ON DENSITY OF POLYNOMIALS IN ALGEBRA OF HOLOMORPHIC FUNCTIONS OF EXPONENTIAL TYPE ON LINEAR LIE GROUP

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Abstract. The question of the density of the algebra of polynomials (regular functions) in the algebra of holomorphic functions of exponential type on a complex Lie group arose in the study of duality for Hopf algebras of holomorphic functions. It was shown by the author in [J . Lie Theory, **29**:4, 1045–1070 (2019)] that the answer is affirmative in the connected linear case. However, the argument is quite involved and here we present a short proof. It contains two ingredients. The first is the existences of a finite–dimensional faithful holomorphic representation with closed range. To prove it, we use an approach developed by Đjoković. The second is a lower bound for the norm of a one–parameter matrix subgroup, which is based on some elementary linear algebra consideration. The rest of the proof is close to the original one and uses a decomposition of the group into a semidirect product of a simply connected solvable and linearly complex reductive factors.

Keywords: complex Lie group, linear group, holomorphic function of exponential type, submultiplicative weight.

Mathematics Subject Classification: 22E30, 22E45

1. Problem and main result

Following [1, Sect. 5.3.1], we say that a holomorphic function f on a complex Lie group G is of *exponential type* if there is a submultiplicative weight ω such that $|f(g)| \leqslant \omega(g)$ for each $g \in G$. A *submultiplicative weight* is a locally bounded non–negative function such that $\omega(gh) \leqslant \omega(g)\,\omega(h)$ for every $g, h \in G$. By $\mathcal{O}_{exp}(G)$ we denote the set of holomorphic functions of exponential type. Being endowed with the inductive topology and the point–wise multiplication, $\mathcal{O}_{exp}(G)$ is a complete locally convex algebra with jointly continuous multiplication [3, Lm. 5.2].

We recall that a complex Lie group G is said to be *linear* if it admits a finite–dimensional faithful holomorphic representation. Suppose additionally that G is connected. Then G is a connected Stein group and hence admits a canonical structure of complex algebraic affine variety [9, Thm. 2]. Thus we can consider the algebra $\mathcal{R}(G)$ of regular functions (polynomials) on G. It is well known that G is a semidirect product, $B \rtimes L$, where B is simply connected and solvable, and L is connected and linearly complex reductive; see, e.g., [6, Thm. 16.3.7]. Moreover, L admits a unique structure of affine algebraic group [9, App.] and B, while it may be non–algebraic, is biholomorphically equivalent to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Hence G can be identified with the affine variety $\mathbb{C}^n \times L$ and $\mathcal{R}(G)$ with $\mathcal{R}(B) \otimes \mathcal{R}(L)$, where $\mathcal{R}(B) = \mathcal{R}(\mathbb{C}^n)$. Although the decomposition in the form $B \rtimes L$ is not unique, see, e.g., [6, Exm. 16.3.12],

it is easy to see that the structure of affine variety is independent of this decomposition and coincides with the canonical structure given in [9].

Our aim is to give a short proof of the following result.

Theorem 1. [3, Cor. 5.11] *Let G be a connected linear complex Lie group. Then $\mathcal{R}(G)$ is contained and dense in $\mathcal{O}_{exp}(G)$.*

This result is a key step in proving the holomorphic reflexivity of the algebra of holomorphic functions on a connected linear Lie group; see [4] and a discussion in the second paragraph of the Introduction in [3]. But the proof given in [3] is based on a complete description of $\mathcal{O}_{exp}(G)$, which is quite complicated. In particular, it needs the concept of exponential radical and some facts on analysis on nilpotent Lie groups contained in [2]. Here we suggest a shorter argument.

2. PROOF

We first prove some auxiliary assertions.

Proposition 2. *A connected linear complex Lie group admits a finite–dimensional faithful holomorphic representation with closed range.*

The variant of this result for real Lie groups and continuous (automatically differentiable) representations is proved in [7, Thm. 9]; see also [5, Prop. 5] or [6, Thm. 16.2.10]. In the complex case, we need a lemma, which is not straightforward in contrast to the real case.

Note that when G is connected, the commutator subgroup (G, G) is normal and integral; see, e.g., [6, Prop. 11.2.4]. If, in addition, G is linear, then (G, G) is closed [8, Prop. 4.37]. Hence, $G/(G, G)$ is a complex Lie group.

Lemma 3. *Let G be a connected linear complex Lie group. Then $G/(G, G)$ admits a finite–dimensional faithful holomorphic representation with closed range.*

Proof. By [8, Thm. 4.38(iii)], the group $G/\operatorname{Rad}(G, G)$ is linear and hence, it is a Stein group. Then Lemma 8 in [9] implies that $G/(G, G)$ is also a Stein group. Being an abelian Stein group, $G/(G, G)$ is a product of finitely many copies of $\mathbb{C}$ and $\mathbb{C}^{\times}$ (the group of units) [10, Prop. 4]; see also [11, Thm. XIII.5.9]. We note that the representation of $\mathbb{C}$ given by $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ is faithful and holomorphic and has closed range. Also, the tautological representation of $\mathbb{C}^{\times}$ evidently has closed range. Therefore, $G/(G, G)$ also admits a finite–dimensional faithful holomorphic representation with closed range. The proof is complete. $\square$

For technical reasons, in what follows we sometimes use homomorphisms to the general linear group $\operatorname{GL}(\mathbb{C}, m)$ instead of m–dimensional representations.

Proof of Proposition 2. We argue as in the proof of [5, Prop. 5] but with the use of Lemma 3. The idea is to take two representations, the first being faithful and the second having closed range, with additional assumptions to ensure that their sum satisfies both properties.

Let $\pi \colon \mathbb{C} \to \operatorname{GL}(\mathbb{C}, m)$ be a faithful holomorphic homomorphism. By Lemma 3, there is a faithful holomorphic homomorphism $G/(G, G) \to \operatorname{GL}(\mathbb{C}, n)$ with closed range. Denote the composition of the quotient map $G \to G/(G, G)$ with this homomorphism by σ. We let

$$\rho \colon G \to \operatorname{GL}(\mathbb{C}, m) \times \operatorname{GL}(\mathbb{C}, n), \ g \mapsto (\pi(g), \sigma(g)).$$

It is clear that ρ is faithful and holomorphic. To show that the range of ρ is closed, we apply [5, Thm. 1] (instead of the last result, which concerns real Lie groups, a generalization concerning general locally compact groups [12, Thm.] can be applied). We let $H := \operatorname{GL}(\mathbb{C}, m) \times \{1\}$.

Then by [5, Thm. 1] it suffices to show that $\rho(G)H$ and $\rho(G) \cap H$ are closed and that $\rho(G)$ normalizes H.

It is clear that $\rho(G)H = \mathrm{GL}(\mathbb{C}, m) \times \sigma(G)$. This set is closed because $\sigma(G)$ is closed. Also, $\rho(G) \cap H = \pi((G, G)) \times \{1\}$. Treating $\mathrm{GL}(\mathbb{C}, m)$ as a subgroup of $\mathrm{GL}(\mathbb{R}, 2m)$, we can apply [5, Prop. 2], which then implies that $\pi((G, G))$ is closed in $\mathrm{GL}(\mathbb{C}, m)$. Hence, $\rho(G) \cap H$ is closed. The fact that $\rho(G)$ normalizes H is trivial. Thus, $\rho(G)$ is closed. The proof is complete. $\square$

Let $\| \cdot \|$ denote the Hilbert space operator norm on the algebra of complex $m \times m$ matrices and

$$\omega(a) := \max\{\|a\|, \|a^{-1}\|\}, a \in \mathrm{GL}(\mathbb{C}, m). \tag{1}$$

Lemma 4. *Let $m \in \mathbb{N}$ and $\pi \colon \mathbb{C} \to \mathrm{GL}(\mathbb{C}, m)$ be a faithful holomorphic homomorphism with closed range. Then there is $C > 0$ such that $|z| \leqslant C\,\omega(\pi(z))$ for every $z \in \mathbb{C}$.*

Proof. Since π is holomorphic, it is not hard to see that there is a generator, i.e., a matrix a such that $\pi(z) = \exp(za)$ for every $z \in \mathbb{C}$. We can assume that a is in a Jordan normal form. The following three cases may occur.

(1) Suppose that a is diagonal and all the eigenvalues are collinear over $\mathbb{R}$. This means that there is $\lambda \in \mathbb{C}$ such that the eigenvalues are $t_1\lambda, \ldots, t_m\lambda$ for some $t_1, \ldots, t_m \in \mathbb{R}$.

Let $\theta \colon \mathbb{R} \to \mathbb{C} \colon x \mapsto ix\bar{\lambda}$. Since the eigenvalues of $\exp(ix\bar{\lambda}a)$ have modulus 1 for every $x \in \mathbb{R}$, we can treat $\pi \circ \theta$ as a continuous homomorphism from $\mathbb{R}$ to $\mathbb{T}^m$. Since the range of π is closed and θ is topologically injective, the range of $\pi \circ \theta$ is also closed. Moreover, it is compact because $\mathbb{T}^m$ is compact. Hence, we have a continuous bijective homomorphism from a σ–compact group onto a compact group. It is easy to see from the Baire category theorem that every such homomorphism is a topological isomorphism; see, e.g., [12, Lm.]. Since $\mathbb{R}$ is not compact, we get a contradiction and thus this case is impossible.

(2) Suppose that a is diagonal and there are two eigenvalues λ_1 and λ_2 that are not collinear over $\mathbb{R}$. Then

$$\omega(\pi(z)) \geqslant \exp(\max\{|\operatorname{Re}(\lambda_1 z)|, |\operatorname{Re}(\lambda_2 z)|\}) \geqslant \max\{|\operatorname{Re}(\lambda_1 z)|, |\operatorname{Re}(\lambda_2 z)|\}$$

for every $z \in \mathbb{C}$. Since λ_1 and λ_2 that are not collinear, the formula on the right–hand side defines a norm on $\mathbb{C}$ as a vector space over $\mathbb{R}$. Since all such norms are equivalent, there is $C > 0$ such that $|z| \leqslant C\omega(\pi(z))$ for every $z \in \mathbb{C}$ and this completes the proof of the assertion of the lemma in this case.

(3) Suppose that a is not diagonal. Taking a suitable Jordan block, we can assume without loss of generality that the upper–left corner of a is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with some $\lambda \in \mathbb{C}$. Then the upper–left corner of $\exp(za)$ has the form $\exp(z\lambda)\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. Considering the upper–right entry, we have

$$\omega(\pi(z)) \geqslant \max\{|z\exp(\lambda z)|, |z\exp(-\lambda z)|\} = |z|\exp|\operatorname{Re}(\lambda z)| \geqslant |z|$$

for every $z \in \mathbb{C}$. Thus the assertion of the lemma holds also in this case. The proof is complete. $\square$

To prove Theorem 1 we need the following notation and terminology. Let G be a complex Lie group and $\upsilon \colon G \to (0, +\infty)$ a locally bounded function. Then we consider the Banach space

$$\mathcal{O}_\upsilon(G) := \left\{ f \text{ is holomorphic on } G \text{ and } |f|_\upsilon := \sup_{g \in G} \upsilon(g)^{-1}|f(g)| < \infty \right\}. \tag{2}$$

We also let $\mathcal{O}_{\upsilon^\infty}(G) := \bigcup_{n \in \mathbb{N}} \mathcal{O}_{\upsilon^n}(M)$ and endow it with the inductive topology.

We say that a function η_1 on G is *dominated* by a function η_2 if there are K, $\alpha > 0$ such that

$$\eta_1(g) \leqslant K \, \eta_2(g)^{\alpha} \quad \text{for all } g \in G.$$

If, in addition, η_2 is dominated by η_1, then η_1 and η_2 is said to be *equivalent*.

Recall also that if G is compactly generated, i.e., generated by a relatively compact neighbourhood U of the identity, then

$$\eta(g) := \min\{2^n : g \in U^n\}, \tag{3}$$

where $U^0 = \{1\}$) defines a submultiplicative weight, which is called a *word weight*. If, in addition, $U^{-1} = U$, then η is *symmetric*, i.e., $\eta(g^{-1}) = \eta(g)$ for every g.

Proof of Theorem 1. Fix a decomposition $G = B \rtimes L$ as above. Let η and η_1 be symmetric word weights on G and L, respectively. The restriction of η to B is also denoted by η. Since G is a semidirect product, it follows from [3, Prop. 4.2] that the functions $(b, l) \to \eta(bl)$ and $(b, l) \to \eta(b)\eta_1(l)$ are equivalent on $B \times L$. Then by [3, Prop. 5.5(B)], we have $\mathcal{O}_{\eta^{\infty}}(G) \cong \mathcal{O}_{\eta^{\infty}}(B) \,\widehat{\otimes}\, \mathcal{O}_{\eta_1^{\infty}}(L)$, where $\widehat{\otimes}$ denotes the complete projective tensor product. Moreover, since η and η_1 are symmetric word weights, it follows from [1, Thm. 5.3] that $\mathcal{O}_{exp}(G) = \mathcal{O}_{\eta^{\infty}}(G)$ and $\mathcal{O}_{exp}(L) = \mathcal{O}_{\eta_1^{\infty}}(L)$ as locally convex algebras. Therefore, $\mathcal{O}_{exp}(G) \cong \mathcal{O}_{\eta^{\infty}}(B) \,\widehat{\otimes}\, \mathcal{O}_{exp}(L)$.

Since L is connected and linearly complex reductive, $\mathcal{O}_{exp}(L) = \mathcal{R}(L)$ [3, Thm. 5.9]. Since $\mathcal{O}_{\eta^{\infty}}(B) \otimes \mathcal{O}_{exp}(L)$ is dense in $\mathcal{O}_{\eta^{\infty}}(B) \,\widehat{\otimes}\, \mathcal{O}_{exp}(L)$ and $\mathcal{R}(G) = \mathcal{R}(B) \otimes \mathcal{R}(L)$, to complete the proof it suffices to show that $\mathcal{R}(B)$ is contained and dense in $\mathcal{O}_{\eta^{\infty}}(B)$.

Proposition 2 implies that there are $m \in \mathbb{N}$ and a faithful holomorphic homomorphism $\pi : G \to \mathrm{GL}(\mathbb{C}, m)$ with closed range. Since B is simply connected and solvable, it can be represented as an iterated semidirect product of subgroups $F_1, \ldots, F_n$ each of which is isomorphic to $\mathbb{C}$. For every j fix an isomorphism $\mathbb{C} \to F_j$ and denote by ρ_j its composition with the embedding $F_j \to G$. Then $\pi \circ \rho_j : \mathbb{C} \to \mathrm{GL}(\mathbb{C}, m)$ is also a faithful holomorphic homomorphism with closed range .

Define ω as in (1). It is well known that every submultiplicative weight is dominated by every word weight; see, e.g., [1, Thm. 5.3]. Since $\omega \circ \pi$ is a submultiplicative weight on G, this means that there exist $K > 0$ and $\alpha > 0$ such that $\omega(\pi(g)) \leqslant K\eta(g)^{\alpha}$ for all $g \in G$. Applying Lemma 4, we conclude that there is $K' > 0$ such that

$$|z| \leqslant K'\eta(\rho_j(z))^{\alpha} \qquad \text{for all } j = 1, \ldots, n \text{ and } z \in \mathbb{C}. \tag{4}$$

Since G is biholomorphically equivalent to $F_1 \times \cdots F_n \times L$, every element g of G can be identified with $(z_1, \ldots, z_n, l)$, where $z_j \in \mathbb{C}$ and $l \in L$. It follows from (4) that the function $g \mapsto z_j$ is in $\mathcal{O}_{\eta^{\infty}}(B)$ for every j and so the whole $\mathcal{R}(B)$, which can be identified with $\mathbb{C}[z_1, \ldots, z_n]$, is contained in $\mathcal{O}_{\eta^{\infty}}(B)$.

Finally, note that every $f \in \mathcal{O}_{\eta^{\infty}}(B)$ is an entire function in the variables $z_1, \ldots, z_n$. Using the bound in Lemma 4, we conclude that f can be approximated in the topology of $\mathcal{O}_{\eta^{\infty}}(B)$ by partial sums of its Taylor series. Thus, $\mathcal{R}(B)$ is dense in $\mathcal{O}_{\eta^{\infty}}(B)$. The proof is complete. $\square$

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