)-\gamma_1\theta_x-\gamma_2P_x=0,\\ c\theta_t+dP_t-\kappa\theta_{xx}-\gamma_1\psi_{tx}=0,\\ d\theta_t+rP_t-\hbar P_{xx}-\gamma_2\psi_{tx}=0\end{cases} \tag{1.2}$$

subject to the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi^0(x), & \psi(x,0) = \psi^0(x), & \theta(x,0) = \theta^0(x) & \text{for} \quad x \in (0,L), \\ P(x,0) = P^0(x), & \varphi_t(x,0) = \varphi^1(x), & \psi_t(x,0) = \psi^1(x) & \text{for} \quad x \in (0,L), \end{cases} \tag{1.3}$$

and a Dirichlet boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi(0,t) = \theta(0,t) = P(0,t) = 0, & t > 0, \\ \varphi(L,t) = \psi(L,t) = \theta(L,t) = P(L,t) = 0, & t > 0. \end{cases} \tag{1.4}$$

We assume that the symmetric matrix $\Lambda = \begin{pmatrix} c & d \\ d & r \end{pmatrix}$ is positive definite, that is,

$$\delta = cr - d^2 > 0. \tag{1.5}$$

We note that the above relation implies

$$c\theta^2 + 2d\theta P + rP^2 > 0 \tag{1.6}$$

for $\theta, P \neq 0$. We also observe that condition (1.5) is needed to stabilize the system, when the diffusion effects are added to thermal effects.

In our opinion, the concept of the mass diffusion introduced into Timoshenko equations could have very significant physical effects other than body deformations. For example, recent studies focused on the effect of mass diffusion on the damping ratio in microbeam resonators, see, for instance, [9]. Moreover, the mass diffusion introduces a new critical thickness in addition to the conventional critical thickness of thermoelastic damping.

The explanations above indicate that the mass diffusion plays an important role in the clarification of the thermomechanical behaviour of Timoshenko model. To the best of the authors' knowledge, no theoretical or numerical simulation of the mass diffusion effects on the thermal vibration of the Timoshenko beam was done. And the goal of this work is to examine the effect of mass diffusion alongside the effect of temperature on the behaviour of the Timoshenko beam.

The paper is organized as follows. In Section 2 we demonstrate the well-posedness of our problem using the semi-group technique. We establish a general stability under a suitable conditions in Section 3.

2. Well-posedness

In this section we prove that the considered problem is well-posed. We introduce a new variable

$$z(x,\rho,t) = \varphi_t(x, t - \tau(t)\rho), \qquad x \in (0,1), \quad \rho \in (0,1), \quad t > 0,$$

and get following equation:

$$\tau(t) z_t(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0, \qquad (x,\rho,t) \in (0,1) \times (0,1) \times (0,+\infty).$$

Then problem (1.2) can be rewritten as

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + \mu_1 \varphi_t(x,t) + \mu_2 z(x,1,t) = 0, \\ \rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \gamma_1 \theta_x - \gamma_2 P_x = 0, \\ c\theta_t + dP_t - \kappa \theta_{xx} - \gamma_1 \psi_{tx} = 0, \\ d\theta_t + rP_t - \hbar P_{xx} - \gamma_2 \psi_{tx} = 0, \\ \tau(t) z_t(x,\rho,t) + (1 - \tau'(t)\rho) z_\rho(x,\rho,t) = 0, \qquad (x,\rho,t) \in (0,L) \times (0,1) \times (0,+\infty), \end{cases} \tag{2.1}$$

subject to the initial conditions

$$\begin{cases} \varphi(x,0)=\varphi^0(x), & \psi(x,0)=\psi^0(x), \quad \theta(x,0)=\theta^0(x) \quad \text{for} \quad x\in(0,L), \\ P(x,0)=P^0(x), & \varphi_t(x,0)=\varphi^1(x), \quad \psi_t(x,0)=\psi^1(x) \quad \text{for} \quad x\in(0,L), \\ z\left(x,0,t\right)=\varphi_t\left(x,t\right), & x\in\left(0,L\right), \quad t>0, \\ z\left(x,\rho,0\right)=f_0\left(x,1-\rho\tau(0)\right), & \left(x,\rho\right)\in\left(0,L\right)\times\left(0,1\right), \end{cases} \tag{2.2}$$

and the Dirichlet boundary conditions

$$\begin{cases} \varphi(0,t)=\psi(0,t)=\theta(0,t)=P(0,t)=0 & \text{for} \quad t>0, \\ \varphi(L,t)=\psi(L,t)=\theta(L,t)=P(L,t)=0 & \text{for} \quad t>0. \end{cases} \tag{2.3}$$

Here the function $\tau(t)$ is supposed to satisfy the condition

$$0<\tau_0\leqslant\tau(t)\leqslant\overline{\tau} \qquad \text{for all} \quad t>0. \tag{2.4}$$

$$\tau'(t)\leqslant m<1 \qquad \text{for all} \quad t>0, \tag{2.5}$$

$$\tau\in W^{2,\infty}[0,+\infty), \tag{2.6}$$

We are going to study the well-posedness of the above problem. Namely, we provide sufficient conditions that ensuring the well-posedness. In order to do this, we follow procedures from recent paper [9], in which the Thimoshenko problem with a frictional damping was studied.

We rewrite system (2.1), (2.2), (2.3) as a first order system in order to apply the semigroup approach. Namely, we let $U(t) = (\varphi(t), v(t), \psi(t), \phi(t), \theta(t), P(t), z(t))^T$, and rewrite (2.1), (2.2), (2.3) as

$$\begin{cases} U' = \mathscr{A}U, \\ U\left(0\right)=U_0 := \left(\varphi^0(x),\varphi^1(x),\psi^0(x),\psi^1(x),\theta^0(x),P^0(x),f_0\left(x,1-\rho\tau(0)\right)\right)^T, \end{cases}$$

where $\mathscr{A}$ is an operator defined as

$$\mathscr{A}\begin{pmatrix}\varphi\\ v\\ \psi\\ \phi\\ \theta\\ p\\ z\end{pmatrix} = \begin{pmatrix} v \\ \dfrac{K}{\rho_1\left(\varphi_{xx}+\psi_x\right)}-\dfrac{\mu_1}{\rho_1 v}-\dfrac{\mu_2}{\rho_1 z\left(\cdot,1\right)} \\ \phi \\ \rho_2^{-1}\left(\alpha\psi_{xx}-k\left(\varphi_x+\psi\right)+\gamma_1\theta_x+\gamma_2 P_x\right) \\ -\delta^{-1}\left(\left(d\gamma_2-r\gamma_1\right)\phi_x-r\kappa\theta_{xx}+d\hbar P_{xx}\right) \\ -\delta^{-1}\left(\left(d\gamma_1-c\gamma_2\right)\phi_x+d\kappa\theta_{xx}-c\hbar P_{xx}\right) \\ \dfrac{(\tau'(t)\rho-1)z_\rho}{\tau(t)} \end{pmatrix}$$

on the domain

$$D\left(\mathscr{A}\right)=\left\{\left(\varphi,v,\psi,\phi,\theta,p,z\right)^T\in H:\ v\equiv z\left(\cdot,0\right),\ \text{in}\ \left(0,L\right)\right\},$$

where

$$\begin{aligned} H := &\left(H^2\left(0,L\right)\cap H_0^1\left(0,L\right)\right)\times H_0^1\left(0,L\right)\times\left(H^2\left(0,L\right)\cap H_0^1\left(0,L\right)\right)\times H_0^1\left(0,L\right) \\ &\times\left(H^2\left(0,L\right)\cap H_0^1\left(0,L\right)\right)\times\left(H^2\left(0,L\right)\cap H_0^1\left(0,L\right)\right)\times L^2\left(\left(0,L\right);H^1\left(0,L\right)\right). \end{aligned}$$

The energy space $\mathscr{H}$ is defined as

$$\begin{aligned} \mathscr{H} :=&H_0^1\left(0,L\right)\times L^2\left(0,L\right)\times H_0^1\left(0,L\right)\times L^2\left(0,L\right)\times L^2\left(0,L\right) \\ &\times L^2\left(0,L\right)\times L^2\left(\left(0,L\right);L^2\left(0,L\right)\right). \end{aligned}$$

For $U_j = (\varphi_j, v_j, \psi_j, \phi_j, \theta_j, P_j, z_j)^T \in \mathscr{H}$ and $j = 1, 2$, and a positive constant ξ obeying

$$\frac{\mu_2}{\sqrt{1-m}}\leqslant\xi\leqslant 2\mu_1-\frac{\mu_2}{\sqrt{1-m}}, \qquad \mu_2<\sqrt{1-m}\mu_1, \tag{2.7}$$

the inner product in $\mathscr{H}$ is defined as

$$(U_1,U_2)_{\mathscr{H}} = \int_0^L \Big[\rho_1 v_1 \overline{v_2} + \rho_2 \phi_1 \overline{\phi_2} + \alpha \psi_{1,x} \overline{\psi_{2,x}} + k\left(\varphi_{1,x}+\psi_1\right)\overline{\left(\varphi_{2,x}+\psi_2\right)}$$

$$+\Lambda \begin{pmatrix}\theta_1\\ P_1\end{pmatrix} \cdot \left(\overline{\theta_2 P_2}\right)\Big]\, dx + \xi\tau(t)\int_0^L\int_0^1 z\left(x,\rho\right)\overline{z_1}\left(x,\rho\right) d\rho dx.$$

The solvability result is formulated in the following theorem.

Theorem 2.1. *Assume that* $\mu_2 < \sqrt{1-m}\mu_1$, *then for any* $U_0 \in \mathscr{H}$ *there exists a unique solution* $U \in C\left([0,+\infty), \mathscr{H}\right)$ *of system* (2.1), (2.2), (2.3). *Moreover, if* $U_0 \in D\left(\mathscr{A}\right)$, *then*

$$U \in C\left([0,+\infty), D\left(\mathscr{A}\right)\right) \cap C^1\left([0,+\infty), \mathscr{H}\right).$$

Proof. We use the semigroup approach in the proof. In other words, we show how the operator $\mathscr{A}$ produces a C_0-semigroup in $\mathscr{H}$. We are going to demonstrate that the operator $B(t) = \mathscr{A} - \beta(t)I$ is dissipative with

$$\beta(t) = \frac{\sqrt{\tau'(t)^2+1}}{2\tau(t)}.$$

Indeed, for $U = \left(\varphi,u,\psi,v,\theta,q,z\right)^T \in D(\mathscr{A})$, we have

$$\begin{aligned}\langle \mathscr{A}U,U\rangle_{\mathscr{H}} =& k\int_0^L \left(\varphi_x+\psi\right)_x v\, dx + \int_0^L \left(\alpha\psi_x + \gamma_1\theta + \gamma_2 P\right)_x \phi\, dx\\ &- \mu_2 \int_0^L z\left(x,1\right) v\, dx - k\int_0^L \left(\varphi_x+\psi\right)\phi\, dx + \alpha\int_0^L \phi_x\psi_x\, dx\\ &+ k\int_0^L \left(v_x+\phi\right)\left(\varphi_x+\psi\right) dx + \int_0^L \Lambda\begin{pmatrix}\Theta\\ \Phi\end{pmatrix}\cdot\begin{pmatrix}\overline{\theta}\\ \overline{P}\end{pmatrix} dx\\ &+ \xi\int_0^L\int_0^1 (\tau'(t)\rho-1) z\left(x,\rho\right) z_\rho\left(x,\rho\right)\, d\rho dx,\end{aligned} \tag{2.8}$$

where

$$\Theta = -\delta^{-1}\left(\left(d\gamma_2 - r\gamma_1\right)\phi_x - r\kappa\theta_{xx} + d\hbar P_{xx}\right), \qquad \Phi = -\delta^{-1}\left(\left(d\gamma_1 - c\gamma_2\right)\phi_x + d\kappa\theta_{xx} - c\hbar P_{xx}\right).$$

The last term in the right-hand side of (2.8) can be rewritten as

$$\begin{aligned}\int_0^L\int_0^1 (\tau'(t)\rho-1) z\left(x,\rho\right) z_\rho\left(x,\rho\right) d\rho dx =& \int_0^L\int_0^1 (\tau'(t)\rho-1)\frac{1}{2}\frac{\partial}{\partial\rho} z^2\left(x,\rho\right) d\rho dx\\ =& \frac{1}{2}\int_0^L \left\{z^2\left(x,1\right)\left(\tau'(t)-1\right) - z^2\left(x,0\right)\right\} dx\\ &- \frac{\tau'(t)}{2}\int_0^L\int_0^1 z^2\left(x,\rho\right) d\rho dx.\end{aligned}$$

As a result, (2.8) becomes

$$\begin{aligned}\langle \mathscr{A}U,U\rangle_{\mathscr{H}} =& -k\int_0^L \theta_x^2\,dx-\hbar\int_0^L P_x^2dx-\mu_1\int_0^L v^2\,dx\\ &-\mu_2\int_0^L z\left(x,1\right)v\,dx-\frac{\xi\tau'(t)}{2}\int_0^L\int_0^1 z^2\left(x,\rho\right)\,d\rho dx\\ &+\frac{\xi}{2}\int_0^L(\tau'(t)-1)z^2\left(x,1\right)\,dx-\frac{\xi}{2}\int_0^L v^2\left(x\right)\,dx.\end{aligned}\tag{2.9}$$

By using Young inequality and (2.9) we obtain

$$\begin{aligned}\langle \mathscr{A}U,U\rangle_{\mathscr{H}} \leqslant& -k\int_0^L \theta_x^2dx-\hbar\int_0^L P_x^2dx+\left(-\mu_1+\frac{\mu_2}{2\sqrt{1-m}}-\frac{\xi}{2}\right)\int_0^L v^2\left(x\right)\,dx\\ &+\left(\frac{\mu_2\sqrt{1-m}}{2}-\xi\frac{(1-m)}{2}\right)\int_0^L z^2\left(x,1\right)\,dx+\beta(t)\langle \mathscr{A}U,U\rangle_{\mathscr{H}}.\end{aligned}$$

In view of condition (2.7) we have

$$-\mu_1+\frac{\mu_2}{2\sqrt{1-m}}-\frac{\xi}{2}\leqslant 0,\qquad \frac{\mu_2\sqrt{1-m}}{2}-\xi\frac{(1-m)}{2}\leqslant 0.$$

Hence, the operator $\mathscr{A}$ is dissipative.

Now we are going to prove that the operator $\lambda I-\mathscr{A}$ is surjective for $\lambda>0$. We take an element $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathscr{H}$ and we seek a solution $U=(\varphi,v,\psi,\phi,\theta,p,z)^T\in D\left(\mathscr{A}\right)$ to the equation

$$\lambda U-\mathscr{A}U=F$$

or equivalently

$$\begin{cases}\lambda\varphi-v=f_1,\\ \lambda v-\dfrac{K}{\rho_1}\left(\varphi_{xx}+\psi_x\right)+\dfrac{\mu_1}{\rho_1}v+\dfrac{\mu_2}{\rho_1}z\left(\,\cdot\,,1\right)=f_2,\\ \lambda\psi-\phi=f_3,\\ \lambda\rho_2\phi-\alpha\psi_{xx}+k\left(\varphi_x+\psi\right)-\gamma_1\theta_x-\gamma_2P_x=\rho_2f_4,\\ \lambda\delta\theta+\left(d\gamma_2-r\gamma_1\right)\phi_x-rk\theta_{xx}+d\hbar P_{xx}=\delta f_5\\ \lambda\delta P+\left(d\gamma_1-c\gamma_2\right)\phi_x+d\kappa\theta_{xx}-c\hbar P_{xx}=\delta f_6\\ \lambda z-\dfrac{(\tau'(t)\rho-1)}{\tau(t)}z_\rho=f_7.\end{cases}\tag{2.10}$$

Assume that we have found φ and ψ with the needed regularity. Then the first and third equations in (2.10) give

$$\begin{cases}v=\lambda\varphi-f_1,\\ \phi=\lambda\psi-f_3.\end{cases}\tag{2.11}$$

It is clear that $v\in H_0^1\left(0,1\right)$, and $\phi\in H_0^1\left(0,1\right)$. Moreover, we can find z as

$$z\left(x,0\right)=v\left(x\right)\quad\text{for}\quad x\in\left(0,1\right).$$

Following the lines of [6], by using the last equation in (2.10) we obtain

$$z(x,\rho) = v(x)\, e^{-\lambda\rho\tau(t)} + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^\rho f_7(x,s)\, e^{\lambda s\tau(t)}\, ds \quad \text{if} \quad \tau'(t) = 0,$$

and

$$z(x,\rho) = v(x) e^{r_\rho(t)} + e^{r_\rho(t)} \int_0^1 \frac{f_7(x,s)\tau(t)}{1-\tau'(t)s} e^{-r_s(t)}\, ds \quad \text{if} \quad \tau'(t) \neq 0,$$

where

$$r_\rho(t) = \lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho).$$

Using (2.11), we then get

$$z(x,\rho) = \lambda\varphi(x)\, e^{-\lambda\rho\tau(t)} - f_1 e^{-\lambda\rho\tau(t)} + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^1 f_7(x,s)\, e^{\lambda s\tau(t)}\, ds \quad \text{if} \quad \tau'(t) = 0,$$

and

$$z(x,\rho) = \lambda\varphi e^{r_\rho(t)} - f_1 e^{r_\rho(t)} + e^{r_\rho(t)} \int_0^1 \frac{f_7(x,s)\tau(t)}{1-\tau'(t)s} e^{-r_s(t)}\, ds \quad \text{if} \quad \tau'(t) \neq 0.$$

By the above identities we have

$$z(x,1) = g(t)\varphi(x) + z_0(x),$$
$$g(t) = \begin{cases} \lambda e^{-\lambda\tau(t)} & \text{if} \quad \tau'(t) = 0, \\ \lambda e^{r_\rho(t)} & \text{if} \quad \tau'(t) \neq 0. \end{cases}$$

and

$$z_0(x) = \begin{cases} -f_1 e^{-\lambda\tau(t)} + \tau(t) e^{-\lambda\tau(t)} \int_0^1 f_7(x,s)\, e^{\lambda s\tau(t)}\, ds & \text{if} \quad \tau'(t) = 0, \\ -f_1 e^{r_\rho(t)} + e^{r_\rho(t)} \int_0^1 \frac{f_7(x,s)\tau(t)}{1-\tau'(t)s} e^{-r_s(t)}\, ds & \text{if} \quad \tau'(t) \neq 0, \end{cases} \tag{2.12}$$

where $x \in (0,L)$. According to the above formula, z_0 depends only on f_i, $i = 1,\ldots,7$. The following system can be satisfied by employing (2.10) and (2.11) with the functions φ, ψ, θ and p:

$$\begin{cases} \left(\lambda^2 + \frac{\mu_1}{\rho_1}\lambda + g(t)\frac{\mu_2}{\rho_1}\right)\varphi - \frac{K}{\rho_1}(\varphi_{xx} + \psi_x) = f_2 + \left(\lambda + \frac{\mu_1}{\rho_1}\right) f_1 - \frac{\mu_2}{\rho_1} z_0(x), \\ \lambda^2\rho_2\psi - \alpha\psi_{xx} + k(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = \rho_2(\lambda f_3 + f_4), \\ \lambda\delta\theta + (d\gamma_2 - r\gamma_1)\psi_x - r\kappa\theta_{xx} + d\hbar P_{xx} = \delta f_5 + (d\gamma_2 - r\gamma_1) f_{3,x}, \\ \lambda\delta P + (d\gamma_1 - c\gamma_2)\psi_x + d\kappa\theta_{xx} - c\hbar P_{xx} = \delta f_6 + (d\gamma_1 - c\gamma_2) f_{3,x}. \end{cases} \tag{2.13}$$

Solving system (2.13) is equivalent to finding

$$(\varphi,\psi,\theta,p) \in H^2(0,L) \cap H_0^1(0,L) \times H^2(0,1) \cap H_0^1(0,L) \times H^1(0,L) \times H_0^1(0,L)$$

such that

$$\begin{cases} \int_0^L \left(\left(\lambda^2\rho_1 + \mu_1\lambda + g(t)\mu_2\right)\varphi w + K\left(\varphi_x + \psi\right) w_x\right)\, dx \\ \qquad\qquad = \int_0^L \left(\rho_1 f_2 + \left(\lambda\rho_1 + \mu_1\right) f_1 - \mu_2 z_0(x)\right) w\, dx, \\ \int_0^L \left(\lambda^2\rho_2\psi - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) - \gamma_1\theta_x - \gamma_2 P_x\right)\chi dx = \int_0^L \rho_2\left(f_4 + \lambda f_3\right)\chi\, dx, \\ \int_0^L \left(\lambda\delta\theta + \left(d\gamma_2 - r\gamma_1\right)\psi_x - r\kappa\theta_{xx} + d\hbar P_{xx}\right) w_1\, dx = \int_0^L \left(\delta f_5 + \left(d\gamma_2 - r\gamma_1\right) f_{3,x}\right) w_1\, dx, \\ \int_0^L \left(\lambda\delta P + \left(d\gamma_1 - c\gamma_2\right)\psi_x + d\kappa\theta_{xx} - c\hbar P_{xx}\right)\chi_1 = \int_0^L \left(\delta f_6 + \left(d\gamma_1 - c\gamma_2\right) f_{3,x}\right)\chi_1\, dx, \end{cases} \tag{2.14}$$

for all $(w, \chi, w_1, \chi_1) \in H_0^1(0, L) \times H_0^1(0, L) \times H^1(0, L) \times H_0^1(0, L)$. Hence, problem (2.14) is equivalent to

$$\zeta\left(\left(\varphi, \psi, \theta, p\right), \left(w, \chi, w_1, \chi_1\right)\right) = l\left(w, \chi, w_1, \chi_1\right), \tag{2.15}$$

where a bilinear from

$$\zeta : \left(H_0^1(0, L) \times H_0^1(0, L) \times H^1(0, L) \times H_0^1(0, L)\right)^2 \to \mathbb{R}$$

and a linear form

$$l : H_0^1(0, L) \times H_0^1(0, L) \times H^1(0, L) \times H_0^1(0, L) \to \mathbb{R}$$

are defined as

$$\begin{aligned} \zeta\left(\left(\varphi, \psi, \theta, p\right), \left(w, \chi, w_1, \chi_1\right)\right) = & \int_0^L \left(\left(\lambda^2\rho_1 + \mu_1\lambda + g(t)\mu_2\right)\varphi w + K\left(\varphi_x + \psi\right) w_x\right)\, dx \\ & \int_0^L \left(\lambda^2\rho_2\psi - \alpha\psi_{xx} + k\left(\varphi_x + \psi\right) - \gamma_1\theta_x - \gamma_2 P_x\right)\chi\, dx \\ & \int_0^L \left(\lambda\delta\theta + \left(d\gamma_2 - r\gamma_1\right)\psi_x - r\kappa\theta_{xx} + d\hbar P_{xx}\right) w_1 dx \\ & \int_0^L \left(\lambda\delta P + \left(d\gamma_1 - c\gamma_2\right)\psi_x + d\kappa\theta_{xx} - c\hbar P_{xx}\right)\chi_1, \end{aligned}$$

and

$$\begin{aligned} l\left(w, \chi, w_1, \chi_1\right) = & \int_0^L \left(\rho_1 f_2 + \left(\lambda\rho_1 + \mu_1\right) f_1 - \mu_2 z_0(x)\right) w\, dx + \int_0^L \rho_2\left(f_4 + \lambda f_3\right)\chi\, dx \\ & + \int_0^L \left(\delta f_5 + \left(d\gamma_2 - r\gamma_1\right) f_{3,x}\right) w_1\, dx + \int_0^L \left(\delta f_6 + \left(d\gamma_1 - c\gamma_2\right) f_{3,x}\right)\chi_1\, dx, \end{aligned}$$

where $z_0(x)$ satisfies the equation in (2.12). The continuous and coercive character of ζ is easily verified, and l is continuous, so applying the Lax-Milgram theorem, we deduce that for all

$$(w, \chi, w_1, \chi_1) \in H_0^1(0, L) \times H_0^1(0, L) \times H^1(0, L) \times H_0^1(0, L)$$

problem (2.15) possesses a unique solution

$$(\varphi, \psi, \theta, p) \in H_0^1(0, L) \times H_0^1(0, L) \times H^1(0, L) \times H_0^1(0, L).$$

Applying the classical elliptic regularity, by (2.14) we find that

$$(\varphi, \psi, \theta, p) \in H^2(0, L) \times H^2(0, L) \times H^1(0, L) \times H_0^1(0, L).$$

The operator $\lambda I - \mathscr{A}$ is hence surjective for each $\lambda > 0$. Now the statement of the theorem follows from the Hille-Yosida theorem. The proof is complete. □

3. Exponential stability for $\mu_2 < \sqrt{1-m}\mu_1$.

In this section we show the exponential stability of system (2.1), (2.2), (2.3) under the assumption $\sqrt{1-d}\mu_1 > \mu_2$ and the condition of nonequal wave speeds of propagation

$$\frac{k}{\rho_1} \neq \frac{\alpha}{\rho_2}. \tag{3.1}$$

Our approach is based on an appropriate Lyapunov functional using the energy technique, which results in the needed exponential decay.

We first observe that $(\varphi, \psi, \theta, p, z)$ satisfies the same system (2.1), (2.2) and (2.3) and ξ still satisfies

$$\frac{\mu_2}{\sqrt{1-m}} \leqslant \xi \leqslant 2\mu_1 - \frac{\mu_2}{\sqrt{1-m}}. \tag{3.2}$$

The functional energy of the problem(2.1), (2.2) reads as

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \alpha\psi_x^2 + \kappa\left(\varphi_x + \psi\right)^2 + c\theta^2 + 2d\theta P + rP^2\right] dx \\ &+ \frac{\xi\tau(t)}{2}\int_0^L\int_0^1 z^2(x, \rho, t)\, d\rho dx. \end{aligned} \tag{3.3}$$

We multiply the first equation in (2.1) by ψ_t, the second equation by ψ_t, the third equation in (2.1) by θ, and the fourth equation in (2.1) by q. Then we integrate by parts and we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \alpha\psi_x^2 + \kappa\left(\varphi_x + \psi\right)^2 + c\theta^2 + 2d\theta P + rP^2\right] dx \\ &= -\kappa\int_0^L \theta_x^2\, dx - \hbar\int_0^L P_x^2\, dx - \mu_1\int_0^L \varphi_t^2(x, t)\, dx - \mu_2\int_0^L \varphi_t(x, t)\, z(x, 1, t)\, dx. \end{aligned} \tag{3.4}$$

We multiply the last equation in (2.1) by ξz and z and integrate the result over $(0, L) \times (0, 1)$ with respect to ρ and x respectively. This gives

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_0^L\int_0^1 \tau(t)z^2(x,\rho,t)\,d\rho dx =& \xi\int_0^L\int_0^1 (\tau'(t)\rho-1)zz_\rho(x,\rho,t)\,d\rho dx \\ &+\frac{\xi}{2}\tau'(t)\int_0^L\int_0^1 z^2(x,\rho,t)\,d\rho dx \\ =&\frac{\xi}{2}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}(\tau'(t)\rho-1)z^2(x,\rho,t)\,d\rho dx \\ =&\frac{\xi}{2}\int_0^L \left(z^2(x,0,t)-z^2(x,1,t)\right)\,dx \\ &+\frac{\xi\tau'(t)}{2}\int_0^L z^2(x,1,t)\,dx.\end{aligned} \tag{3.5}$$

By (3.3), (3.4) and (3.5) we find

$$\begin{aligned}\frac{dE(t)}{dt} =& -\kappa\int_0^L \theta_x^2\,dx - \hbar\int_0^L P_x^2\,dx - \left(\mu_1-\frac{\xi}{2}\right)\int_0^L \varphi_t^2(x,t)\,dx \\ &+\left(-\frac{\xi}{2}+\frac{\xi\tau'(t)}{2}\right)\int_0^L z^2(x,1,t)\,dx - \mu_2\int_0^L \varphi_t(x,t)\,z(x,1,t)\,dx.\end{aligned} \tag{3.6}$$

Using the Young inequality, we rewrite (3.6) as

$$\begin{aligned}\frac{dE(t)}{dt} \leqslant& -\kappa\int_0^L \theta_x^2\,dx - \hbar\int_0^L P_x^2\,dx - \left(\mu_1-\frac{\xi}{2}-\frac{\mu_2}{2\sqrt{1-m}}\right)\int_0^L \varphi_t^2(x,t)\,dx \\ &+\left(\frac{\xi}{2}(\tau'(t)-1)+\frac{\mu_2\sqrt{1-m}}{2}\right)\int_0^L z^2(x,1,t)\,dx.\end{aligned}$$

In view of (3.2), (2.4), (2.5) and (2.6) we conclude there exists $C > 0$ such that

$$\frac{dE(t)}{dt} \leqslant -\kappa\int_0^L \theta_x^2\,dx - \hbar\int_0^L P_x^2\,dx - C\left\{\int_0^L \varphi_t^2(x,t)\,dx + \int_0^L z^2(x,1,t)\,dx\right\}.$$

According to the last inequality, the function E does not increase in t.

Now we are in position to formulate our main result.

Theorem 3.1. *Assume (1.1), (3.1), (2.5), (2.6) and $\mu_2 < \sqrt{1-m}\mu_1$. Then, for any solution to problem (1.2), (1.3), (1.4) there are two positive constants C and γ independent of t such that*

$$E(t) \leqslant Ce^{-\gamma t} \quad \textit{for all} \quad t \geqslant 0.$$

To establish the exponential decay of the solution, it is sufficient to construct a functional $\mathscr{L}(t)$, which is equivalent to the energy $E(t)$ and satisfies

$$\frac{d\mathscr{L}(t)}{dt} \leqslant -\Lambda \mathscr{L}(t) \quad \text{for all} \quad t \geqslant 0$$

with some constant $\Lambda > 0$.

In order to find such functional, we first introduce another functional defined as

$$\mathcal{I}(t) = \int_0^L (\rho_1 \varphi_t \varphi + \rho_2 \psi_t \psi)\, dx + \frac{\mu_1}{2} \int_0^L \varphi^2\, dx. \tag{3.7}$$

We then have the following estimate.

Lemma 3.1. *Assume that conditions (1.6) and (3.1) hold and $(\varphi, \varphi_t, \psi, \psi_t, \theta, P)$ is the solution to problem (2.1), (2.2), (2.3). If $\varepsilon_1 > 0$, we then have the estimate*

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) \leqslant & \rho_1 \int_0^L \varphi_t^2\, dx + \rho_2 \int_0^L \psi_t^2\, dx + \left(-k + \frac{\mu_2 C' \varepsilon_1}{2}\right) \int_0^L \varphi_x^2\, dx \\ & - \frac{\alpha}{2} \int_0^L \psi_x^2\, dx + C_1 \int_0^L \theta_x^2\, dx + C_2 \int_0^L P_x^2\, dx - k \int_0^L \psi^2\, dx \\ & - 2k \int_0^L \psi \varphi_x\, dx + \frac{\mu_2}{2\varepsilon_1} \int_0^L z^2(x,1,t)\, dx.\end{aligned} \tag{3.8}$$

Proof. We calculate the derivative of $\mathcal{I}(t)$:

$$\frac{d}{dt}\mathcal{I}(t) = \rho_1 \int_0^L \left(\varphi_t^2 + \varphi \varphi_{tt}\right) dx + \rho_2 \int_0^L \left(\psi_t^2 + \psi \psi_{tt}\right) dx + \mu_1 \int_0^L \varphi \varphi_t\, dx.$$

It follows from (2.1) and (2.1) that

$$\begin{aligned}\frac{d}{dt} \int_0^L \rho_1 \varphi_t \varphi\, dx = & \rho_1 \int_0^L \varphi_t^2\, dx - k \int_0^L \varphi_x^2\, dx - k \int_0^L \psi \varphi_x\, dx \\ & - \mu_1 \int_0^L \varphi \varphi_t\, dx - \mu_2 \int_0^L \varphi z(x,1,t)\, dx,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\frac{d}{dt} \int_0^L \rho_2 \psi_t \psi\, dx = & \rho_2 \int_0^L \psi_t^2\, dx + \int_0^L (\alpha \psi_{xx} - k(\varphi_x + \psi) + \gamma_1 \theta_x + \gamma_2 P_x)\, \psi\, dx \\ = & \rho_2 \int_0^L \psi_t^2\, dx - k \int_0^L \psi \varphi_x dx - k \int_0^L \psi^2\, dx - \alpha \int_0^L \psi_x^2\, dx - \int_0^L (\gamma_1 \theta + \gamma_2 P)\, \psi_x\, dx.\end{aligned}$$

Summing (3.9) and (3.8), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) =&\rho_1\int_0^L \varphi_t^2\,dx+\rho_2\int_0^L \psi_t^2\,dx-\alpha\int_0^L \psi_x^2\,dx-\mu_2\int_0^L \varphi z\left(x,1,t\right)\,dx\\ &-\int_0^L\left(\gamma_1\theta+\gamma_2 P\right)\psi_x\,dx-k\int_0^L \psi^2\,dx-k\int_0^L \varphi_x^2\,dx-2k\int_0^L \psi\varphi_x\,dx.\end{aligned}\tag{3.10}$$

Using the Young and PoincarГ© inequalities, we arrive at

$$-\int_0^L\left(\gamma_1\theta+\gamma_2 P\right)\psi_x\,dx\leqslant C_1\int_0^L \theta_x^2 dx+C_2\int_0^L P_x^2\,dx+\frac{\alpha}{2}\int_0^L \psi_x^2\,dx,\tag{3.11}$$

$$-\mu_2\int_0^L \varphi z\left(x,1,t\right)\,dx\leqslant\frac{\mu_2 C'\varepsilon_1}{2}\int_0^L \varphi_x^2\,dx+\frac{\mu_2}{2\varepsilon_1}\int_0^L z^2\left(x,1,t\right)\,dx.\tag{3.12}$$

Substituting (3.11), (3.12) into (3.10), we arrive at (3.14). The proof is complete. □

Now, we define one more functional:

$$\mathcal{J}(t)=\rho_2\int_0^L \psi_t\omega\,dx,\tag{3.13}$$

where

$$-\gamma_1\omega_x=c\theta+dP,\qquad \omega(0)=\omega(L)=0.$$

Lemma 3.2. *Let the assumptions of Lemma (3.1) hold true. Then the functional $\mathcal{J}$ defined by (3.13) satisfies*

$$\begin{aligned}\frac{d\mathcal{J}}{dt}(t)\leqslant&-\frac{\rho_2}{2}\int_0^L \psi_t^2\,dx+\frac{\alpha}{16}\int_0^L \psi_x^2\,dx+\frac{k}{8}\int_0^L \varphi_x^2\,dx+\frac{k}{4}\int_0^L \psi^2\,dx\\ &+\frac{k}{2}\int_0^L \psi\varphi_x\,dx+C_3\int_0^L \theta_x^2\,dx+C_4\int_0^L P_x^2\,dx.\end{aligned}\tag{3.14}$$

Proof. We take the derivative of (4.7) and we get

$$\frac{d\mathcal{J}}{dt}(t)=\int_0^L \rho_2\psi_t\omega_t\,dx+\int_0^L \rho_2\psi_{tt}\omega\,dx:=\mathcal{J}_1(t)+\mathcal{J}_2(t).\tag{3.15}$$

Employing (3.15) and Young inequality, we find:

$$\begin{aligned}\mathcal{J}_1(t):=&\int_0^L \rho_2\psi_t\omega_t\,dx=-\frac{\rho_2}{\gamma_1}\int_0^L \psi_t\partial_x^{-1}\left(\kappa\theta_{xx}+\gamma_1\psi_{xt}\right)\,dx\\ =&-\rho_2\int_0^L \psi_t^2\,dx-\frac{\rho_2\kappa}{\gamma_1}\int_0^L \psi_t\theta_x\,dx\leqslant-\frac{\rho_2}{2}\int_0^L \psi_t^2\,dx+C^{(1)}\int_0^L \theta_x^2\,dx.\end{aligned}$$

It follows from (3.15) that

$$\begin{aligned}\mathcal{J}_2(t) :=& \int_0^L \rho_2 \psi_{tt} \omega \, dx = \int_0^L \left(\alpha \psi_{xx} - k\left(\varphi_x + \psi\right) + \gamma_1 \theta_x + \gamma_2 P_x\right) \omega \, dx \\ =& - \alpha \int_0^L \omega_x \psi_x \, dx - k \int_0^L \left(\varphi_x + \psi\right) \omega \, dx + \int_0^L \omega \left(\gamma_1 \theta_x + \gamma_2 P_x\right) \, dx.\end{aligned}$$

By using Young and PoincarΓ©'s inequalities, we arrive at

$$-\alpha \int_0^L \omega_x \psi_x \, dx \leqslant \frac{\alpha}{16} \int_0^L \psi_x^2 \, dx + C^{(2)} \int_0^L \theta_x^2 \, dx + C^{(3)} \int_0^L P_x^2 \, dx. \tag{3.16}$$

$$\int_0^L \omega \left(\gamma_1 \theta_x + \gamma_2 P_x\right) \, dx \leqslant C^{(4)} \int_0^L \theta_x^2 \, dx + C^{(5)} \int_0^L P_x^2 \, dx, \tag{3.17}$$

and

$$\begin{aligned}-k \int_0^L \left(\varphi_x + \psi\right) \omega \, dx \leqslant& \frac{k}{8} \int_0^L \left(\varphi_x + \psi\right)^2 \, dx + C^{(6)} \int_0^L \theta_x^2 \, dx + C^{(7)} \int_0^L P_x^2 \, dx \\ =& \frac{k}{8} \int_0^L \varphi_x^2 \, dx + \frac{k}{8} \int_0^L \psi^2 \, dx + \frac{k}{4} \int_0^L \psi \varphi_x \, dx \\ &+ C^{(6)} \int_0^L \theta_x^2 \, dx + C^{(7)} \int_0^L P_x^2 \, dx \\ \leqslant& \frac{k}{8} \int_0^L \varphi_x^2 \, dx + \frac{k}{4} \int_0^L \psi^2 \, dx + \frac{k}{2} \int_0^L \psi \varphi_x \, dx \\ &+ C^{(6)} \int_0^L \theta_x^2 \, dx + C^{(7)} \int_0^L P_x^2 \, dx.\end{aligned} \tag{3.18}$$

Substituting (3.16), (3.17) and (3.18) into (3.15), we obtain (3.14) with

$$C_3 = c^{(1)} + c^{(2)} + c^{(4)} + c^{(6)}, \qquad C_4 = c^{(3)} + c^{(5)} + c^{(7)}.$$

The proof is complete. □

Our next step is to define a Lyapunov functional $\mathscr{L}(t)$ (t) and prove that it is equivalent to an energy functional E.

Lemma 3.3. *Under the assumptions of Lemma 3.1, there exists a constant $\beta_0 > 0$ such that*

$$\left(N - \beta_0\right) E(t) \leqslant \mathscr{L}(t) \leqslant \left(N + \beta_0\right) E(t), \quad \textit{for all} \quad t \geqslant 0, \tag{3.19}$$

where $\mathscr{L}(t)$ is a Lyapunov functional defined by

$$\mathscr{L}(t) = NE(t) + \mathcal{I}(t) + 4\mathcal{J}(t), \tag{3.20}$$

and $N > \beta_0$ is a sufficiently large constant.

Proof. Young, PoincarГ©, and Cauchy-Schwarz inequalities show that

$$|\mathcal{I}(t)| \leqslant \frac{\rho_1}{2}\int_0^L \varphi_t^2\,dx + \frac{\rho_2}{2}\int_0^L \psi_t^2\,dx + \frac{\rho_1 L^2}{2}\int_0^L \psi_x^2\,dx + \frac{\mu_1}{2}\int_0^L \varphi^2\,dx,$$

$$|\mathcal{J}(t)| \leqslant \frac{\rho_2}{2}\int_0^L \psi_t^2\,dx + c_1\int_0^L \theta^2\,dx + c_2\int_0^L P^2\,dx.$$

Hence, there exists a constant $\beta_0 > 0$ such that

$$|\mathscr{L}(t) - NE(t)| = |\mathcal{I}(t) + 4\mathcal{J}(t)| \leqslant \beta_0 E(t),$$

and this implies estimate (3.19). The proof is complete. □

Theorem 3.2. *Let the assumptions of Lemma 3.1 hold. Then there exist positive constants υ_0, υ_1 such that the energy functional satisfies*

$$E(t) \leqslant \upsilon_1 E(0) e^{-\upsilon_0 t} \quad \textit{for all} \quad t \geqslant 0. \tag{3.21}$$

Proof. It follows from (3.7), (3.14) and (3.20) that for each $t > 0$ we have the inequality

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) \leqslant &-(\kappa N - C_1 - 4C_3)\int_0^L \theta_x^2\,dx - (\hbar N - C_2 - 4C_4)\int_0^L P_x^2\,dx\\ &-\rho_2\int_0^L \psi_t^2\,\mathrm{d}x - \left(\frac{k}{2} - \frac{\mu_2 C'\varepsilon_1}{2}\right)\int_0^L \varphi_x^2 dx - \frac{\alpha}{4}\int_0^L \psi_x^2\,dx\\ &-(CN - \rho_1)\int_0^1 \varphi_t^2 dx - \left(CN - \frac{\mu_2}{2\varepsilon_1}\right)\int_0^1 z^2(x,1,t)\,dx.\end{aligned}$$

We choose ε_1 small enough such that

$$\frac{k}{2} \geqslant \frac{\mu_2 C'\varepsilon_1}{2}.$$

Then we choose N large enough such that

$$N > \sup\left\{\frac{C_1 - 4C_3}{\kappa}, \frac{C_2 - 4C_4}{\hbar}, \frac{\rho_1}{C}, \frac{\mu_2}{2C\varepsilon_1}\right\}.$$

Then there exists a positive constant ς such that

$$\frac{d}{dt}\mathscr{L}(t) \leqslant -\varsigma E(t)$$

and by using (3.19) it yields

$$\frac{d}{dt}\mathscr{L}(t) \leqslant -\zeta\mathscr{L}(t)$$

with some positive constant ζ. Now estimate (3.21) follows by using (3.19) and this completes the proof. □

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