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INDUCTIVE METHODS FOR HARDY INEQUALITY ON TREES

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Abstract. We study the Hardy inequality on at most countable rooted tree. The main known criteria in the lower–triangle case for this inequality are two Arcozzi–Rochberg–Sawyer criteria and the capacity criterion. In the survey we show that these two criteria are connected with the criteria for the Hardy inequalities for the sequences, for the Hardy inequality on an interval of the real axis and for the trace inequalities with the Riesz potentials. We provide the examples from the literature, when the trace inequality or another statement is characterized in terms of the validity of the Hardy inequality on the tree. We simplify two known proofs of the Arcozzi–Rochberg–Sawyer criteria, which are based on the Marcinkiewicz interpolation theorem and on the capacity criterion. We provide new proofs for Arcozzi–Rochberg–Sawyer criteria, which are based on the induction in the tree, the inductive formula for the capacity and the formula of integration by parts. The latter of the proofs is written for the Hardy inequality on the tree with a boundary and for the Hardy inequality over the family of all binary cubes. In the diagonal case this proof provides an optimal constant p, which coincides with the Bennett constant in the Hardy inequality for the sequences. In the general case we provide a few new inductive criteria for the validity of the Hardy inequality in terms of the existence of a family of functions satisfying an inductive relation. One of these criteria is applied in the proof of a theorem containing additional equivalent conditions for the validity of the Hardy inequality on the trees in the diagonal case.

Keywords: two–weight inequality, rooted tree, Hardy inequality.

Mathematics Subject Classification: 05C05, 31C20, 47A30

1. Introduction

Let T be at most countable tree. This means that an antireflexive ($x \not\sim x$) and symmetric ($x \sim y \Leftrightarrow y \sim x$) relation $\sim$ is given on at most countable set $T \neq \varnothing$ and this relation is so that for each $x, y \in T$ there exists a unique set $(x_i)_{i=0}^n$ ($n \geqslant 0$) of mutually disjoint points in T with the properties

$$x_0 = x \quad \& \quad x_n = y \quad \& \quad x_i \sim x_{i+1} \quad (0 \leqslant i < n).$$

The set $(x_i)_{i=0}^n$ is denoted by $[x, y]$.

Choosing a point o in T, we obtain a rooted tree (T, o). The relation

$$x \leqslant y \quad \Leftrightarrow \quad x \in [o, y]$$

defines a partial order $\leqslant$ on T with the least element o. On functions $f : T \to \mathbb{R}$ we define a Hardy operator $\mathcal{I}$ by the formula

$$\mathcal{I}f(x) = \sum_{[o,x]} f = \sum_{w \in [o,x]} f(w).$$

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The present paper is devoted to *Hardy inequality on a tree*:

$$(\exists A \geqslant 0)\ (\forall f: T \to [0,\infty)) \quad \Big(\sum_T u[\mathcal{I}f]^q\Big)^{1/q} \leqslant A\Big(\sum_T v f^p\Big)^{1/p}. \tag{1.1}$$

It is determined by the numbers $1 < p < \infty$ and $1 < q < \infty$ and by the functions $u : T \to [0,\infty)$ and $v : T \to (0,\infty)$. The choice $f = \chi_{\{o\}}$ indicates that (1.1) can hold only as

$$\sum_T u < \infty,$$

and this is assumed in what follows. Apart of the notations T, $x \sim y$, $[x, y]$, o, $x \leqslant y$, $\mathcal{I}f$, p, q, u and v, without additional comments in the paper we use the following notations:

$$\begin{aligned}
&p' = p/(p-1) \quad \text{(is an adjoint exponent)},\\
&P_x = [o,x] = \big\{w \in T\colon w \leqslant x\big\} \quad \text{(are predecessors of } x \text{ or } x),\\
&R_x = \big\{y \in T\colon x \sim y \ \&\ x \leqslant y\big\} \quad \text{(are descendants of } x),\\
&S_x = \big\{y \in T\colon x \leqslant y\big\} \quad \text{(are descendants of } x \text{ or } x),\\
&U(x) = \sum_{S_x} u, \qquad V(x) = \sum_{P_x} v^{1-p'}, \qquad B(x) = \sum_{S_x} U^{p'} v^{1-p'},\\
&C = \sup_T B^{1/p'} U^{-1/q'}, \qquad D = \sup_T U^{1/q} V^{1/p'},\\
&\mathcal{E} = \big\{E \subset T\colon (\forall x \in E)\ S_x \subset E\big\}.
\end{aligned}$$

For a set $E \subset T$ by $E^{\min}$ we denote the set of all minimal elements in E, while by χ_E we denote the characteristic function of E. The indeterminate form $0 \cdot \infty$ (including the indeterminate forms $0/0$ and ∞/∞) is treated as $0 \cdot \infty = \infty \cdot 0 = 0$ as it is conventional in the theory of Hardy inequalities. In particular, $C = 0$ if $u \equiv 0$. Sometimes inequality (1.1) will be considered as $p, q \in (0,\infty)$ when the definitions P_x, R_x, S_x, $U(x)$, $\mathcal{E}$, $E^{\min}$ and χ_E make sense.

For a tree $T = \mathbb{N} = \{1, 2, \dots\}$ with the adjacency relation

$$n \sim k \quad \Leftrightarrow \quad n - k = \pm 1$$

and the root $o = 1$ inequality (1.1) becomes

$$\Big(\sum_{n=1}^{\infty} u_n\Big(\sum_{k=1}^{n} a_k\Big)^q\Big)^{1/q} \leqslant A\Big(\sum_{n=1}^{\infty} v_n a_n^p\Big)^{1/p}, \tag{1.2}$$

where $u_n = u(n) \geqslant 0$, $v_n = v(n) > 0$, and $A \geqslant 0$ is independent of the numbers $a_n = f(n) \geqslant 0$. Inequality (1.2) possesses a continuous analogue

$$\Big(\int_0^b\Big(\int_0^x f\,dt\Big)^q w_1(x)\,dx\Big)^{1/q} \leqslant A\Big(\int_0^b f^p(x) w_2(x)\,dx\Big)^{1/p}, \tag{1.3}$$

where $0 < b \leqslant \infty$, functions w_1, w_2 and f are measurable on $(0, b)$, $w_1 \geqslant 0$, $w_2 > 0$, while the number $A \geqslant 0$ is independent of $f \geqslant 0$. Inequalities (1.2) and (1.3) often appear in the analysis and their theory is well–developed.

Estimate (1.1) is most studied in the lower–triangle case $1 < p \leqslant q < \infty$. The main results are capacity criterion (Lemma 4.3) and the following Arcozzi–Rochberg–Sawyer criteria:

$$\text{if} \quad 1 < p \leqslant q < \infty, \quad \text{then} \quad (1.1) \Leftrightarrow C < \infty; \tag{1.4a}$$

$$\text{if} \quad 1 < p < q < \infty, \quad \text{then} \quad (1.1) \Leftrightarrow D < \infty. \tag{1.4b}$$

The criterion for $p = q$ and a binary tree T from work [1] by the author can be regarded as a multiplicative form of criterion (1.4a). Criterion (1.4a) was first proved in [2, Thm. 3] by estimating the distribution function

$$t \mapsto \sum_{\{\mathcal{I}f>t\}} u$$

of the function $\mathcal{I}f$ in terms of so–called good–λ inequalities. Later the same authors for the case $p = q$ and a tree with a boundary $T \cup \partial T$ provided simpler arguing by using Marcinkiewicz interpolation theorem (see [3, § 3], [4, 5.4.1]) and the capacity criterion [4, Thm. 43]. In work [5] criterion (1.4a) for $p = q = 2$, $v \equiv 1$ and one–dimensional binary tree T was reproved by means of Bellman functions method. Criterion (1.4b) was first proved in [2, Thm. 4] by reducing to criterion (1.4a) by means of a good–λ inequality. Criterion (1.4b) was reproved in preprint [6] by using work [7].

The author does not know a convincing comparison of the described results with the previous ones. In works [6]–[8] and book [9] the capacity criterion is given without the capacity terminology and without mentioning capacity criteria for the trace inequalities. Inequality (1.1) for $p = q$ and binary tree T was employed in a series of works by the author, see [1], [10], [11] and the references in [11], under the name "discrete weight inequality", but it has never been considered in the general context of trees. The comments on the capacity criterion and criteria (1.4) in works [2], [4], [12] are more appropriate, but they are mostly moved to the theory of spaces of analytic functions, and in our opinion, they should be completed. The latter is the first aim of this paper.

The second aim of the paper is to try the induction in the tree for Hardy inequality (1.1). In this method one first formulates a statement depending on $x \in T$ on a rooted tree (S_x, x), which for $x = o$ coincides with the required result. Then in this statement one makes an induction in decreasing the length of the chain $[o, x]$. The base of the induction is that the statement is true for sufficiently long chains $[o, x]$, which is ensured by some approximation of the original problem. The induction transition shows that the discussed statement is true for x if it is true for all elements of the set R_x. The inductive transition is usually based on an easily verified identity

$$S_x = \{x\} \cup \bigcup_{y \in R_x} S_y \quad \text{(the unions are disjunctive).}$$

For instance, the arguing with the Bellman functions in [13, § 1] can be written in the inductive form. A.A. Vasil'eva in [7] used the induction in tree for (1.1) in combination with the inductive formula for the capacity from [8], see formula (4.4) below.

The third aim of the paper is to provide new proofs of criteria (1.4). We hope that this improves the coherence of the theory of the Hardy inequality and it can be useful in close situations. As an example of the latter, we mention the Hardy inequality on Cartesian products of one–dimensional binary trees [12], [14], which are not trees.

A strictly upper–triangle case $1 < q < p < \infty$ in the paper is not studied due to its features and it requires an independent study. In this case the criteria of validity of inequalities (1.1)–(1.3) and the trace inequalities are usually of form of convergence for some series or integral (and not the finiteness of some supremum as in (1.4)) and they often admit generalizations for the case $p > 1$, $0 < q < p$, see [15, Thm. 1] and publications [16]–[20].

The structure of the work is as follows. In Section 2 we describe the criteria for inequalities (1.2), (1.3) and for trace inequalities adjacent to the capacity criterion and criteria (1.4). In Section 3 we prove three inductive criteria of validity of inequality (1.1) and after that a theorem on the diagonal case $p = q$ is established. The theorem states the equivalence of inequality (1.1), a condition from work [1] (in a simplified form), the condition $C < \infty$ and a new condition of inductive type. In Section 4 we simplify two known proofs of criterion (1.4a) and give new proofs for criteria (1.4); sometimes we also provide known simple calculations. In concluding

Section 5 we prove versions of criteria (1.4) for a tree with a boundary $T \cup \partial T$ and for a binary family $\mathcal{D}$; we also establish a couple of statements from Section 2.

2. On criteria adjacent to capacity criterion and Arcozzi–Rochberg–Sawyer criteria

We first briefly describe the history of Hardy inequalities (1.2) and (1.3) in the lower–triangle case $p \leqslant q$. A detailed exposition was given in book [19].

A simplest Hardy inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \leqslant A^2 \sum_{n=1}^{\infty} a_n^2$$

follows from the inverse Hölder inequality and Hilbert inequality (1906), which states the boundedness of a form with a matrix $\left(\frac{1}{n+k}\right)_{n,k=1}^{\infty}$ in the space of sequences ℓ_2. A long story of an independent derivation of this inequality with an optimal constant was provided in the Appendix to [19]. In classical book [21] there were obtained particular cases of inequality (1.2) (Theorems 326 and 339) and of inequality (1.3) (Theorems 327, 330 and 340).

The problem of characterization of pair of functions (w_1, w_2) with property (1.3) in the lower–triangle case $1 < p \leqslant q < \infty$ is resolved by the criterion

$$(1.3) \quad \Leftrightarrow \quad \sup_{r\in(0,b)}\left(\int_r^b w_1(x)\,dx\right)^{1/q}\left(\int_0^r w_2^{1-p'}(x)\,dx\right)^{1/p'} < \infty. \tag{2.1}$$

For $p = q = 2$ and $w_2 \equiv 1$ this criterion was proved in [22], while for $p = q$ this was done in [23], [24] and also in a series of published and unpublished works at the same time, see [19, Ch. 4]. In full generality criterion (2.1) was proved by Walsh [25] and the formulation of the problem covers both inequalities (1.2) and (1.3). Walsh represented the kernel $\chi_{\{x>t\}}$ of the Hardy operator as the produce of two kernels and applied a version of the Schur test. Analogue of (2.1) for inequality (1.2) reads as

$$(1.2) \quad \Leftrightarrow \quad D = \sup_{n\in\mathbb{N}}\left(\sum_{k=n}^{\infty} u_k\right)^{1/q}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{1/p'} < \infty. \tag{2.2}$$

Criteria (2.1) and (2.2) were reproved by a series of authors.

Later other equivalent conditions ensuring the validity of inequalities (1.2) and (1.3) were found. For instance, in [26] the following criterion was given:

$$(1.2) \quad \Leftrightarrow \quad C < \infty \quad \Leftrightarrow \quad D < \infty \quad \Leftrightarrow \quad D_1 < \infty, \tag{2.3}$$

$$D_1 = \sup_{n\in\mathbb{N}}\left(\sum_{k=1}^{n} u_k V^q(k)\right)^{1/q} V^{-1/p}(n) \quad \text{for} \quad V(n) = \sum_{k=1}^{n} v_k^{1-p'}.$$

For inequality (1.3) with $p = q$ the analogues of conditions $C < \infty$ and $D_1 < \infty$ were used in works [27] and [24], respectively.

Concerning the Hardy inequality on tree (1.1), usually it implicitly or directly appears in discretization of the embedding of the Sobolev space (or a similar space of analytic functions) into the weighted space L_q. As the theory of such embeddings developed, on the one hand, there appeared criteria similar to the criteria for the validity of inequality (1.1), and on the other hand, there appeared theorems appeared involving (1.1) explicitly. Let us dwell on these two aspects in the lower–triangle case.

1) We consider model embeddings of the aforementioned type (called trace inequalities):

$$(\forall u \in C_0^\infty(\mathbb{R}^n)) \quad \Big(\int\limits_{\mathbb{R}^n} |u|^q \, d\mu\Big)^{1/q} \leqslant A_0 \Big(\int\limits_{\mathbb{R}^n} |\nabla_l u|^p \, dx\Big)^{1/p}, \tag{2.4}$$

$$(\forall f \in L_p(\mathbb{R}^n)) \quad \Big(\int\limits_{\mathbb{R}^n} |I_l f|^q \, d\mu\Big)^{1/q} \leqslant A_1 \Big(\int\limits_{\mathbb{R}^n} |f|^p \, dx\Big)^{1/p}. \tag{2.5}$$

Here $n \geqslant 2$, μ is a measure in $\mathbb{R}^n$ (a non–negative countably additive function on the Borel σ–algebra, which is finite on compact sets), A_i are constants, $l \in [1, n/p)$ is integer, $\nabla_l u$ is the set of all partial derivatives of the function u of order l, dx is the Lebesgue measure,

$$I_l f(x) = \int\limits_{\mathbb{R}^n} |x-y|^{l-n} f(y)\, dy \quad \text{(is the Riesz potential).}$$

In (2.5) we mean that for each $0 \leqslant f \in L_p(\mathbb{R}^n)$ the integral $I_l f$ converges almost everywhere with respect to the measure μ and this allows us to define I_l on $L_p(\mathbb{R}^n)$.

Sobolev integral representation [20] yields the implication (2.5) $\Rightarrow$ (2.4). The opposite implication can be proved by using the density of $C_0^\infty(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$, a smooth cut–off function and Mikhlin theorem on Fourier multipliers [20]. This is why we shall consider trace inequality (2.5) and for the generality we assume that $n \geqslant 1$, while the number $l \in (0, n/p)$ is real.

In 60s 70s of XXth century Mazya, Adams and Dahlberg established that condition (2.5) is equivalent to each of the following conditions:

$$(\forall K) \quad \Big(\int\limits_{\mathbb{R}^n} (I_l \mu_K)^{p'} \, dx\Big)^{1/p'} \leqslant A_2 \mu(K)^{1/q'}, \tag{2.6}$$

$$(\forall f \in L_p(\mathbb{R}^n)) \quad \sup_{t>0} t\mu(\{x\colon |I_l f(x)| \geqslant t\})^{1/q} \leqslant A_3 \Big(\int\limits_{\mathbb{R}^n} |f|^p \, dx\Big)^{1/p}, \tag{2.7}$$

$$(\forall K) \quad \mu(K)^{1/q} \leqslant A_4 \left(\inf \left\{ \int\limits_{\mathbb{R}^n} f^p \, dx \colon f \geqslant 0 \ \&\ (I_l f)\big|_K \geqslant 1 \right\} \right)^{1/p}, \tag{2.8}$$

see Theorem 7.2.1 in [28]. Here K is a compact set in $\mathbb{R}^n$, $\mu_K(X) = \mu(K \cap X)$ and

$$I_l \mu_K(x) = \int\limits_{\mathbb{R}^n} |x-y|^{l-n} \, d\mu_K(y).$$

The infimum in (2.8) is called the capacity of the set K, and the criterion (2.5) $\Leftrightarrow$ (2.8) is similar to the capacity criterion from Lemma 4.3. As $p < q$ in (2.8), by Adams theorem ([20] or [28]) we can restrict ourselves by closed balls K and this is similar to criterion (1.4b).

The results of the previous paragraph were applied in work [29] for a capacity characterization of the embedding of the Dirichlet space on the circle $|z| < 1$ into the space $L_2(\mu)$ and for describing the multipliers of the Dirichlet space.

In 80s and 90s of XXth century there appeared new criteria for the validity of conditions (2.5)–(2.8) in the complicated case $p = q$. In work [30] Kerman and Sawyer showed that in condition (2.6) one can restrict himself by balls K. In [16, Sect. 2] there was reproduced the proof from [28] for the equivalence of conditions (2.5)–(2.8) (as $p = q$) and there was given a simpler than in [31] derivation of the equivalence of conditions (2.5)–(2.8) and each of the following conditions:

$$\int\limits_K (I_l\mu_K)^{p'}\,dx \leqslant A_5^{p'}\mu(K) \quad \text{for each ball} \quad K, \tag{2.9}$$

$$I_l[(I_l\mu)^{p'}] \leqslant A_6^{p'} I_l\mu < \infty \quad \text{almost everywhere in} \quad (\mathbb{R}^n, dx), \tag{2.10}$$

$$\text{condition (2.5) is true for the measure } (I_l\mu)^{p'}dx \text{ instead of } d\mu, \tag{2.11}$$

see also [20, Sect. 11.5]. In [16, Sect. 3] by means of the Wolff inequality it was shown that conditions (2.5)–(2.11) are equivalent to the estimate

$$\sum_{Q\in\mathcal{D}\colon\, Q\subset P} \mu(Q)^{p'}\ell_Q^{p'(l-n)+n} \leqslant A_7^{p'}\mu(P) \quad \text{for each cube} \quad P\in\mathcal{D}. \tag{2.12}$$

Here $\mathcal{D}$ is the family of all binary cubes in $\mathbb{R}^n$:

$$\mathcal{D} = \big\{Q\subset\mathbb{R}^n\colon Q = [0,2^a)^n + 2^a\vec{a} \quad \text{for some} \quad a\in\mathbb{Z} \quad \text{and} \quad \vec{a}\in\mathbb{Z}^n\big\},$$

and $\ell_Q = 2^a$ is the side length of the cube Q.

It will be shown in Section 5 that as $\operatorname{supp}\mu\subset K\in\mathcal{D}$, condition (2.12) is an analogue of condition $C\big|_{p=q}<\infty$ for one Hardy inequality on a tree with a boundary $T\cup\partial T$ for $T=\mathcal{D}(K)$, where

$$\mathcal{D}(K) = \big\{Q\in\mathcal{D}\colon Q\subset o = K\big\} \tag{2.13}$$

and $Q_1 \leqslant Q_2 \Leftrightarrow Q_1 \supset Q_2$, and this is why (2.5) and (2.12) are equivalent to this Hardy inequality. Similarly, for general μ conditions (2.5) and (2.12) are equivalent to some Hardy inequality on $\mathcal{D}$. Thus, as close predecessors of Arcozzi–Rochberg–Sawyer criteria (1.4) we can regard criterion (2.3), Adams theorem and the equivalence between (2.5) and a group of similar conditions: (2.6), Kerman and Sawyer condition, (2.9), (2.10) and (2.12).

2) The author knows the following situations, in which the Hardy inequality on a tree appeared explicitly in studies and not independently but as being involved in some other criteria.

2a) In [4, Rem. 35] for a Riesz potential on a bounded Ahlfors regular metric space X there was mentioned an analogue of a statement in a few lines above on the equivalence of the trace inequality and the Hardy inequality on $T\cup\partial T$. As T one takes a binary decomposition of the space X in [32], which is similar to the family $\mathcal{D}(K)$ for the case $X=K\in\mathcal{D}$.

2b) In [2, Prop. 5], [3, Thms. 20, 23] and [17, Thm. 2.5] there was established an equivalence between the Hardy inequality and embeddings of spaces of analytic functions into the spaces $L_q(\mu)$ (such measures μ are called Carleson ones). In preprint [12] there is a similar statement on the equivalence between the Carleson property of a measure for the Dirichlet space in a bi–circle and the Hardy inequality on the product $\mathcal{D}([0,1))\times\mathcal{D}([0,1))$.

2c) In Theorem 5.4.6 in [9] for the domains $\Omega\subset\mathbb{R}^n$ of class GRD, which are characterized by the presence of skeleton–tree for the domain, there was obtained the equivalence between the embedding $W^1(X(\Omega),Y(\Omega))\subset Z(\Omega)$ and a corresponding embedding of type of the Hardy inequality on the tree. Here $X(\Omega)$, $Y(\Omega)$ and $Z(\Omega)$ are Banach functional spaces with certain properties, while $W^1(X(\Omega),Y(\Omega))$ is the generalization of the Sobolev space $W_p^1(\Omega)$. In the used trees the neighbouring vertices ($x\sim y$) are supposed to be joined by a segment and this allowed one to consider differential equations on such trees [33]. Such trees are usually called trees, metric trees or geometric trees.

2d) In a series of works by the author, see the references in [11], there was obtained an equivalence between the Hardy inequality and a few properties related with straightening of Lipschitz domains. As examples we mention [1, Thm. 19(iii)], [10, Thm. 22] and [11, Thm. 5]. In [1] the Hardy inequality was taken over the tree $T=\mathcal{D}([0,1)^n)$, while the Hardy inequalities in [10], [11] are equivalent to the Hardy inequalities over the trees $T=\mathcal{D}([0,1)^n+\vec{a})$, $\vec{a}\in\mathbb{Z}^n$.

3. Inductive criteria and diagonal case

Here we provide three theorems, in which we show the equivalence of inequality (1.1) to the existence of the family of functions $Q_x(r,s)$, $x \in T$, obeying the inductive condition. As in the method of Bellman functions [13] (which are independent of x), the equivalence is almost tautological and a difficulty in applying the theorems is to construct such families. Each of the theorems is preceded by a corresponding lemma.

Lemma 3.1. *Let $E \subset T$. Then the sets S_x ($x \in E^{\min}$) are mutually disjoint and the identity holds:*

$$E = \bigcup_{x \in E^{\min}} (S_x \cap E) \quad (\textit{the union is disjunctive}). \tag{3.1}$$

For each $x \in T$ we have

$$S_x = \{x\} \cup \bigcup_{y \in R_x} S_y \quad (\textit{the unions are disjunctive}). \tag{3.2}$$

Proof. We argue by contradiction. Suppose that $y \in S_{x_1} \cap S_{x_2}$, where $x_1, x_2 \in E^{\min}$ and $x_1 \neq x_2$. Then $x_1, x_2 \in [o, y]$, and hence either $x_1 \leqslant x_2$ or $x_2 \leqslant x_1$. In view of $x_1 \neq x_2$ the former contradicts the minimality of the element x_2, while the latter does that of the element x_1.

To confirm (3.1), it remains to observe that as $y \in E$ the least element x of the set $[o, y] \cap E$ belongs to $E^{\min}$ and $y \in S_x \cap E$.

Let $x, y \in T$ and $[o, y] = (y_i)_{i=0}^n$. It is obvious that the condition $y \in (S_x \setminus \{x\})^{\min}$ is equivalent to the condition $n \neq 0$ & $x = y_{n-1}$, which is equivalent to the condition $y \in R_x$. This is why relation (3.2) is implied by (3.1). The proof is complete. □

Theorem 3.1. *Let $p, q \in (0, \infty)$. Then Hardy inequality* (1.1) *is equivalent to the existence of a family of functions*

$$Q_x : [0,\infty) \times [0,\infty) \to [0,\infty) \quad (x \in T),$$

such that for all $x \in T$, $r \geqslant 0$, $\rho \geqslant 0$ and $s_y \geqslant 0$ ($\sigma = \sum_{y \in R_x} s_y < \infty$) we have

$$u(x)(r+\rho)^q + \sum_{y \in R_x} Q_y(r+\rho, s_y) \leqslant Q_x(r, v(x)\rho^p + \sigma). \tag{3.3}$$

Proof. Let (1.1) hold. We let

$$Q_x(r,s) = \sup_{\substack{f: S_x \to [0,\infty) \\ \sum_{S_x} v f^p \leqslant s}} \sum_{S_x} u\big[r + \mathcal{I}[\chi_{S_x} f]\big]^q.$$

Here by $\chi_{S_x} f$ we denote (a bit incorrectly) the continuation of the function f by zero on the set $T \setminus S_x$. By (1.1) we have $Q_x(r,s) < \infty$.

We choose arbitrary $x \in T$, $r \geqslant 0$, $\rho \geqslant 0$, $s_y \geqslant 0$ with $\sigma = \sum\limits_{y \in R_x} s_y < \infty$ and functions $\varphi_y : S_y \to [0,\infty)$ ($y \in R_x$) with $\sum\limits_{S_y} v\varphi_y^p \leqslant s_y$. In accordance with (3.2) we define the function $f : S_x \to [0, \infty)$ by the formula

$$f(z) = \begin{cases} \rho & \text{as} \quad z = x, \\ \varphi_y(z) & \text{as} \quad z \in S_y \ (y \in R_x). \end{cases}$$

Then

$$\sum_{S_x} v f^p = v(x)\rho^p + \sum_{y \in R_x} \sum_{S_y} v\varphi_y^p \leqslant v(x)\rho^p + \sigma,$$

$$\begin{aligned} Q_x(r, v(x)\rho^p + \sigma) &\geqslant \sup_{\{\varphi_y\}} \sum_{S_x} u\big[r + \mathcal{I}[\chi_{S_x} f]\big]^q \\ &= u(x)(r+\rho)^q + \sup_{\{\varphi_y\}} \sum_{y \in R_x} \sum_{S_y} u\big[r + \rho + \mathcal{I}[\chi_{S_y}\varphi_y]\big]^q \\ &= u(x)(r+\rho)^q + \sum_{y\in R_x} Q_y(r+\rho, s_y). \end{aligned}$$

This completes the proof of (3.3).

And vice versa, let there exist functions Q_x with property (3.3). Then (3.3) is also true for the function $\chi_F u$ instead of u, where the set $F \subset T$ is finite. If we confirm (1.1) for a function $\chi_F u$ (with a constant independent of F), then the passage to the limit with using a monotonic exhausting $F_1 \subset F_2 \subset \cdots$ of the tree T will show that (1.1) holds also for the function u. Hence, without loss of generality we suppose that the set $\{u \neq 0\}$ is finite.

We take $f : T \to [0, \infty)$ such that $\sum_T v f^p < \infty$. If the length of the chain $[o, x]$ is sufficiently large, then the inequality

$$\sum_{S_x} u[\mathcal{I}f]^q \leqslant Q_x\Big(\mathcal{I}f(x) - f(x), \sum_{S_x} v f^p\Big) \tag{3.4}$$

is trivially satisfied since its left hand side vanishes due to the finiteness of the set $\{u \neq 0\}$. Arguing by induction in decreasing the length of the chain $[o, x]$, we can suppose that (3.4) holds for all $y \in R_x$ instead of x. This assumption covers the case $R_x = \varnothing$. Let

$$r = \mathcal{I}f(x) - f(x) \quad \& \quad \rho = f(x) \quad \& \quad s_y = \sum_{S_y} v f^p.$$

Then

$$\sigma = \sum_{y\in R_x} s_y \leqslant \sum_T v f^p < \infty.$$

In view of (3.2)–(3.4) we obtain

$$\begin{aligned} \sum_{S_x} u[\mathcal{I}f]^q &= u(x)(r+\rho)^q + \sum_{y\in R_x}\sum_{S_y} u[\mathcal{I}f]^q \\ &\leqslant u(x)(r+\rho)^q + \sum_{y\in R_x} Q_y(\mathcal{I}f(y) - f(y), s_y) \\ &\leqslant Q_x(r, v(x)\rho^p + \sigma), \end{aligned}$$

since $\mathcal{I}f(y) - f(y) = \mathcal{I}f(x) = r + \rho$. Thus, we have proved (3.4) for a given x and hence by induction for all $x \in T$.

By (3.4) for $x = o$ we have

$$\sum_T u[\mathcal{I}f]^q \leqslant Q_o\Big(0, \sum_T v f^p\Big).$$

The invariance of estimate (1.1) with respect to multiplication of f by a positive constant shows that (1.1) holds with a constant $A = Q_o(0,1)^{1/q}$. The proof is complete. □

Lemma 3.2. *Let $p, q \in (1, \infty)$. Then Hardy inequality (1.1) is equivalent to the existence of $A \geqslant 0$ such that*

$$(\forall g : T \to [0,\infty)) \quad \Big(\sum_T v^{1-p'}[\mathcal{J}g]^{p'}\Big)^{1/p'} \leqslant A\Big(\sum_T u g^{q'}\Big)^{1/q'}, \tag{3.5}$$

where $\mathcal{J}g(x) = \sum_{S_x} ug$. The best possible constants A in (1.1) and (3.5) are equal.

This result is well–known, see [2] or [4, $p = q$]. The proof is easily made on the base of the identity

$$\sum_T u[\mathcal{I}f]g = \sum_T f\mathcal{J}g,$$

where f and g vanish outside some finite set. This identity allows us to interpret $\mathcal{J}$ as the adjoint operator for $\mathcal{I}$ with respect to the pairings mentioned in the identity. Also it should be taken into consideration that conditions (1.1) and (3.5) are equivalent to the same conditions for $\mathbb{R}$–valued functions f and g.

Theorem 3.2. *Let $p, q \in (1, \infty)$. Then Hardy inequality* (1.1) *is equivalent to the existence of non–negative functions $Q_x(r, s)$, $x \in T$, defined for $r, s \geqslant 0$ with the restriction $r \leqslant U^{1/q}(x)s^{1/q'}$, non–increasing in r and such that for all $x \in T$, $\rho \geqslant 0$, $r_y \geqslant 0$ and $s_y \geqslant 0$, $y \in R_x$, with the properties*

$$r_y \leqslant U^{1/q}(y)s_y^{1/q'} \quad \& \quad \sigma_r = \sum_{y\in R_x} r_y < \infty \quad \& \quad \sigma = \sum_{y\in R_x} s_y < \infty \tag{3.6}$$

the condition holds

$$v^{1-p'}(x)(u(x)\rho + \sigma_r)^{p'} + \sum_{y\in R_x} Q_y(r_y, s_y) \leqslant Q_x(u(x)\rho + \sigma_r, u(x)\rho^{q'} + \sigma). \tag{3.7}$$

By the Hölder inequality and (3.2) we get

$$\begin{aligned} u(x)\rho + \sigma_r &\leqslant u(x)\rho + \sum_{y\in R_x} U^{1/q}(y)s_y^{1/q'} \\ &\leqslant \Big(u(x) + \sum_{R_x} U\Big)^{1/q} (u(x)\rho^{q'} + \sigma)^{1/q'} = U^{1/q}(x)(u(x)\rho^{q'} + \sigma)^{1/q'}, \end{aligned}$$

and this is why the condition $\sigma_r < \infty$ in (3.6) follows from other two conditions, while the value of the function Q_x in (3.7) is well–defined.

Proof. Let (1.1) hold. Then by Lemma 3.2 we have (3.5) and the functions

$$Q_x(r, s) = \sup_{\substack{g: S_x \to [0,\infty) \\ \mathcal{J}g(x) \geqslant r \ \& \ \sum_{S_x} ug^{q'} \leqslant s}} \sum_{S_x} v^{1-p'}[\mathcal{J}g]^{p'}$$

are finite. In view of $r \leqslant U^{1/q}(x)s^{1/q'}$ the set of admissible functions g contains a function $g \equiv r/U(x)$ and this is why it is non–empty. This set does not enlarge as r grows and this is why the functions $r \mapsto Q_x(r, s)$ do not increase.

Let x, ρ, r_y and s_y be the same as in (3.6). For the functions $\psi_y : S_y \to [0, \infty)$ with the properties $\mathcal{J}\psi_y(y) \geqslant r_y$ and $\sum_{S_y} u\psi_y^{q'} \leqslant s_y$ we let

$$g(z) = \begin{cases} \rho & \text{as} \quad z = x, \\ \psi_y(z) & \text{as} \quad z \in S_y \ (y \in R_x). \end{cases}$$

Then, by (3.2),

$$\begin{aligned} &\mathcal{J}g(x) = u(x)\rho + \sum_{y\in R_x} \mathcal{J}\psi_y(y) \geqslant u(x)\rho + \sigma_r, \\ &\sum_{S_x} ug^{q'} = u(x)\rho^{q'} + \sum_{y\in R_x}\sum_{S_y} u\psi_y^{q'} \leqslant u(x)\rho^{q'} + \sigma, \\ &Q_x(u(x)\rho + \sigma_r, u(x)\rho^{q'} + \sigma) \geqslant \sup_{\{\psi_y\}} \sum_{S_x} v^{1-p'}[\mathcal{J}g]^{p'} \end{aligned}$$

$$\geqslant v^{1-p'}(x)(u(x)\rho+\sigma_r)^{p'}+\sup_{\{\psi_y\}}\sum_{y\in R_x}\sum_{S_y}v^{1-p'}[\mathcal{J}\psi_y]^{p'}$$
$$=v^{1-p'}(x)(u(x)\rho+\sigma_r)^{p'}+\sum_{y\in R_x}Q_y(r_y,s_y).$$

This completes the proof of (3.7).

And vice versa, let there exist functions Q_x with required properties. Take $g : T \to [0,\infty)$ such that the set $\{g \neq 0\}$ is finite. Then for sufficiently long chains $[o,x]$ we have

$$\sum_{S_x}v^{1-p'}[\mathcal{J}g]^{p'}\leqslant Q_x\biggl(\mathcal{J}g(x),\sum_{S_x}ug^{q'}\biggr), \tag{3.8}$$

since the left hand side vanishes, while the right hand side is well–defined by the Hölder inequality:

$$\mathcal{J}g(x)\leqslant U^{1/q}(x)\bigl(\sum_{S_x}ug^{q'}\bigr)^{1/q'}.$$

Arguing by induction, we suppose that (3.8) holds on the elements of the set R_x instead of x. We let

$$\rho=g(x)\quad \&\quad r_y=\mathcal{J}g(y)\quad \&\quad s_y=\sum_{S_y}ug^{q'}.$$

By the above stated facts we have (3.6) and thus

$$\mathcal{J}g(x)=u(x)g(x)+\sum_{y\in R_x}\mathcal{J}g(y)=u(x)\rho+\sigma_r,$$
$$\begin{aligned}\sum_{S_x}v^{1-p'}[\mathcal{J}g]^{p'}&=v^{1-p'}(x)(u(x)\rho+\sigma_r)^{p'}+\sum_{y\in R_x}\sum_{S_y}v^{1-p'}[\mathcal{J}g]^{p'}\\&\leqslant Q_x(u(x)\rho+\sigma_r,u(x)\rho^{q'}+\sigma)\quad\text{(due to (3.8) and (3.7))}\\&=Q_x\biggl(\mathcal{J}g(x),\sum_{S_x}ug^{q'}\biggr).\end{aligned}$$

By induction (3.8) holds for all $x\in T$.

Taking $x=o$ in (3.8), we find:

$$\sum_{T}v^{1-p'}[\mathcal{J}g]^{p'}\leqslant Q_o\biggl(\mathcal{J}g(o),\sum_{T}ug^{q'}\biggr)\leqslant Q_o\biggl(0,\sum_{T}ug^{q'}\biggr),$$

and this gives (3.5) with a constant $A=Q_o(0,1)^{1/p'}$ for a given function g. The approximation provides (3.5) completely and this proves (1.1) by Lemma 3.2. □

Lemma 3.3. *For $q\in[1,\infty)$ the estimates*

$$\biggl(\sum_{i=0}^{n}a_i\biggr)^q\geqslant\sum_{j=0}^{n}\biggl(\sum_{i=0}^{j}a_i\biggr)^{q-1}a_j\qquad(n\geqslant 0\ \&\ a_0,\dots,a_n\geqslant 0), \tag{3.9}$$

$$\biggl(\sum_{i=0}^{n}a_i\biggr)^q\leqslant q\sum_{j=0}^{n}\biggl(\sum_{i=0}^{j}a_i\biggr)^{q-1}a_j, \tag{3.10}$$

$$(a+b)^q\leqslant a^q+q2^{q-1}(a^{q-1}b+b^q)\quad(a,b\geqslant 0), \tag{3.11}$$

hold true.

Inequality (3.10) is usually called a formula of integration by parts, while inequality (3.11) is called a binomial estimate in [28].

Proof. For $a,b \geqslant 0$

$$a^q + (a+b)^{q-1}b \leqslant (a+b)^{q-1}a + (a+b)^{q-1}b = (a+b)^q,$$

and by induction this gives (3.9). By finite increments formula

$$(a+b)^q = a^q + q\xi^{q-1}b \leqslant a^q + q(a+b)^{q-1}b,$$

where $a \leqslant \xi \leqslant a+b$. By induction this yields formula (3.10), while (3.11) follows from the inequality

$$(a+b)^{q-1} \leqslant \max\{1, 2^{q-2}\}(a^{q-1}+b^{q-1}).$$

□

Theorem 3.3. *Let $p \in (0,\infty)$ and $q \in [1,\infty)$. Then Hardy inequality* (1.1) *is equivalent to the existence of a set of functions*

$$Q_x : [0,\infty) \times [0,\infty) \to [0,\infty), \quad (x \in T),$$

such that for all $x \in T$, $r \geqslant 0$, $\rho \geqslant 0$ and $s_y \geqslant 0$ ($\sigma = \sum_{y \in R_x} s_y < \infty$) we have

$$U(x)(r+\rho)^{q-1}\rho + \sum_{y \in R_x} Q_y(r+\rho, s_y) \leqslant Q_x(r, v(x)\rho^p + \sigma). \tag{3.12}$$

Proof. For $f : T \to [0,\infty)$, $x \in T$ and $[o,x] = (x_i)_{i=0}^n$ we have

$$[\mathcal{I}f]^q(x) = \left(\sum_{i=0}^n f(x_i)\right)^q \leqslant q \sum_{P_x} [\mathcal{I}f]^{q-1} f,$$

$$\sum_T u[\mathcal{I}f]^q \leqslant q \sum_{w,x \in T\colon w \leqslant x} u(x)[\mathcal{I}f]^{q-1}(w)f(w) = q\sum_T U[\mathcal{I}f]^{q-1}f$$

owing to (3.10) and the Fubini theorem. In combination with a similar arguing and inequality (3.9) we see that (1.1) is equivalent to the condition

$$(\exists A \geqslant 0)\ (\forall f : T \to [0,\infty)) \quad \left(\sum_T U[\mathcal{I}f]^{q-1}f\right)^{1/q} \leqslant A\left(\sum_T vf^p\right)^{1/p}.$$

Now the proof of Theorem 3.3 can be done completely similarly to the proof of Theorem 3.1 with applying the functions

$$Q_x(r,s) = \sup_{\substack{f:S_x \to [0,\infty) \\ \sum_{S_x} vf^p \leqslant s}} \sum_{S_x} U\big[r + \mathcal{I}[\chi_{S_x} f]\big]^{q-1} f$$

in the first part of the proof and the condition

$$\sum_{S_x} U[\mathcal{I}f]^{q-1}f \leqslant Q_x\left(\mathcal{I}f(x) - f(x), \sum_{S_x} vf^p\right)$$

in the second part. The proof is complete. □

Now we apply Theorem 3.1 to the diagonal case in Hardy inequality (1.1). The author did not succeed to generalize the arguing from the proof of the next theorem to the case $p < q$. In Section 4 the induction is made twice for $p < q$ outside the framework of Theorems 3.1–3.3.

Theorem 3.4. *Let $p = q > 1$. Then the following conditions are equivalent.*
(i) Hardy inequality (1.1) *holds true.*
(ii) There exist $\beta > 0$ and $\beta_1 \geqslant 1$ such that as

$$\pi_x(y) = \prod_{S_x \cap P_y} \left(1 + \beta\{U/v\}^{p'-1}\right) \quad (y \in S_x)$$

we have

$$\sum_{S_x} u\pi_x \leqslant \beta_1 U(x)$$

for all $x \in T$.

(iii) $C < \infty$.

(iv) There exist $E : T \to [0, \infty)$ *and* $\varepsilon > 0$ *such that*

$$u(x) + \sum_{R_x} E \leqslant E(x)\big(1 + \varepsilon\{E(x)/v(x)\}^{p'-1}\big)^{1-p}$$

for each $x \in T$.

The criterion (i) ⇔ (ii) simplifies Theorem 16 from [1], where π_x was defined in a slightly different way and an additional condition $\sup_T(U/v) < \infty$ was applied. We note that

$$\text{(ii)} \Rightarrow \sup_T(U/v) \leqslant ((\beta_1 - 1)/\beta)^{p-1}$$

due to the relations

$$\pi_x(y) \geqslant 1 + \beta \sum_{S_x \cap P_y} \{U/v\}^{p'-1} \geqslant 1 + \beta\{U(x)/v(x)\}^{p'-1}. \tag{3.13}$$

The criterion (i) ⇔ (iii) is a particular case of criterion (1.4a).

The criterion (i) ⇔ (iv) is new.

Proof. Let us show that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

The proof of the implication (i) ⇒ (ii) reproduces work [1]. Let (1.1) hold. The case $A = 0$ (as $u \equiv 0$) is trivial and this is why we suppose that $A > 0$. We denote

$$\beta = (2A/p)^{-p'} \quad \& \quad \beta_1 = 2^p.$$

We choose $x \in T$. We fix a predecessor $y' \in S_x$ of an element $y \in S_x \setminus \{x\}$ by the condition $y \in R_{y'}$. Using the convention $\pi_x(x') = 1$, we let

$$f(y) = \begin{cases} \pi_x^{1/p}(y) - \pi_x^{1/p}(y') \\ \quad = \pi_x^{1/p}(y')\big[\big(1 + \beta\{U(y)/v(y)\}^{p'-1}\big)^{1/p} - 1\big] & \text{as} \quad y \in S_x, \\ 0 & \text{as} \quad y \in T \setminus S_x. \end{cases}$$

By (1.1), the concavity of the function $s \mapsto s^{1/p}$ and the Fubini theorem,

$$\mathcal{I}f(y) = \sum_{S_x \cap P_y} f = \pi_x^{1/p}(y) - 1 \quad (y \in S_x),$$

$$\begin{aligned}
\sum_{S_x} u\pi_x = \sum_{S_x} u[1 + \mathcal{I}f]^p &\leqslant 2^{p-1}U(x) + 2^{p-1}\sum_{S_x} u[\mathcal{I}f]^p \\
&\leqslant 2^{p-1}U(x) + 2^{p-1}A^p \sum_{S_x} vf^p,
\end{aligned}$$

$$\begin{aligned}
f^p(y) &\leqslant \pi_x(y')(\beta/p)^p\{U(y)/v(y)\}^{p'}, \\
(p/\beta)^p \sum_{S_x} vf^p &\leqslant \sum_{y \in S_x} v(y)\pi_x(y')\{U(y)/v(y)\}^{p'} \\
&= \sum_{y \in S_x} \pi_x(y')\{U(y)/v(y)\}^{p'-1} \sum_{z \in S_y} u(z) \\
&= \beta^{-1} \sum_{z \in S_x} u(z) \sum_{y \in S_x \cap P_z} (\pi_x(y) - \pi_x(y'))
\end{aligned}$$

$$= \beta^{-1} \sum_{z\in S_x} u(z)(\pi_x(z)-1) \leqslant \beta^{-1} \sum_{S_x} u\pi_x,$$

$$\sum_{S_x} u\pi_x \leqslant 2^{p-1} U(x) + (1/2) \sum_{S_x} u\pi_x.$$

If the set $\{u \neq 0\}$ is finite, we get $\sum_{S_x} u\pi_x \leqslant \beta_1 U(x)$. The general case is reduced to that partial one as in the proof of Theorem 3.1. The proof of the implication (i) $\Rightarrow$ (ii) is complete.

Let (ii) hold. Then for each $x \in T$ by (3.13) and the Fubini theorem we get

$$\beta_1 U(x) \geqslant \sum_{S_x} u\pi_x \geqslant U(x) + \beta \sum_{y\in S_x} u(y) \sum_{S_x\cap P_y} \{U/v\}^{p'-1}$$
$$= U(x) + \beta \sum_{S_x} U^{p'} v^{1-p'} = U(x) + \beta B(x).$$

Hence, $C \leqslant ((\beta_1 - 1)/\beta)^{1/p'} < \infty$. We have shown that (ii) $\Rightarrow$ (iii).

Let (iii) hold and hence

$$E(x) \equiv U(x) + C^{-p'} \sum_{S_x} U^{p'} v^{1-p'} \leqslant 2U(x).$$

By (3.2) and the convexity of the function $s \mapsto s^{1-p}$ we find

$$u(x) + \sum_{R_x} E = U(x) + C^{-p'} \sum_{S_x\setminus\{x\}} U^{p'} v^{1-p'}$$
$$= E(x) - C^{-p'} U^{p'}(x) v^{1-p'}(x)$$
$$\leqslant E(x)\big(1 - (2C)^{-p'} \{E(x)/v(x)\}^{p'-1}\big)$$
$$\leqslant E(x)\big(1 + \varepsilon\{E(x)/v(x)\}^{p'-1}\big)^{1-p},$$

where $\varepsilon > 0$ is arbitrary for $C = 0$ ($\Rightarrow u \equiv 0$) and

$$\varepsilon = (2C)^{-p'}/(p-1)$$

as $0 < C < \infty$. This proves the implication (iii) $\Rightarrow$ (iv).

Let condition (iv) hold. We let

$$A = \varepsilon^{-1/p'}, \qquad Q_x(r,s) = E(x) r^p + A^p s.$$

For $r \geqslant 0$ and $\rho \geqslant 0$ by the Hölder inequality we have

$$E(x)(r+\rho)^p = \Big\{1 \cdot E^{1/p}(x) r + \varepsilon^{1/p'} \{E(x)/v(x)\}^{1/p} \cdot A v^{1/p}(x)\rho\Big\}^p$$
$$\leqslant \big(1 + \varepsilon\{E(x)/v(x)\}^{p'-1}\big)^{p-1} \big(E(x) r^p + A^p v(x) \rho^p\big).$$

As $\sigma = \sum_{y\in R_x} s_y < \infty$, by (iv) we have

$$u(x)(r+\rho)^p + \sum_{y\in R_x} Q_y(r+\rho, s_y) = u(x)(r+\rho)^p + \sum_{y\in R_x} \big(E(y)(r+\rho)^p + A^p s_y\big)$$
$$= \Big(u(x) + \sum_{R_x} E\Big)(r+\rho)^p + A^p\sigma$$
$$\leqslant E(x) r^p + A^p v(x)\rho^p + A^p\sigma = Q_x(r, v(x)\rho^p + \sigma).$$

We have obtained inequality (3.3). By Theorem 3.1 the inequality (1.1) holds true. This completes the proof of the implication (iv) $\Rightarrow$ (i) and of the theorem. □

4. Arcozzi–Rochberg–Sawyer criteria for T

For the proof (which is either known and simplified or new) criteria (1.4) we shall need four lemmas.

Lemma 4.1. *For $1 < p \leqslant q < \infty$ and $q/p' \leqslant r < \infty$ we let*
$$s = \frac{p-1}{q-1}\frac{q-r}{r}.$$
For a function $f : T \to [0,\infty)$ and a point $x \in T$ we denote
$$E_x(y) = \sum_{S_x \cap P_y \setminus \{x\}} f \qquad (y \in S_x), \qquad F_x(y) = \sum_{S_x \cap P_y} f = E_x(y) + f(x).$$
Then for each $\varepsilon > 0$ the estimate
$$\Big(\sum_{S_x} uF_x^q\Big)^{p/q} \leqslant \sum_{y\in R_x}\Big(\sum_{S_y} uF_y^q\Big)^{p/q} + \varepsilon \frac{U^{p'(1/q-s/q')}}{v^{p'-1}}\Big(\sum_{S_x} uE_x^r\Big)^{p/r} + c_1(p,q,\varepsilon)\{U^{p/q}+v\}f^p \quad (4.1)$$
holds, where the functions U, v and f are taken at the point x.

Proof. We denote
$$M = \sum_{S_x} uE_x^q, \qquad N = \sum_{S_x} uE_x^r.$$
If $M \leqslant Uf^q$, then by the Minkowski inequality
$$\Big(\sum_{S_x} uF_x^q\Big)^{1/q} \leqslant \Big(\sum_{S_x} uE_x^q\Big)^{1/q} + \Big(\sum_{S_x} uf^q\Big)^{1/q} \leqslant 2(Uf^q)^{1/q},$$
$$\Big(\sum_{S_x} uF_x^q\Big)^{p/q} \leqslant 2^p U^{p/q} f^p. \quad (4.2)$$
If $M > Uf^q$, by binomial estimate (3.11) we obtain
$$F_x^q \leqslant E_x^q + q2^{q-1}(E_x^{q-1}f + f^q),$$
$$\sum_{S_x} uF_x^q \leqslant M + q2^{q-1}\Big(f\sum_{S_x} uE_x^{q-1} + Uf^q\Big),$$
$$(a+b)^{p/q} \leqslant a^{p/q} + (p/q)a^{p/q-1}b \quad (a > 0 \;\&\; b \geqslant 0),$$
$$\Big(\sum_{S_x} uF_x^q\Big)^{p/q} \leqslant M^{p/q} + p2^{q-1}M^{p/q-1}\Big(f\sum_{S_x} uE_x^{q-1} + Uf^q\Big)$$
$$\leqslant M^{p/q} + p2^{q-1}\Big(fM^{p/q-1}\sum_{S_x} uE_x^{q-1} + U^{p/q}f^p\Big).$$
By the Hölder inequality and condition $q/p' \leqslant r$ we have
$$M^{p/q-1}\sum_{S_x} uE_x^{q-1} \leqslant M^{p/q-1}\Big(\sum_{S_x} uE_x^q\Big)^{(q-p)/q}\Big(\sum_{S_x} uE_x^{q/p'}\Big)^{p/q} = \Big(\sum_{S_x} uE_x^{q/p'}\Big)^{p/q}$$
$$\leqslant U^{p/q-(p-1)/r}N^{(p-1)/r} = U^{1/q-s/q'}N^{(p-1)/r}.$$
In view of (4.2) we confirm that in each case
$$\Big(\sum_{S_x} uF_x^q\Big)^{p/q} \leqslant M^{p/q} + p2^q\big(U^{1/q-s/q'}N^{(p-1)/r}f + U^{p/q}f^p\big).$$

It is obvious that $E_x(x) = 0$ and $E_x = F_y$ on S_y $(y \in R_x)$ and this is why

$$M^{p/q} = \Big(\sum_{y \in R_x} \sum_{S_y} u F_y^q \Big)^{p/q} \leqslant \sum_{y \in R_x} \Big(\sum_{S_y} u F_y^q \Big)^{p/q}$$

by the Jensen inequality (the embedding $\ell_{p/q} \subset \ell_1$ of the spaces of sequences, see [21, Thm. 19]). By the Young inequality

$$p 2^q U^{1/q - s/q'} N^{(p-1)/r} f \leqslant \varepsilon U^{p'(1/q - s/q')} v^{1-p'} N^{p/r} + c(p, q, \varepsilon) v f^p.$$

The comparison of latter three estimates proves (4.1). □

Lemma 4.2. *Let $1 < p \leqslant q < \infty$ and $u_E : T \to [0, \infty)$. We denote by $U_E(x)$, $B_E(x)$ and C_E the numbers $U(x)$, $B(x)$ and C from the introduction constructed from the function u_E instead of u. Let $U_E \leqslant U$ and $C < \infty$. Then*

$$B_E(o) \leqslant q C^{p'} U_E^{p'/q'}(o), \qquad C_E \leqslant q^{1/p'} C.$$

This statement for $p = q$ and a tree with a boundary under the condition $u_E \leqslant u$ was proved in [4, 5.6] by means of the distribution function of the maximal function. We are going to treat the case $p \leqslant q$ and to apply Lemma 3.1 instead of the maximal function that is a bit simpler.

Proof. By the condition $U_E \leqslant U$, the Fubini theorem, formula (3.1), the definitions of the number C and the set $\{U_E/U > t\}$ and the embedding $\ell_1 \subset \ell_{p'/q'}$ we obtain

$$\begin{aligned} B_E(o) &= \sum_{x \in T} U^{p'}(x) v^{1-p'}(x) \int\limits_{0 < t < U_E(x)/U(x)} d(t^{p'}) \\ &= p' \int\limits_0^1 t^{p'-1}\, dt \sum_{\{U_E/U > t\}} U^{p'} v^{1-p'} \\ &= p' \int\limits_0^1 t^{p'-1}\, dt \sum_{x \in \{U_E/U > t\}^{\min}} \ \sum_{S_x \cap \{U_E/U > t\}} U^{p'} v^{1-p'} \\ &\leqslant p' C^{p'} \int\limits_0^1 t^{p'-1}\, dt \sum_{\{U_E/U > t\}^{\min}} U^{p'/q'} \\ &\leqslant p' C^{p'} \int\limits_0^1 t^{p'-1-p'/q'}\, dt \Big(\sum_{\{U_E/U > t\}^{\min}} U_E \Big)^{p'/q'}. \end{aligned}$$

By the first statement in Lemma 3.1 the expression in round brackets does not exceed $U_E(o)$ and after the integration this gives the first statement of the lemma.

The second statement of the lemma is obtained by applying the first statement to the rooted trees (S_x, x) for all possible x. The proof is complete. □

Lemma 4.3. *Let $0 < p < \infty$. For $x \in T$ and $E \in \mathcal{E}$ we denote*

$$\Omega_x(E) = \big\{ f : T \to [0, \infty) \colon f\big|_{T \setminus S_x} \equiv 0 \ \& \ (\mathcal{I} f)\big|_{S_x \cap E} \geqslant 1 \big\},$$

$$\operatorname{cap}(E) = \inf \Big\{ \sum_T v f^p \colon f \in \Omega_o(E) \Big\}.$$

Then for $0 < p \leqslant q < \infty$ *Hardy inequality* (1.1) *is equivalent to the existence of* $\alpha \geqslant 0$ *such that*

$$(\forall E \in \mathcal{E}) \quad \Big(\sum_E u\Big)^{1/q} \leqslant \alpha \operatorname{cap}(E)^{1/p}. \tag{4.3}$$

For best possible constants in (1.1) *and* (4.3) *we have* $\alpha \leqslant A \leqslant 2^{2+1/p}\alpha$.

This result is similar to the capacity criterion (2.5) $\Leftrightarrow$ (2.8). A version of this lemma for $1 \leqslant p \leqslant q \leqslant \infty$ was proved in [7, Lm. 2.5] by reducing to Theorem 3.1 from [8], where the case of metric trees was considered. As in work [8], we employ the Mazya method of cut–offs [20].

Proof. Let (1.1) hold. Then for each $E \in \mathcal{E}$ and $f \in \Omega_o(E)$ we have

$$\Big(\sum_E u\Big)^{1/q} \leqslant \Big(\sum_E u[\mathcal{I}f]^q\Big)^{1/q} \leqslant A\Big(\sum_T vf^p\Big)^{1/p}.$$

Taking $\inf_f$, we prove (4.3) with a constant $\alpha \leqslant A$.

And vice versa, let condition (4.3) hold. For $f : T \to [0, \infty)$ we let

$$\begin{aligned} E_k &= \big\{x \in T \colon \mathcal{I}f(x) > 2^k\big\} \quad \text{as} \quad k \in \mathbb{Z},\\ F_k &= E_k \setminus E_{k+1},\\ f_k &= \chi_{F_{k-1} \cup F_k^{\min}} f,\\ G_k &= \bigcup_{x \in F_k^{\min}} S_x. \end{aligned}$$

It is obvious that $F_k \subset G_k \in \mathcal{E}$. If we establish the inequality $\mathcal{I}f_k \geqslant 2^{k-1}$ on the set $F_k^{\min}$, then by the monotonicity of the function $\mathcal{I}f_k$ it turns out to be true also on G_k.

We take $x \in F_k^{\min}$ and we let

$$W = E_{k-1} \cap P_x.$$

It is obvious that $x \in W$. If $w \in W \setminus \{x\}$, then

$$\mathcal{I}f(w) \leqslant \mathcal{I}f(x) \leqslant 2^{k+1},$$

and hence $w \notin E_k$ by the minimality of x in F_k. Hence, $w \in F_{k-1}$ and $W \subset F_{k-1} \cup F_k^{\min}$.

We denote by w the least element in W. Then $\mathcal{I}f(w) - f(w) \leqslant 2^{k-1}$, which is trivial for $w = o$, while for $w \neq o$ it follows from the minimality of w in W. This implies

$$\begin{aligned} &\mathcal{I}f_k(x) \geqslant \sum_W f = \mathcal{I}f(x) - (\mathcal{I}f(w) - f(w)) > 2^k - 2^{k-1} = 2^{k-1},\\ &2^{1-k} f_k \in \Omega_o(G_k),\\ &\sum_{F_k} u \leqslant \sum_{G_k} u \leqslant \alpha^q \operatorname{cap}(G_k)^{q/p} \leqslant \alpha^q \Big(\sum_T v[2^{1-k} f_k]^p\Big)^{q/p},\end{aligned}$$

$$\begin{aligned} \sum_T u[\mathcal{I}f]^q &= \sum_k \sum_{F_k} u[\mathcal{I}f]^q \leqslant \sum_k 2^{(k+1)q} \sum_{F_k} u\\ &\leqslant 2^{2q}\alpha^q \sum_k \Big(\sum_{F_{k-1}\cup F_k^{\min}} vf^p\Big)^{q/p}\\ &\leqslant 2^{2q}\alpha^q \Big(\sum_k \sum_{F_{k-1}\cup F_k^{\min}} vf^p\Big)^{q/p} \leqslant 2^{2q+q/p}\alpha^q \Big(\sum_T vf^p\Big)^{q/p}. \end{aligned}$$

We have applied the embedding $\ell_1 \subset \ell_{q/p}$ and the fact that the sets F_k are mutually disjoint. This proves (1.1) with a constant $A \leqslant 2^{2+1/p}\alpha$. The proof is complete. □

Lemma 4.4. *Let $1 < p < \infty$. For $x \in T$ and $E \in \mathcal{E}$ we denote*

$$\operatorname{cap}_x(E) = \inf\left\{\sum_{S_x} v f^p \colon f \in \Omega_x(E)\right\}.$$

Then the identities hold:

$$\operatorname{cap}_x(E) = \begin{cases} v(x) & \textit{as} \quad x \in E, \\ \left(v^{1-p'}(x) + \sigma^{1-p'}\right)^{1-p} & \textit{as} \quad x \notin E, \end{cases} \tag{4.4}$$

$$\operatorname{cap}(S_x \cap E) = \left(\mathcal{V}(x) + \operatorname{cap}_x(E)^{1-p'}\right)^{1-p}, \tag{4.5}$$

$$\operatorname{cap}(S_x \cap E) = \begin{cases} V^{1-p}(x) & \textit{as} \quad x \in E, \\ \left(V(x) + \sigma^{1-p'}\right)^{1-p} & \textit{as} \quad x \notin E, \end{cases} \tag{4.6}$$

where

$$\sigma = \sum_{y \in R_x} \operatorname{cap}_y(E), \qquad \mathcal{V}(x) = \sum_{P_x \setminus \{x\}} v^{1-p'}.$$

The number $\operatorname{cap}_x(E) \in [0, v(x)]$ is a capacity $\operatorname{cap}(S_x \cap E)$ calculated for the tree (S_x, x) instead of (T, o).

Formula (4.4) was proved in [7, Prop. 2.6] without taking into consideration the case $x \in E$ by reducing to Theorem 4.5 in [8], where the case of metric trees was considered. For us it is easier to prove (4.4) than discussing the notation in [7]. The result of Theorem 30 in [4] looks similar to formula (4.6) with $x \notin E$, but it was formulated with mistakes.

Proof. As $S_x \cap E = \varnothing$ all capacities in (4.4)–(4.6) are zero and these formulas hold in an obvious interpretation. This is why we suppose that $S_x \cap E \neq \varnothing$.

If $x \in E$, then $f(x) \geqslant 1$ and $\sum\limits_{S_x} v f^p \geqslant v(x)$ as $f \in \Omega_x(E)$ and hence $\operatorname{cap}_x(E) \geqslant v(x)$. The choice $f = \chi_{\{x\}}$ indicates that the equality holds true.

Let $x \notin E$, $f \in \Omega_x(E)$ and $\rho = f(x)$. If $0 \leqslant \rho < 1$, then

$$\frac{\chi_{S_y} f}{1-\rho} \in \Omega_y(E) \quad (\forall y \in R_x),$$

$$\sigma = \sum_{y \in R_x} \operatorname{cap}_y(E) \leqslant (1-\rho)^{-p} \sum_{y \in R_x} \sum_{S_y} v f^p = (1-\rho)^{-p} \sum_{S_x \setminus \{x\}} v f^p,$$

$$\rho + 1 - \rho \leqslant \rho + \sigma^{-1/p}\Bigg(\sum_{S_x \setminus \{x\}} v f^p\Bigg)^{1/p} \leqslant \left(v^{1-p'}(x) + \sigma^{1-p'}\right)^{1/p'}\Bigg(v(x)\rho^p + \sum_{S_x \setminus \{x\}} v f^p\Bigg)^{1/p}$$

by (3.2) and the Hölder inequality. As $\rho \geqslant 1$, the obtained estimate is trivial and this gives inequality "$\geqslant$" in (4.4).

Let $x \notin E$ and $\varphi_y \in \Omega_y(E)$ $(y \in R_x)$. We let

$$\rho = \frac{v^{1-p'}(x)}{v^{1-p'}(x) + \sigma^{1-p'}},$$

$$f(z) = \begin{cases} \rho & \text{as} \quad z = x, \\ (1-\rho)\varphi_y(z) & \text{as} \quad z \in S_y \ (y \in R_x), \\ 0 & \text{as} \quad z \notin S_x. \end{cases}$$

If $z \in S_x \cap E$, then $z \in S_y \cap E$ for some $y \in R_x$ and

$$\mathcal{I} f(z) = \rho + (1-\rho)\mathcal{I}\varphi_y(z) \geqslant 1.$$

Hence, $f \in \Omega_x(E)$ and

$$\begin{aligned}\operatorname{cap}_x(E) \leqslant \inf_{\{\varphi_y\}} \sum_T v f^p = \inf_{\{\varphi_y\}} \Big\{ v(x)\rho^p + (1-\rho)^p \sum_{y \in R_x} \sum_{S_y} v\varphi_y^p \Big\} \\ = v(x)\rho^p + (1-\rho)^p \sigma \\ = \big(v^{1-p'}(x) + \sigma^{1-p'}\big)^{1-p}.\end{aligned}$$

This proves identity (4.4).

Let $f \in \Omega_o(S_x \cap E)$ and $\rho = \sum_{P_x \setminus \{x\}} f$. Then

$$\rho \leqslant \mathcal{V}^{1/p'}(x) \Big(\sum_{P_x \setminus \{x\}} v f^p \Big)^{1/p} \quad \text{(Hölder inequality).} \tag{4.7}$$

If $0 \leqslant \rho < 1$, then $\chi_{S_x} f/(1-\rho) \in \Omega_x(E)$ and hence,

$$\begin{aligned}\sum_{S_x} v f^p \geqslant (1-\rho)^p \operatorname{cap}_x(E), \\ \rho + 1 - \rho \leqslant \mathcal{V}^{1/p'}(x) \Big(\sum_{P_x \setminus \{x\}} v f^p \Big)^{1/p} + \operatorname{cap}_x(E)^{-1/p} \Big(\sum_{S_x} v f^p \Big)^{1/p} \\ \leqslant \big(\mathcal{V}(x) + \operatorname{cap}_x(E)^{1-p'}\big)^{1/p'} \Big(\sum_{P_x \setminus \{x\}} v f^p + \sum_{S_x} v f^p \Big)^{1/p} \\ \leqslant \big(\mathcal{V}(x) + \operatorname{cap}_x(E)^{1-p'}\big)^{1/p'} \Big(\sum_T v f^p \Big)^{1/p}\end{aligned}$$

by the Hölder inequality. As $\rho \geqslant 1$, the obtained inequality follows from (4.7) and this gives inequality "$\geqslant$" in (4.5).

In order to confirm the reverse inequality, we take $\varphi \in \Omega_x(E)$. We let

$$\rho = \frac{\mathcal{V}(x)}{\mathcal{V}(x) + \operatorname{cap}_x(E)^{1-p'}},$$

$$f(y) = \begin{cases} \rho \mathcal{V}^{-1}(x) v^{1-p'}(y) & \text{as} \quad y \in P_x \setminus \{x\}, \\ (1-\rho)\varphi(y) & \text{as} \quad y \in S_x, \\ 0 & \text{as} \quad y \notin P_x \cup S_x. \end{cases}$$

If $y \in S_x \cap E$, then

$$\begin{aligned}\mathcal{I} f(y) = \sum_{P_x \setminus \{x\}} f + (1-\rho) \sum_{S_x \cap P_y} \varphi \geqslant \rho + 1 - \rho = 1, \\ \sum_T v f^p = \sum_{P_x \setminus \{x\}} v f^p + \sum_{S_x} v f^p = \rho^p \mathcal{V}^{1-p}(x) + (1-\rho)^p \sum_{S_x} v \varphi^p, \\ \operatorname{cap}(S_x \cap E) \leqslant \inf_\varphi \sum_T v f^p = \rho^p \mathcal{V}^{1-p}(x) + (1-\rho)^p \operatorname{cap}_x(E) = \big(\mathcal{V}(x) + \operatorname{cap}_x(E)^{1-p'}\big)^{1-p}.\end{aligned}$$

This establishes identity (4.5), while (4.6) follows from (4.4) and (4.5). The proof is complete. □

The following Arcozzi–Rochberg–Sawyer criteria were commented in the Introduction, see (1.4). The insufficiency of the condition $D < \infty$ for the validity of Hardy inequality (1.1) with $p = q$ was proved in [2] (and in the references in this work) and in [8, Exm. 5.3].

Theorem 4.1. *Under the assumptions and notations of the Introduction we have:*
(a) if $1 < p \leqslant q < \infty$*, then* (1.1) $\Leftrightarrow C < \infty$*;*
(b) if $1 < p < q < \infty$*, then* (1.1) $\Leftrightarrow D < \infty$.

Proof. (a) Let $1 < p \leqslant q < \infty$. If (1.1) holds, then by Lemma 3.2 condition (3.5) holds. We substitute the function $g = \chi_{S_x}$ into (3.5). As $y \in S_x$, we have $\mathcal{J}g(y) = \sum_{S_y} ug = U(y)$ and

$$B^{1/p'}(x) = \Big(\sum_{S_x} U^{p'} v^{1-p'}\Big)^{1/p'} \leqslant \Big(\sum_{T} v^{1-p'} [\mathcal{J}g]^{p'}\Big)^{1/p'}$$
$$\leqslant A\Big(\sum_{T} ug^{q'}\Big)^{1/q'} = AU^{1/q'}(x) \quad \Rightarrow \quad C \leqslant A < \infty.$$

This proof is standard [2].

And vice versa, let $C < \infty$. We provide three ways of deriving inequality (1.1).

First way. We begin with induction. Multiplying u or v by a positive constant, we can achieve $C = 1$. It is sufficient to confirm obtained modified Hardy inequality (1.1) for functions $f : T \to [0, \infty)$, which are non–zero only on a finite set.

We let $r = \max\{1, q - 1\}$. Then $q/p' \leqslant r < q$ and $s > 0$ in the notations of Lemma 4.1. It is obvious that there exist $\varepsilon_1(p, s) > 0$ and $\varepsilon_2(p, s) > 0$ such that

$$\varepsilon_1 \tau + (1 + \varepsilon_2 \tau)^{p-1} \leqslant (1 - \tau)^{-s}, \quad 0 \leqslant \tau \leqslant 1. \tag{4.8}$$

Let c_1 be a constant from inequality (4.1) corresponding to the value $\varepsilon = \varepsilon_1$. We are going to show that for each $x \in T$

$$G_x = \Big(\sum_{S_x} uF_x^q\Big)^{p/q} + B^{-s}(x)\Big(\sum_{S_x} uF_x^r\Big)^{p/r} \leqslant A^p \sum_{S_x} vf^p, \tag{4.9}$$
$$A^p = 2c_1 + (1 + \varepsilon_2^{-1})^{p-1}.$$

Taking $x = o$, we obtain inequality (1.1) since $F_o = \mathcal{I}f$.

We observe that all series involved in (4.9) converge due to $U(o) < \infty$ and the finiteness of the set $\{f \neq 0\}$. The second term in (4.9) is hinted at by an interpolation between the first term and the result of Lemma 5.1.

We shall prove estimate (4.9) by the induction in decreasing the length of the chain $[o, x]$. If this length is sufficiently large, then $G_x = 0$ due to the finiteness of the set $\{f \neq 0\}$ and we can suppose that (4.9) holds for all $y \in R_x$ instead of x. We then also suppose that $B(x) > 0$ since $u\big|_{S_x} \equiv 0$ and $G_x = 0$ as $B(x) = 0$.

We let $\tau = U^{p'}(x)v^{1-p'}(x)/B(x)$. It follows from $C = 1$ that

$$U^{p'(1/q - s/q')}(x)B^{s+1}(x) \leqslant U^{p'(1/q-s/q')+p'(s+1)/q'}(x) = U^{p'}(x),$$
$$\varepsilon \frac{U^{p'(1/q-s/q')}(x)}{v^{p'-1}(x)} \leqslant \varepsilon_1 \tau B^{-s}(x).$$

For each $t > 0$ by the Minkowski ($r \geqslant 1$) and Hölder inequalities we have

$$\Big(\sum_{S_x} uF_x^r\Big)^{p/r} \leqslant (1 + t)^{p-1}\Big(\sum_{S_x} uE_x^r\Big)^{p/r} + \underbrace{(1 + 1/t)^{p-1} U^{p/r}(x) f^p(x)}_{H}.$$

As $t = \varepsilon_2\tau$, in view of (4.1) and (4.8) we obtain

$$G_x \leqslant \sum_{y\in R_x}\Big(\sum_{S_y} uF_y^q\Big)^{p/q} + (1-\tau)^{-s}B^{-s}(x)\Big(\sum_{S_x} uE_x^r\Big)^{p/r} + c_1\big\{U^{p/q}(x)+v(x)\big\}f^p(x) + B^{-s}(x)H. \tag{4.10}$$

It is obvious that

$$(1-\tau)B(x) = B(x) - U^{p'}(x)v^{1-p'}(x) = \sum_{R_x} B.$$

If $p/r \leqslant 1$, then by the embedding $\ell_{p/r} \subset \ell_1$ and the property $s > 0$ we get

$$\Big(\sum_{S_x} uE_x^r\Big)^{p/r} = \Big(\sum_{y\in R_x}\sum_{S_y} uF_y^r\Big)^{p/r} \leqslant \sum_{y\in R_x}\Big(\sum_{S_y} uF_y^r\Big)^{p/r},$$

$$\Big(\sum_{R_x} B\Big)^{-s}\Big(\sum_{S_x} uE_x^r\Big)^{p/r} \leqslant \sum_{y\in R_x} B^{-s}(y)\Big(\sum_{S_y} uF_y^r\Big)^{p/r}. \tag{4.11}$$

As $p/r > 1$, the Hölder inequality implies

$$\Big(\sum_{S_x} uE_x^r\Big)^{p/r} \leqslant \Big(\sum_{R_x} B^\sigma\Big)^{s/\sigma}\sum_{y\in R_x} B^{-s}(y)\Big(\sum_{S_y} uF_y^r\Big)^{p/r},$$

where

$$\sigma = \frac{sr}{p-r} = \frac{p-1}{q-1}\frac{q-r}{p-r} \geqslant 1.$$

The embedding $\ell_1 \subset \ell_\sigma$ again gives estimate (4.11).

By inequalities (4.10), (4.11) and (4.9) (for points in R_x) we obtain

$$\begin{aligned} G_x &\leqslant \sum_{y\in R_x} G_y + c_1\big\{U^{p/q}(x)+v(x)\big\}f^p(x) + B^{-s}(x)H \\ &\leqslant A^p\sum_{S_x\setminus\{x\}} vf^p + c_1\big\{U^{p/q}(x)+v(x)\big\}f^p(x) + B^{-s}(x)H. \end{aligned}$$

The estimates $U^{p'}v^{1-p'} \leqslant B \leqslant U^{p'/q'}$ show that $U^{p/q} \leqslant v$ and

$$\begin{aligned} B^{-s}(x)H &\leqslant B^{-s}(x)\left(\frac{1+\varepsilon_2}{\varepsilon_2\tau}\right)^{p-1}U^{p/r}(x)f^p(x) \\ &= (1+\varepsilon_2^{-1})^{p-1}B^{p-1-s}(x)U^{p/r-p}(x)v(x)f^p(x) \\ &\leqslant (1+\varepsilon_2^{-1})^{p-1}v(x)f^p(x), \end{aligned}$$

since $p-1-s = (p-1)q'/r'$. This proves (4.9) and inequality (1.1).

Second way. In [3, Sect. 3] and [4, 5.4.1] the implication $C < \infty \Rightarrow$ (1.1) for $p = q$ was proved by applying the Marcinkiewicz interpolation theorem to the maximal operator. As in the case of Lemma 4.2, here we consider the case $p \leqslant q$ and apply Lemma 3.1 instead of the maximal operator.

Let

$$g: T \to \mathbb{R}, \qquad \sum_T u|g| < \infty, \qquad t > 0,$$

$$Jg(x) = U^{-1}(x)\sum_{S_x} ug, \qquad E = \big\{x \in T\colon J|g|(x) > t\big\}.$$

Similarly to the proof of Lemma 4.2 we have

$$\sum_E U^{p'}v^{1-p'} = \sum_{x\in E^{\min}}\sum_{S_x\cap E} U^{p'}v^{1-p'} \leqslant C^{p'} \sum_{x\in E^{\min}} U^{p'/q'}(x) \leqslant C^{p'}\Big(\sum_{E^{\min}} U\Big)^{p'/q'}$$
$$\leqslant C^{p'}t^{-p'/q'}\Big(\sum_{x\in E^{\min}}\sum_{S_x} u|g|\Big)^{p'/q'} \leqslant C^{p'}t^{-p'/q'}\Big(\sum_T u|g|\Big)^{p'/q'}.$$

By $|Jg| \leqslant J|g|$, the operator J is an operator of weak type $(1, p'/q')$ with respect to the space T with the measures

$$X \mapsto \sum_X u \quad \& \quad X \mapsto \sum_X U^{p'}v^{1-p'}.$$

In view of $\sup_T |Jg| \leqslant \sup_T |g|$ it is also an operator of type (∞,∞). By the Marcinkiewicz theorem [34, App. B], J is an operator of type (q', p') and hence

$$(\forall g : T \to [0,\infty)) \quad \Big(\sum_T U^{p'}v^{1-p'}[Jg]^{p'}\Big)^{1/p'} \leqslant c(p,q,C)\Big(\sum_T ug^{q'}\Big)^{1/q'}.$$

This property coincides with (3.5) and implies (1.1) by Lemma 3.2.

Third way. V.G. Maz'ya posed a problem, see [4], to give a capacity proof of the implication $C < \infty \Rightarrow$ (1.1). For $p = q$ it was solved in [4, Thm. 43] by means of a predecessor of our Lemma 4.2 and the formula

$$\operatorname{cap}(E) = \sup_{u:T\to[0,\infty):\ \{u\neq 0\}\subset E} \frac{U(o)}{C^p}.$$

We are going to show how to avoid using this formula.

For $E \in \mathcal{E}$ we let $u_E = \chi_E u$. We take $f \in \Omega_o(E)$, that is, a function $f : T \to [0,\infty)$ such that $\mathcal{I}f \geqslant 1$ on E. By the Fubini theorem, the Hölder inequality and Lemma 4.2 we find

$$U_E(o) = \sum_T u_E \leqslant \sum_T u_E \mathcal{I}f = \sum_T U_E f$$
$$\leqslant \Big(\sum_T U_E^{p'}v^{1-p'}\Big)^{1/p'}\Big(\sum_T vf^p\Big)^{1/p}$$
$$= B_E^{1/p'}(o)\Big(\sum_T vf^p\Big)^{1/p} \leqslant q^{1/p'}CU_E^{1/q'}(o)\Big(\sum_T vf^p\Big)^{1/p},$$
$$U_E^{1/q}(o) \leqslant \alpha\Big(\sum_T vf^p\Big)^{1/p} \quad \text{for} \quad \alpha = q^{1/p'}C.$$

Taking $\inf_f$ gives condition (4.3) and this implies (1.1) by Lemma 4.3.

(b) Let $1 < p < q < \infty$. Let (1.1) hold. For a function $f = \chi_{P_x}v^{1-p'}$ we have

$$\mathcal{I}f = \sum_{P_x} f = V(x)$$

on the set S_x. This implies

$$U^{1/q}(x)V(x) \leqslant \Big(\sum_T u[\mathcal{I}f]^q\Big)^{1/q} \leqslant A\Big(\sum_T vf^p\Big)^{1/p} = AV^{1/p}(x),$$

and $D \leqslant A < \infty$. This proof is standard [2]. In the same way we can substitute $g = \chi_{S_x}$ into Lemma 3.2 and we obtain that

$$\mathcal{J}g(w) = \sum_{S_w} ug = U(x) \quad \text{as} \quad w \in P_x,$$

$$U(x)V^{1/p'}(x) \leqslant \Big(\sum_T v^{1-p'}[\mathcal{J}g]^{p'}\Big)^{1/p'} \leqslant A\Big(\sum_T ug^{q'}\Big)^{1/q'} = AU^{1/q'}(x).$$

This again yields $D \leqslant A < \infty$.

And vice versa, let $D < \infty$. We apply the induction in combination with Lemmas 4.3 and 4.4 that is similar to works [6], [7], in which the arguing was more complicated.

We denote

$$\alpha = \left(\frac{q}{q-p}\right)^{1/p'} D.$$

We take $E \in \mathcal{E}$. As in Theorem 3.1, we can suppose that the set $\{u \neq 0\}$ is finite. Then, for sufficiently long chains $[o,x]$,

$$\Big(\sum_{S_x \cap E} u\Big)^{1/q} \leqslant \alpha \operatorname{cap}(S_x \cap E)^{1/p}. \tag{4.12}$$

Suppose that (4.12) holds for the elements of the set R_x instead of x. By formula (4.5) this means that

$$(\forall y \in R_x) \quad \sum_{S_y \cap E} u \leqslant \alpha^q h(s_y),$$

where $s_y = \operatorname{cap}_y(E)$ and

$$h(s) = \big(\mathcal{V}(y) + s^{1-p'}\big)^{-q/p'} = \big(V(x) + s^{1-p'}\big)^{-q/p'}.$$

If $x \in E$, then $\operatorname{cap}(S_x \cap E) = V^{1-p}(x)$ by (4.6) and hence (4.12) holds due to inequality $D \leqslant \alpha$. Let $x \notin E$. Then

$$\sum_{S_x \cap E} u = \sum_{y \in R_x} \sum_{S_y \cap E} u \leqslant \alpha^q \sum_{y \in R_x} h(s_y).$$

As $s > 0$,

$$\frac{d}{ds}\frac{h(s)}{s} = s^{-2}\big(V(x) + s^{1-p'}\big)^{-q/p'-1}\left(\frac{q-p}{p}s^{1-p'} - V(x)\right).$$

Therefore, if

$$\sigma = \sum_{y \in R_x} s_y \leqslant \sigma_0 = \Big(\frac{q-p}{pV(x)}\Big)^{p-1},$$

then

$$s_y \leqslant \sigma \quad \Rightarrow \quad h(s_y) \leqslant s_y h(\sigma)/\sigma \quad \Rightarrow \quad \alpha^q \sum_{y \in R_x} h(s_y) \leqslant \alpha^q h(\sigma),$$

see [21, Sect. 3.14] about this arguing. If $\sigma > \sigma_0$, then

$$\sum_{S_x \cap E} u \leqslant U(x) \leqslant D^q V^{-q/p'}(x) = \alpha^q h(\sigma_0)$$
$$\leqslant \alpha^q h(\sigma) = \alpha^q \operatorname{cap}(S_x \cap E)^{q/p} \quad \text{(due to (4.6))}.$$

By the induction, relation (4.12) has been established for $x \in T$. For $x = o$ we get condition (4.3) and this proves (1.1) by Lemma 4.3. □

5. Arcozzi–Rochberg–Sawyer criteria for $T \cup \partial T$ and $\mathcal{D}$

We denote by ∂T the set of all sequences

$$x = (x_i)_{i=0}^{\infty} \subset T\colon \quad x_i \neq x_j \ (i \neq j) \quad \& \quad x_0 = o \quad \& \quad x_i \sim x_{i+1} \ (i \geqslant 0).$$

For $x \in \overline{T} = T \cup \partial T$ and a function $f : T \to [0, \infty)$ we let

$$P_{x,\overline{T}} = \begin{cases} P_x & \text{as} \quad x \in T, \\ x & \text{as} \quad x \in \partial T, \end{cases}$$
$$\mathcal{I} f(x) = \sum_{P_{x,\overline{T}}} f.$$

For $x \in T$ we let

$$S_{x,\partial T} = \{y \in \partial T\colon x \in y\},$$
$$S_{x,\overline{T}} = \{y \in \overline{T}\colon x \in P_{y,\overline{T}}\} = S_x \cup S_{x,\partial T}.$$

Usually the set $\overline{T}$ is equipped with a topology and one considers Borel measures on it. For our purposes it is sufficient to consider a σ–algebra $\mathfrak{S}$ in $\overline{T}$ generated by all sets $S_{x,\overline{T}}$. If the set $\{f \neq 0\}$ is finite, then the function $\mathcal{I} f$ is $\mathfrak{S}$–measurable and hence the same is true for each function f. It follows from formula (3.2) and a similar formula

$$S_{x,\partial T} = \bigcup_{y \in R_x} S_{y,\partial T} \quad \text{(the union is disjunctive)} \tag{5.1}$$

that

$$\{x\} = S_x \setminus \bigcup_{y \in R_x} S_y = S_{x,\overline{T}} \setminus \bigcup_{y \in R_x} S_{y,\overline{T}} \in \mathfrak{S}$$

for each $x \in T$. Hence, $T \in \mathfrak{S}$, and the σ–algebra $\mathfrak{S}\big|_T$ consists of all subsets in T, while the σ–algebra $\mathfrak{S}\big|_{\partial T}$ is generated by the sets $S_{x,\partial T}$.

We are going to establish a version of Arcozzi–Rochberg–Sawyer criteria for the set $\overline{T}$ (by means of new arguing); first we need to prove a couple of lemmas. We consider $p, q \in (1, \infty)$. For a finite measure ν defined on $\mathfrak{S}$ we let

$$U_{\overline{T}}(x) = \nu(S_{x,\overline{T}}) \qquad (x \in T),$$
$$B_{\overline{T}}(x) = \sum_{S_x} U_{\overline{T}}^{p'} v^{1-p'} \quad (x \in T),$$
$$C_{\overline{T}} = \sup_T B_{\overline{T}}^{1/p'} U_{\overline{T}}^{-1/q'},$$
$$D_{\overline{T}} = \sup_T U_{\overline{T}}^{1/q} V^{1/p'}.$$

Lemma 5.1. *The identity* $\beta = B_{\overline{T}}^{1/p'}(o)$ *holds, where*

$$\beta = \sup_f \frac{\int_{\overline{T}} \mathcal{I} f \, d\nu}{\left(\sum_T v f^p\right)^{1/p}}.$$

This lemma "resolves" the Hardy inequality on $\overline{T}$ as $p > q = 1$. For trace inequality (2.5) a similar criterion is $A_1 = \left(\int_{\mathbb{R}^n} (I_l \mu)^{p'} \, dx\right)^{1/p'}$ [20].

Proof. By the Fubini theorem and the Hölder inequality

$$\int_{\overline{T}} \mathcal{I}f\,d\nu = \sum_T U_{\overline{T}} f \leqslant \Big(\sum_T U_{\overline{T}}^{p'} v^{1-p'}\Big)^{1/p'} \Big(\sum_T v f^p\Big)^{1/p} = B_{\overline{T}}^{1/p'}(o)\Big(\sum_T v f^p\Big)^{1/p},$$

so that $\beta \leqslant B_{\overline{T}}^{1/p'}(o)$. The choice $f = (U_{\overline{T}}/v)^{p'-1}$ (and a simple regularization in the case $B_{\overline{T}}(o) = \infty$) shows that here the equality holds true. The proof is complete. □

Lemma 5.2. *The following estimates hold:*

$$D_{\overline{T}} \leqslant q^{1/p'} C_{\overline{T}}, \tag{5.2}$$

$$p < q \quad \Rightarrow \quad C_{\overline{T}} \leqslant 2(2^{p'/q'-1} - 1)^{-1/p'} D_{\overline{T}}, \tag{5.3}$$

$$(T, o) = (\mathbb{N}, 1) \quad \Rightarrow \quad C_{\overline{T}} \leqslant (q')^{1/p'} D_{\overline{T}}. \tag{5.4}$$

For $\nu(\partial T) = 0$ an estimate of type (5.3) was established in [2].

Proof. Let $U_{\overline{T}}(x) > 0$. On P_x we have $U_{\overline{T}} \geqslant U_{\overline{T}}(x)$ and this is why

$$V(x) = \sum_{w\in P_x} U_{\overline{T}}^{p'}(w) v^{1-p'}(w) \int\limits_{0<t<1/U_{\overline{T}}(w)} d(t^{p'}) = p' \int\limits_0^{1/U_{\overline{T}}(x)} t^{p'-1}\,dt \sum_{E_t} U_{\overline{T}}^{p'} v^{1-p'},$$

where $E_t = \big\{w \in P_x \colon U_{\overline{T}}(w) < 1/t\big\}$. This set contains x and has the least element w, for which

$$\sum_{E_t} U_{\overline{T}}^{p'} v^{1-p'} \leqslant B_{\overline{T}}(w) \leqslant C_{\overline{T}}^{p'} U_{\overline{T}}^{p'/q'}(w) \leqslant C_{\overline{T}}^{p'} t^{-p'/q'},$$

$$V(x) \leqslant p' C_{\overline{T}}^{p'} \int\limits_0^{1/U_{\overline{T}}(x)} t^{p'/q-1}\,dt = q C_{\overline{T}}^{p'} U_{\overline{T}}^{-p'/q}(x),$$

and this proves (5.2).

Let $p < q$ and $U_{\overline{T}}(x) > 0$. For $k \geqslant 0$ we denote

$$Y_k = \big\{y \in S_x \colon U_{\overline{T}}(y)/U_{\overline{T}}(x) \in (2^{-k-1}, 2^{-k}]\big\}.$$

The sets $S_{y,\overline{T}} \subset S_{x,\overline{T}}$ ($y \in Y_k^{\min}$) are mutually disjoint (as in Lemma 3.1). This by formula (3.1) yields

$$U_{\overline{T}}(x) \geqslant \sum_{y\in Y_k^{\min}} U_{\overline{T}}(y) \geqslant 2^{-k-1} U_{\overline{T}}(x)\operatorname{card} Y_k^{\min},$$

$$B_{\overline{T}}(x) = \sum_{k=0}^{\infty} \sum_{Y_k} U_{\overline{T}}^{p'} v^{1-p'} \leqslant \sum_{k=0}^{\infty} 2^{k+1} \max_{y\in Y_k^{\min}} \sum_{S_y\cap Y_k} U_{\overline{T}}^{p'} v^{1-p'}.$$

For $y \in Y_k^{\min} \subset Y_k$ and $z_1, z_2 \in S_y \cap Y_k$ we have

$$2^{-k-1} + 2^{-k-1} < \frac{U_{\overline{T}}(z_1)}{U_{\overline{T}}(x)} + \frac{U_{\overline{T}}(z_2)}{U_{\overline{T}}(x)} = \frac{\nu(S_{z_1,\overline{T}} \cup S_{z_2,\overline{T}}) + \nu(S_{z_1,\overline{T}} \cap S_{z_2,\overline{T}})}{U_{\overline{T}}(x)}$$

$$\leqslant 2^{-k} + \frac{\nu(S_{z_1,\overline{T}} \cap S_{z_2,\overline{T}})}{U_{\overline{T}}(x)}.$$

Hence, $S_{z_1,\overline{T}} \cap S_{z_2,\overline{T}} \neq \varnothing$ and thus, $z_1 \leqslant z_2$ or $z_2 \leqslant z_1$, that is, the set $S_y \cap Y_k$ is linearly ordered. For each $z \in S_y \cap Y_k$

$$\sum_{S_y\cap Y_k\cap P_z} U_{\overline{T}}^{p'} v^{1-p'} \leqslant (2^{-k} U_{\overline{T}}(x))^{p'} V(z)$$

$$\begin{aligned}&\leqslant (2^{-k}U_{\overline{T}}(x))^{p'} D_{\overline{T}}^{p'} U_{\overline{T}}^{-p'/q}(z)\\ &\leqslant (2^{-k}U_{\overline{T}}(x))^{p'} D_{\overline{T}}^{p'} (2^{-k-1}U_{\overline{T}}(x))^{-p'/q}\\ &= 2^{p'/q}2^{-kp'/q'} D_{\overline{T}}^{p'} U_{\overline{T}}^{p'/q'}(x).\end{aligned}$$

Taking the supremums in z and y, we see that

$$B_{\overline{T}}(x) \leqslant 2^{p'/q+1} D_{\overline{T}}^{p'} U_{\overline{T}}^{p'/q'}(x) \sum_{k=0}^{\infty} 2^{k(1-p'/q')} = \frac{2^{p'} D_{\overline{T}}^{p'} U_{\overline{T}}^{p'/q'}(x)}{2^{p'/q'-1}-1}.$$

This proves estimate (5.3). We observe that it can be strengthened since in fact we have $\operatorname{card} Y_k^{\min} \leqslant 2^{k+1}-1$.

Let $(T,o)=(\mathbb{N},1)$ and $U_{\overline{T}}(x)>0$. On S_x we have $U_{\overline{T}} \leqslant U_{\overline{T}}(x)$ and this is why

$$B_{\overline{T}}(x) = \sum_{y\in S_x} v^{1-p'}(y) \int\limits_{0<t<U_{\overline{T}}(y)} d(t^{p'}) = p' \int\limits_0^{U_{\overline{T}}(x)} t^{p'-1}\,dt \sum_{E_t} v^{1-p'},$$

where $E_t = \{y \in S_x \colon U_{\overline{T}}(y) > t\}$. Considering $y \in E_t$ and $\sup_{y\in E_t}$, we have

$$\sum_{E_t\cap P_y} v^{1-p'} \leqslant V(y) \leqslant D_{\overline{T}}^{p'} U_{\overline{T}}^{-p'/q}(y) \leqslant D_{\overline{T}}^{p'} t^{-p'/q},$$

$$B_{\overline{T}}(x) \leqslant p' D_{\overline{T}}^{p'} \int\limits_0^{U_{\overline{T}}(x)} t^{p'/q'-1}\,dt = q' D_{\overline{T}}^{p'} U_{\overline{T}}^{p'/q'}(x),$$

and this proves (5.4). The proof is complete. □

Theorem 5.1. *(a) If $1<p\leqslant q<\infty$, then the Hardy inequality*

$$(\exists A \geqslant 0)\ (\forall f : T \to [0,\infty)) \quad \Big(\int\limits_{\overline{T}} [\mathcal{I}f]^q\,d\nu\Big)^{1/q} \leqslant A\Big(\sum_T v f^p\Big)^{1/p} \tag{5.5}$$

is equivalent to $C_{\overline{T}}<\infty$. At the same time,

$$C_{\overline{T}} \leqslant A \leqslant q^{1/q+(p'-q')(p-1)/(p'q')} r^{(p-1)/q} C_{\overline{T}} \quad (r = p'(q-1) \geqslant q) \tag{5.6}$$

for the best A in (5.5)*. In particular, $C_{\overline{T}} \leqslant A \leqslant pC_{\overline{T}}$ as $p=q$.*

(b) If $1<p<q<\infty$, then (5.5) $\Leftrightarrow$ $D_{\overline{T}}<\infty$*.*

For $p=q$ the equivalence from (a) is contained in [4, Thm. 32].

As $T=\{o\}$, the identity $C_{\overline{T}}=A$ holds. For the tree $(T,o)=(\mathbb{N},1)$ and $\nu(\partial T)=0$ in [26] the estimate $A\leqslant qC_{\overline{T}}$ was obtained and it was pointed our that as $p=q$ it is optimal. An optimal estimate for $p<q$ was found in [35, (55)]. Thus, the inequalities $C_{\overline{T}}\leqslant A\leqslant pC_{\overline{T}}$ in (a) are best possible.

Proof. Let $1<p\leqslant q<\infty$. If (5.5) holds, then

$$(\forall f) \quad \int\limits_{\overline{T}} \mathcal{I}f\,d\nu \leqslant U_{\overline{T}}^{1/q'}(o)\Big(\int\limits_{\overline{T}}[\mathcal{I}f]^q\,d\nu\Big)^{1/q} \leqslant AU_{\overline{T}}^{1/q'}(o)\Big(\sum_T vf^p\Big)^{1/p}$$

by the Hölder inequality. By Lemma 5.1 this yields $B_{\overline{T}}^{1/p'}(o)=\beta\leqslant AU_{\overline{T}}^{1/q'}(o)$. Applying this result to all trees (S_x,x), we obtain $C_{\overline{T}}\leqslant A<\infty$.

And vice versa, let $C_{\overline{T}}<\infty$. Our derivation of (5.5) is similar to the derivation of the implications (2.10) ⇒ (2.11) ⇒ (2.5) in [16] by means of an analogue of formula (3.10) for the Riesz potentials. It is also similar to the proof of inequality (1.3) with $p=q$ in [27].

Let a function $f : T \to [0,\infty)$ be such that $\sum_T vf^p \leqslant 1$ and the set $\{f \neq 0\}$ is finite. For $F = \mathcal{I}f$ and $x \in \overline{T}$ we have

$$F^q(x) \leqslant q \sum_{P_{x,\overline{T}}} F^{q-1} f.$$

For $x \in T$ this estimate was provided in the proof of Theorem 3.3, while for $x = (x_i)_0^\infty \in \partial T$ it should be applied to x_i and then one should pass to the limit as $i \to \infty$. In the same way,

$$F^r(x) \leqslant r \sum_{P_x} F^{r-1} f, \quad x \in T.$$

By the Fubini theorem and Hölder inequality this implies

$$q^{-1} \int_{\overline{T}} F^q \, d\nu \leqslant W = \sum_T U_{\overline{T}} F^{q-1} f \leqslant \Big(\sum_T U_{\overline{T}}^{p'} v^{1-p'} F^r \Big)^{1/p'},$$

$$r^{-1} \sum_T U_{\overline{T}}^{p'} v^{1-p'} F^r \leqslant \sum_T B_{\overline{T}} F^{r-1} f \leqslant C_{\overline{T}}^{p'} \sum_T U_{\overline{T}}^{p'/q'} F^{r-1} f \leqslant C_{\overline{T}}^{p'} \Big(\sup_T U_{\overline{T}}^{1/q} F \Big)^{r-q} W,$$

since $p'/q' - 1 = (r-q)/q$. By the Hölder inequality and estimate (5.2) we have

$$F(x) = \sum_{P_x} f \leqslant \Big(\sum_{P_x} v^{1-p'} \Big)^{1/p'} = V^{1/p'}(x), \qquad \sup_T U_{\overline{T}}^{1/q} F \leqslant D_{\overline{T}} \leqslant q^{1/p'} C_{\overline{T}},$$

$$W \leqslant \big\{ r C_{\overline{T}}^{p'} (q^{1/p'} C_{\overline{T}})^{r-q} W \big\}^{1/p'}, \qquad \int_{\overline{T}} F^q \, d\nu \leqslant qW \leqslant q^{1+(r-q)(p-1)/p'} r^{p-1} C_{\overline{T}}^q.$$

The passage to the limit gives the obtained estimate for all f with $\sum_T vf^p \leqslant 1$ and this implies relations (5.5) and (5.6). The proof of Statement (a) is complete.

Statement (b) follows from (a) and inequalities (5.2) and (5.3).

We note that criterion (2.2) is implied by (a) and inequalities (5.2) and (5.4). The proof is complete. □

We proceed to an interpretation of condition (2.12) in terms of the Hardy inequality.

Lemma 5.3. *The relation*

$$(\forall x \in T_\infty) \quad N(x) = \sum_{R_x \cap T_\infty} N \tag{5.7}$$

holds true, where $T_\infty = \big\{ x \in T \colon S_{x,\partial T} \neq \varnothing \big\}$ *and* $N(x) = \nu(S_{x,\partial T})$.

And vice versa, if the sets $R_x \cap T_\infty$ *are finite and a function* $N : T_\infty \to [0,\infty)$ *satisfies* (5.7), *then there exists a unique finite measure* ν *on the* σ*–algebra* $\mathfrak{S}\big|_{\partial T}$ *such that* $\nu(S_{x,\partial T}) = N(x)$ *as* $x \in T_\infty$.

Proof. The first statement of the lemma is implied by the property

$$(\forall x \in T) \quad S_{x,\partial T} = \bigcup_{y \in R_x \cap T_\infty} S_{y,\partial T} \quad \text{(the union is disjunctive)} \tag{5.8}$$

(corollary from (5.1)) and the countably additivity of the measure ν.

And vice versa, let the sets $R_x \cap T_\infty$ be finite and let (5.7) hold. We let

$$\mathfrak{S}_{\partial T} = \big\{ X \subset \partial T \colon X = \varnothing \quad \text{or} \quad X = S_{x,\partial T} \quad \text{for some} \quad x \in T_\infty \big\}.$$

For $X, Y \in \mathfrak{S}_{\partial T}$ either $X \cap Y = \varnothing$ or there exists $(x_i)_0^\infty \in X \cap Y$. In the second case $X = S_{x_i,\partial T}$ and $Y = S_{x_j,\partial T}$ for some i and j, while x_i and x_j are comparable. Hence, in each case $X \cap Y \in \{\varnothing, X, Y\}$ and the family $\mathfrak{S}_{\partial T}$ is closed with respect to the intersections.

If $X \supsetneq Y \neq \varnothing$, then we necessarily have $i < j$. Applying formula (5.8) to the points $x = x_i, \ldots, x_{j-1}$, we construct a finite expansion $X = \bigcup_{k=1}^n Y_k$ with $Y_1 = Y$ and mutually disjoint $Y_k \in \mathfrak{S}_{\partial T}$. In the terminology from [36, Sect. I.5] this means that $\mathfrak{S}_{\partial T}$ is a semi–ring generating the σ–algebra $\mathfrak{S}\big|_{\partial T}$. This is why the results on the Lebesgue continuation of the measures from [36, Sect. V.3] will prove the lemma if we confirm that there exists a unique countably additive function ν on $\mathfrak{S}_{\partial T}$ with the property $\nu(S_{x,\partial T}) = N(x)$.

We consider the set $Z \subset T_\infty$. Let $x \in T_\infty$ be such that

$$x \notin \bigcup_{z \in Z} (S_z \setminus \{z\}), \tag{5.9a}$$

$$S_{x,\partial T} = \bigcup_{z \in Z\colon S_{z,\partial T} \subset S_{x,\partial T}} S_{z,\partial T} \quad \text{(the union is disjunctive)}, \tag{5.9b}$$

$$N(x) \neq \sum_{z \in Z\colon S_{z,\partial T} \subset S_{x,\partial T}} N(z). \tag{5.9c}$$

We take an arbitrary $y \in R_x \cap T_\infty$. If $y \in S_z \setminus \{z\}$ for some $z \in Z$, then $x \in S_z$ and hence $x = z$ by (5.9a). It follows from (5.9b) that relation (5.9c) becomes the contradiction $N(x) \neq N(x)$. Hence,

$$y \notin \bigcup_{z \in Z} (S_z \setminus \{z\}).$$

Let us show that

$$S_{y,\partial T} = \bigcup_{z \in Z\colon S_{z,\partial T} \subset S_{y,\partial T}} S_{z,\partial T} \quad \text{(the union is disjunctive)}. \tag{5.10}$$

The embedding "$\supset$" is obvious, while the disjunctive property is implied by (5.9b). We consider an arbitrary $(x_i)_0^\infty \in S_{y,\partial T}$. We have $x = x_j$ and $y = x_{j+1}$ for some j, and it follows from (5.9b) that $z = x_k \in Z$ for some k. It follows from (5.9a) that $j \leqslant k$. The identity $x = z$ gives a contradiction as above and this is why $j < k$ and $(x_i)_0^\infty \in S_{z,\partial T} \subset S_{y,\partial T}$. This completes the proof of statement (5.10).

By (5.8), (5.9b) and (5.10) there is a disjunctive union

$$\{z \in Z\colon S_{z,\partial T} \subset S_{x,\partial T}\} = \bigcup_{y \in R_x \cap T_\infty} \{z \in Z\colon S_{z,\partial T} \subset S_{y,\partial T}\}.$$

Suppose that for each $y \in R_x \cap T_\infty$

$$N(y) = \sum_{z \in Z\colon S_{z,\partial T} \subset S_{y,\partial T}} N(z).$$

Then

$$N(x) = \sum_{R_x \cap T_\infty} N = \sum_{y \in R_x \cap T_\infty} \sum_{z \in Z\colon S_{z,\partial T} \subset S_{y,\partial T}} N(z) = \sum_{z \in Z\colon S_{z,\partial T} \subset S_{x,\partial T}} N(z),$$

which contradicts (5.9c). Hence, for some $y \in R_x \cap T_\infty$ instead of x properties (5.9) hold true.

A multiple application of this statement provides $n \geqslant 0$ and $(x_i)_0^\infty \in \partial T$ such that $[o, x] = (x_i)_{i=0}^n$ and (5.9) holds for all x_i ($i \geqslant n$) instead of x. We have $(x_i)_0^\infty \in S_{x,\partial T}$ by $x = x_n$ and this is why $z = x_j \in Z$ for some j by (5.9b). At the same time $j \geqslant n$ due to (5.9a). However $x_{j+1} \in S_z \setminus \{z\}$ in contradiction with (5.9a). Hence, $x \in T_\infty$ can not satisfy all conditions (5.9), that is,

$$\text{(5.9a) \& (5.9b)} \quad \Rightarrow \quad N(x) = \sum_{z \in Z\colon S_{z,\partial T} \subset S_{x,\partial T}} N(z). \tag{5.11}$$

Now we consider different $x, y \in T_\infty$ such that $S_{x,\partial T} = S_{y,\partial T}$. This is possible, for instance, as $(T, o) = (\mathbb{N}, 1)$. Swapping x and y if this is needed, we suppose that $x \notin S_y \setminus \{y\}$. For $Z = \{y\}$ implication (5.11) shows that $N(x) = N(y)$. Hence, the definition

$$\nu(\varnothing) = 0 \quad \& \quad \nu(S_{x,\partial T}) = N(x)$$

of the set function $\nu : \mathfrak{S}_{\partial T} \to [0, \infty)$ is correct.

While checking that ν is countably additive, it is sufficient to consider only non–empty sets. Suppose that we are given a disjunctive union

$$S_{x,\partial T} = \bigcup_{z \in Z} S_{z,\partial T},$$

where $Z \subset T_\infty$, and $x \in T_\infty$ is minimal among all points x with a given $S_{x,\partial T}$. This minimality and the embeddings $S_{z,\partial T} \subset S_{x,\partial T}$ yield that (5.9a) is true. Hence, (5.11) proves the countably additive property of ν and Lemma 5.3. The proof is complete. □

Let a measure μ in $\mathbb{R}^n$ be supported by a binary cube $K \in \mathcal{D}$. We let

$$T = \mathcal{D}(K) \quad \text{(see (2.13))} \quad \& \quad o = K \quad \& \quad N(Q) = \mu(Q) \quad (Q \in T).$$

The set $R_Q \cap T_\infty = R_Q$ consists of 2^n cubes forming a partition of the cube Q and this gives (5.7). By Lemma 5.3, on ∂T a measure ν is defined and it is such that

$$U_{\overline{T}}(Q) = \nu(S_{Q,\overline{T}}) = \nu(S_{Q,\partial T}) = N(Q) = \mu(Q),$$

where we have continued nu by zero on T: $\nu(T) = 0$. For $v(Q) = \ell_Q^{n-pl}$ we have

$$B_{\overline{T}}(P) = \sum_{Q \in \mathcal{D}\colon Q \subset P} \mu(Q)^{p'} \ell_Q^{p'(l-n)+n} \quad (0 < l < n/p = n/q),$$

which is equal to the left hand side in (2.12). By $p'(l-n)+n < 0$ condition (2.12) is equivalent to the same condition (with another constant A_7), where P ranges over the set $\mathcal{D}(K)$. Hence, conditions (2.5) and (2.12) are equivalent to the condition $C_{\overline{T}} < \infty$ and by Theorem 5.1(a), to corresponding Hardy inequality (5.5).

We are going to derive an analogue of Theorem 5.1 for a binary family $\mathcal{D}$, when the "measure is supported on the boundary" and "the root is moved to infinity".

Suppose that we are given $p, q \in (1, \infty)$, a measure μ in $\mathbb{R}^n$ $(n \in \mathbb{N})$ and a function $v : \mathcal{D} \to (0, \infty)$. For a point $x \in \mathbb{R}^n$, a cube $Q \in \mathcal{D}$ and a function $f : \mathcal{D} \to [0, \infty)$ we denote

$$P_{x,\mathcal{D}} = \{Q \in \mathcal{D}\colon x \in Q\}, \qquad P_{Q,\mathcal{D}} = \{P \in \mathcal{D}\colon P \supset Q\},$$

$$U_{\mathcal{D}}(Q) = \mu(Q), \qquad V_{\mathcal{D}}(Q) = \sum_{P_{Q,\mathcal{D}}} v^{1-p'},$$

$$\mathcal{I}_{\mathcal{D}} f(x) = \sum_{P_{x,\mathcal{D}}} f, \qquad B_{\mathcal{D}}(Q) = \sum_{\mathcal{D}(Q)} U_{\mathcal{D}}^{p'} v^{1-p'},$$

$$C_{\mathcal{D}} = \sup_{\mathcal{D}} B_{\mathcal{D}}^{1/p'} U_{\mathcal{D}}^{-1/q'}, \qquad D_{\mathcal{D}} = \sup_{\mathcal{D}} U_{\mathcal{D}}^{1/q} V_{\mathcal{D}}^{1/p'}.$$

The followings result hold true.

Lemma 5.4. *For each $Q \in \mathcal{D}$*

$$\sup_{f\colon \{f \neq 0\} \subset \mathcal{D}(Q)} \frac{\int_Q \mathcal{I}_{\mathcal{D}} f \, d\mu}{\Big(\sum_{\mathcal{D}(Q)} v f^p\Big)^{1/p}} = B_{\mathcal{D}}^{1/p'}(Q).$$

Lemma 5.5. *The estimates*

$$D_{\mathcal{D}} \leqslant q^{1/p'} C_{\mathcal{D}},$$
$$p < q \quad \Rightarrow \quad C_{\mathcal{D}} \leqslant 2(2^{p'/q'-1} - 1)^{-1/p'} D_{\mathcal{D}}$$

are valid.

Theorem 5.2. *(a) If $1 < p \leqslant q < \infty$, then the Hardy inequality*

$$(\exists A \geqslant 0)\ (\forall f : \mathcal{D} \to [0, \infty)) \quad \Big(\int_{\mathbb{R}^n} [\mathcal{I}_{\mathcal{D}} f]^q \, d\mu\Big)^{1/q} \leqslant A \Big(\sum_{\mathcal{D}} v f^p\Big)^{1/p} \tag{5.12}$$

is equivalent to the condition $C_{\mathcal{D}} < \infty$. At the same time

$$C_{\mathcal{D}} \leqslant A \leqslant q^{1/q+(p'-q')(p-1)/(p'q')} r^{(p-1)/q} C_{\mathcal{D}} \quad (r = p'(q-1) \geqslant q)$$

for the best A in (5.12)*. In particular, $C_{\mathcal{D}} \leqslant A \leqslant pC_{\mathcal{D}}$ as $p = q$.*

(b) If $1 < p < q < \infty$, then (5.12) $\Leftrightarrow D_{\mathcal{D}} < \infty$.

The proof of these results is almost identical to the proof of Lemmas 5.1 and 5.2 and Theorem 5.1. The proof of Theorem 5.2 uses the estimate

$$F^s(x) \leqslant s \sum_{P_{x,\mathcal{D}}} F^{s-1} f \quad (s \in \{q, r\} \quad \text{and} \quad x \in \mathbb{R}^n \cup \mathcal{D})$$

for a function $F(x) = \sum_{P_{x,\mathcal{D}}} f$, where the set $\{f \neq 0\}$ is finite. It is a particular case of a similar inequality for the tree $T = \mathcal{D}(K)$, where the cube $K \in \mathcal{D}$ is sufficiently big so that $\mathcal{D}(K) \supset P_{x,\mathcal{D}} \cap \{f \neq 0\}$.

Let $0 < l < n/p = n/q$ and $v(Q) = \ell_Q^{n-lp}$. Then, as above, the number $B_{\mathcal{D}}(P)$ is equal to the left hand side of condition (2.12), and condition (2.12) means that $C_{\mathcal{D}} \leqslant A_7$. By Theorem 5.2(a) conditions (2.5) and (2.12) are equivalent to Hardy inequality (5.12). This equivalence can be used to transform the counterexamples for Adams theorem as $p = q$, see [20] or [30], into counterexamples for Theorem 5.2(b) as $p = q$.

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