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GEOMETRY OF SUB–RIEMANNIAN MANIFOLDS EQUIPPED WITH A SEMIMETRIC QUARTER–SYMMETRIC CONNECTION

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Abstract. On a sub–Riemannian manifold we introduce a semimetric quarter–symmetric connection by defining intrinsic metric connection and two structural endomorphisms preserving the distribution on a sub–Riemannian manifold. We find conditions ensuring the metric property of the introduced connection. We clarify the nature of the structural endomorphisms of semimetric connection consistent with a sub–Riemannian quasi–static structure defined on non-holonomic Kenmotsu manifold and on almost quasi–Sasakian manifold. We find conditions, under which the mentioned manifolds are Einstein manifolds with respect to the quarter–symmetric connection.

Keywords: quarter–symmetric connection, sub–Riemannian quasi–static structure, non–holonomic Kenmotsu manifold, almost quasi–Sasakian manifold.

Mathematics Subject Classification: 53B20

1. Introduction

A lot of works was devoted to studying almost contract metric manifolds equipped with a metric connection with a torsion, in particular, with metric quarter–symmetric connections, see [1], [4], [5], [7]–[11], [23]. E. Cartan was first who considered a linear metric connection with a torsion together with the Levi–Civita connection [13]. The most interest among the connections with a torsion is attracted by a semi–symmetric connection [7], [9], [10], a thorough study of which was carried out by K. Yano in work [22]. A quarter–symmetric connection was defined by S. Golab in 1975 [19].

An interest of researches to the connections with a torsion is mostly motivated by the employing of such connections in theoretical physics [14], [16], [20]. In work [5] it was pointed out that the most of studied connections with a torsion can be described by means of endomorphisms preserving the distributions on almost contact metric manifolds.

In the present work, on a sub–Riemannian manifold of a contact type M, we consider a semi–metric quarter–symmetric connection D_X, which is associated with a triple (∇, N, S), where ∇ is an intrinsic metric connection, while N and S are endomorphisms preserving the distribution D. In what follows we usually omit the term “semi–metric”.

The notion of a contact sub–Riemannian manifold is employed in work [15]. A contact sub–Riemannian manifold is a smooth manifold M of an odd dimension $n = 2m + 1$ with a defined on it maximal non–integrable distribution D of codimension 1. On the distribution D a positive definite metric is given, which defines the scalar product only for the vectors of the distribution. Let η be a differential 1–form generating the distribution $D : \ker(\eta) = D$. Then by means the identities $\eta(\vec{\xi}) = 1$, $i_{\vec{\xi}}\omega = 0$ on the manifold M we define a unique vector

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distribution $\vec{\xi}$, which is transversal to the distribution D. Here $\omega = d\eta$ is a differential 2–form of rank $2m$. Then the manifold M naturally becomes a Riemannian manifold once we suppose that $\vec{\xi}$ is a unit vector field, which is orthogonal to the distribution D with respect to the metric of the manifold M. In the present work we consider a more general notion than the contact sub–Riemannian manifold. We study sub–Riemannian manifolds of a contact type. The manifold M is initially assumed to be Riemannian, the existence of the vector field $\vec{\xi}$ is postulated, while the distribution D can have an arbitrary degree of integrability, in particular, it can be involutive. The idea of generalization of the notion of the contact sub–Riemannian manifold used in work [15] is motivated by the need of using a wide class of almost contact metric structures appearing in specifying the sub–Riemannian structure. The structures of non–holonomic Kenmotsu manifold and of quasi–Sasakian manifolds, which are employed in the present work, are obtained from the structure of the sub–Riemannian manifold of contact type by introducing a special structural endomorphism.

The quarter–symmetric connection D_X, which is studied in this work, is expressed via Levi–Civita connection $\tilde{\nabla}$ by means of the following identity

$$D_X Y = \tilde{\nabla}_X Y + C(X,Y)\vec{\xi} + \eta(X)(N - C - \psi)Y + \eta(Y)(S - C - \psi)X.$$

Here an endomorphism $\psi : TM \to TM$ is defined by the identity $\omega(X,Y) = g(\psi X, Y)$, while N, $S : TM \to TM$ are endomorphisms of the tangent bundle of the manifold M such that

$$N\vec{\xi} = \vec{0}, \qquad N(D) \subset D, \qquad S\vec{\xi} = \vec{0}, \qquad S(D) \subset D.$$

The following relations also hold true:

$$C(X,Y) = \frac{1}{2}(L_{\vec{\xi}}g)(X,Y), \qquad g(CX,Y) = C(X,Y).$$

Such definition of the connection D_X is a very perspective generalization of the connection, which is traditionally defined in an almost contact manifold by the identity

$$D_X Y = \tilde{\nabla}_X Y - \eta(X)\phi Y.$$

The torsion $T(X,Y)$ of the defined connection D_X reads as

$$T(X,Y) = \eta(X)\tilde{N}Y - \eta(Y)\tilde{N}X.$$

Here $\tilde{N} = N - S$.

The work consists of four sections. In each section we determine the nature of the structural endomorphisms N and S in view of the features of the studied structure, the Riemannian structure, sub–Riemannian quasi–static structure, the nature of the non–holonomic Kenmotsu manifold and the nature of the almost quasi–Sasakian manifold. We find the conditions ensuring the metricity of the introduced connection. We find out the nature of the structural endomorphisms of semi–metric connection consistent with the sub–Riemannian quasi–static structure. We study the properties of the semi–metric connection defined on the non–holonomic Kenmotsu manifold and on almost quasi–Sasakian manifold. We find conditions, under which the mentioned manifolds are Einstein ones with respect to the quarter–symmetric connection.

We note that the notion of the structures of the quasi–static manifold, of non–holonomic Kenmotsu manifold and of almost quasi–Sasakian manifold were introduced by the authors of the present work, see [1], [3], [6].

2. Defining quarter–symmetric connection on sub–Riemannian manifolds of contact type

Let M be a Riemannina manifold of an odd dimension $n = 2m + 1$ with a given on it sub–Riemannian structure $(\vec{\xi}, \eta, g, D)$ of contact type, where g is a metric tensor defined on the manifold M, while η and $\vec{\xi}$ are 1–form and unit vector field generating respectively mutually orthogonal distributions D and $D^{\perp}$:

$$\ker(\eta) = D, \qquad D^{\perp} = \langle\vec{\xi}\rangle.$$

We postulate that

$$\omega(\vec{\xi}, \cdot\,) = d\eta(\vec{\xi}, \cdot\,) = 0, \qquad \mathrm{rk}(\omega) \geqslant 2.$$

In what follows we call M a sub–Riemannian manifold.

Throughout the paper we actively use adapted coordinates. A chart $k(x^i)$, $i, j, k = 1, \dots, n$, $a, b, c = 1, \dots, 2m$, of the manifold M is called adapted to the distribution D if $\partial_n = \vec{\xi}$. Here $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = (\partial_1, \dots, \partial_n)$ is the field of the frames defined by the adapted chart.

Let $P : TM \to D$ be the projection defined by the expansion $TM = D \oplus D^{\perp}$ and $k(x^i)$ be an adapted chart. The vector fields $P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^n_a \partial_n$ are linearly independent at each point and in the domain of the corresponding chart they generate a distribution D, which reads $D = \mathrm{Span}(\vec{e}_a)$. With a non–holonomic field of bases $(\vec{e}_i) = (\vec{e}_a, \partial_n)$ we associate a field of cobases $(dx^a, \eta = \theta^n = dx^n + \Gamma^n_a dx^a)$.

For adapted charts $k(x^i)$ and $k'(x^{i'})$ the following coordinate transform formulas hold: $x^a = x^a(x^{a'})$, $x^n = x^{n'} + x^n(x^{a'})$.

A tensor field t of type (p, q) defined on a sub–Riemannian manifold is called admissible (to the distribution D) or transversal if t vanishes as soon as $\vec{\xi}$ or η is among its variables. A coordinate representation of an admissible tensor field in an adapted chart reads as

$$t = t^{a_1 \dots a_p}_{b_1 \dots b_q} \vec{e}_{a_1} \otimes \dots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_q}.$$

The transformation of the entries of an admissible tensor field in the adapted coordinates follows the law $t^a_b = A^a_{a'} A^{b'}_b t^{a'}_{b'}$, where $A^{a'}_a = \frac{\partial x^{a'}}{\partial x^a}$.

The transformation formulas for the entries of admissible field yield that the derivatives $\partial_n t^a_b$ are the entries of an admissible tensor field of the same type. We observe that vanishing of the derivatives $\partial_n t^a_b$ is independent of the choice of the adapted coordinates.

The adapted coordinates play a role of holonomic coordinates for a non–involutive distribution. The identity $[\vec{e}_a, \vec{e}_b] = 2\omega_{ba}\vec{\xi}$ holds true. In particular, this implies a statement, which is important for further consideration: the condition $d\eta(\vec{\xi}, \cdot\,) = 0$ is equivalent to the identity $\partial_n \Gamma^n_a = 0$.

An intrinsic linear connection ∇ [17], [18] on sub–Riemannian manifold is a mapping $\nabla : \Gamma(D) \times \Gamma(D) \longrightarrow \Gamma(D)$ obeying the following conditions:

1) $\nabla_{f_1 X + f_2 Y} = f_1 \nabla_X + f_2 \nabla_Y$,
2) $\nabla_X f Y = (Xf)Y + f\nabla_X Y$,
3) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,

where $\Gamma(D)$ is the module of admissible vector fields, which are vector fields belonging to the distribution D at each point.

The coefficients of intrinsic linear connection are determined by the relation $\nabla_{\vec{e}_a}\vec{e}_b = \Gamma^c_{ab}\vec{e}_c$. The identity $\vec{e}_a = A^{a'}_a \vec{e}_{a'}$, where $A^{a'}_a = \frac{\partial x^{a'}}{\partial x^a}$, implies in a usual way a formula for transforming the coefficients of the intrinsic connection:

$$\Gamma^c_{ab} = A^{a'}_a A^{b'}_b A^c_{c'} \Gamma^{c'}_{a'b'} + A^c_{c'} \vec{e}_a A^{c'}_b .$$

In particular this implies that the derivatives $\partial_n \Gamma^d_{ac}$ are the components of an admissible tensor field.

A torsion and a curvature of the intrinsic connection are respectively admissible tensor fields

$$T(X,Y) = \nabla_X Y - \nabla_Y X - P[X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{P[X,Y]} Z - P[Q[X,Y],Z],$$

where $Q = I - P$, $X,Y,Z \in \Gamma(D)$.

The tensor $R(X,Y)Z$ is called a Schouten curvature tensor of a sub–Riemannian manifold. The entries of the Schouten curvature tensor in the adapted coordinates are determined by the identity

$$R^d_{abc} = 2\vec{e}_{[a}\Gamma^d_{b]c} + 2\Gamma^d_{[a|e|}\Gamma^e_{b]c}.$$

The following proposition holds true.

Proposition 2.1. *On a sub–Riemannian manifold there exists a unique connection* ∇ *with zero torsion such that* $\nabla_X g(Y,Z) = 0$, $X,Y,Z \in \Gamma(D)$.

The proof of the above proposition almost literally reproduces the proof of existence and uniqueness theorem for the Levi–Civita connection.

We call the connection ∇ an intrinsic metric connection. The coefficients of the intrinsic metric connection are given by the formulas

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}).$$

Let $\tilde{\nabla}$ be a Levi–Civita connection.

Proposition 2.2. *The coefficients* $\tilde{\Gamma}^k_{ij}$ *of the Levi–Civita connection* $\tilde{\nabla}$ *of a sub–Riemannian manifold in the adapted coordinates read as*

$$\tilde{\Gamma}^c_{ab} = \Gamma^c_{ab}, \qquad \tilde{\Gamma}^n_{ab} = \omega_{ba} - C_{ab}, \qquad \tilde{\Gamma}^b_{an} = \tilde{\Gamma}^b_{na} = C^b_a + \psi^b_a, \qquad \tilde{\Gamma}^n_{na} = -\partial_n \Gamma^n_a, \qquad \tilde{\Gamma}^a_{nn} = g^{ab}\partial_n \Gamma^n_b,$$

where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} + \vec{e}_c g_{bd} - \vec{e}_d g_{bc}), \qquad \psi^b_a = g^{bc}\omega_{ac}, \qquad C_{ab} = \frac{1}{2}\partial_n g_{ab}, \qquad C^b_a = g^{bc} C_{ac}.$$

Proof. We recall that the endomorphism $\psi : TM \to TM$ is determined by the identity $\omega(X,Y) = g(\psi X, Y)$. Moreover, the following relations hold:

$$C(X,Y) = \frac{1}{2}(L_{\vec{\xi}} g)(X,Y), \quad g(CX,Y) = C(X,Y).$$

The proof is reduced to applying a known formula for the coefficients of the Levi–Civita connection in a non–holonomic frame (A_i) :

$$2\Gamma^m_{ij} = g^{km}(A_i g_{jk} + A_j g_{ik} - A_k g_{ij} + \Omega^l_{kj} g_{li} + \Omega^l_{ki} g_{lj}) + \Omega^m_{ij}.$$

□

The object Ω^m_{ij} is called nonholonomity object of the frame (A_i). The field of the frames $(\vec{e}_i) = (\vec{e}_a, \partial_n)$, which is used in this work, satisfies the identity $\Omega^n_{ab} = 2\omega_{ba}$.

Remark 2.1. *The coefficients* $\tilde{\Gamma}^k_{ij}$ *become simpler if we adopt the condition* $d\eta(\vec{\xi}, X) = 0$. *In this case* $\tilde{\Gamma}^n_{na} = -\partial_n \Gamma^n_a = 0$ *and* $\tilde{\Gamma}^a_{nn} = g^{ab}\partial_n \Gamma^n_b = 0$.

The intrinsic connection provides a parallel transport of admissible vectors (the vectors, which belong to the distribution D) along admissible curves (the curves, which touch the distribution D at each point). At the same time, in order to solve a series of problem, there appears a need to extend the intrinsic connection to a connection on the entire manifold. Sometimes it is sufficient to have an intermediate construction, a connection in the vector bundle (M, π, D). There exist various ways of continuing the intrinsic connection. In papers [1], [4]–[6], [12],

[17], [18] a so–called N–connection ∇^N is discussed. On a sub–Riemannian manifold M the N–connection ∇^N is defined by a pair (∇, N), where ∇ is the intrinsic metric connection, $N : TM \to TM$ is the endomorphism of the tangent bundle of the manifold M such that $N\vec{\xi} = \vec{0}$, $N(D) \subset D$.

We define a semi–metric quarter–symmetric connection D_X on a sub–Riemannian manifold by the following identity:

$$D_X Y = \tilde{\nabla}_X Y + C(X, Y)\vec{\xi} + \eta(X)(N - C - \psi)Y + \eta(Y)(S - C - \psi)X,$$

where the endomorphism $\psi : TM \to TM$ is defined by the identity $\omega(X, Y) = g(\psi X, Y)$, while N, $S : TM \to TM$ are the endomorphisms of the tangent bundle of the manifold M such that $N\vec{\xi} = \vec{0}$, $N(D) \subset D$, $S\vec{\xi} = \vec{0}$, $S(D) \subset D$.

It follows from the definition of the quarter–symmetric connection D_X that its torsion $T(X, Y)$ is given by the identity

$$T(X, Y) = \eta(X)\tilde{N}Y - \eta(Y)\tilde{N}X.$$

Here $\tilde{N} = N - S$.

The following proposition holds true.

Proposition 2.3. *Nonzero coefficients G^k_{ij} of the connection D_X of a sub–Riemannian manifold in the adapted coordinates read as*

$$G^c_{ab} = \tilde{\Gamma}^c_{ab}, \qquad G^n_{ab} = \omega_{ba}, \qquad G^b_{na} = N^b_a, \quad G^b_{an} = S^b_a.$$

In order to prove the proposition it is sufficient to substitute the corresponding basis vectors into the formula for the quarter–symmetric connection D_X. For instance, if $X = \vec{e_a}$, $Y = \vec{e_b}$, then we get

$$G^c_{ab}\vec{e}_c + G^n_{ab}\partial_n = \tilde{\Gamma}^c_{ab}\vec{e}_c + \tilde{\Gamma}^n_{ab}\partial_n + C_{ab}\partial_n.$$

In view of Proposition 2 we obtain $G^c_{ab} = \tilde{\Gamma}^c_{ab}$, $G^n_{ab} = \omega_{ba}$.

We are going to find out under which restrictions for the endomorphisms N, S the connection D_X is a metric one. It is straightforward to confirm that if $N = C$, then $D_n g_{ab} = 0$. We then have

$$D_a g_{nb} = -G^c_{an} g_{cb} - G^n_{ab} = -S^c_a g_{cb} - \omega_{ba} = 0.$$

This implies $S = \psi$.

Proposition 2.4. *The quarter–symmetric connection D_X associated with the triple (∇, C, S) is a metric connection if and only if $S = \psi$.*

Remark 2.2. *Here we consider the case $N = C$ due to the reason that exactly in this case we succeed to obtain a metric connection, which is an independent important fact.*

3. Sub–Riemannian quasi–static manifolds equipped with the semi–metric quarter–symmetric connection

In work [3], on a non–holonomic Kenmotsu manifold, a sub–Riemannian quasi–static structure was introduced and studied. Non–holonomic Kenmotsu manifolds form a special class of sub–Riemannian manifolds of contact type. The case of the sub–Riemannian quasi–static structure defined on a non–holonomic Kenmotsu manifold is a connection with a torsion of a spectial nature. Such connection is defined by an intrinsic connection and a structural endomorphism, which preserves the distributions on the non–holonomic Kenmotsu manifold. It was shown in work [3] that the intrinsic connection is consistent with the metric induced on the distribution on the considered manifold. The nature of the structural endomorphism was found.

A triplet (M, g, ∇) is called a sub–Riemannian quasi–static structure [3] if the identity

$$\varPhi(X,Y,Z) = \nabla_X g(Y,Z) - \nabla_Y g(X,Z) + \tilde{T}(X,Y,Z) - 2\omega(X,Y)\eta(Z) = 0$$

holds, where $\tilde{T}(X,Y,Z) = g(T(X,Y),Z)$, $X,Y,Z \in \Gamma(TM)$. In the adapted coordinates the non–zero entries of the tensor $\tilde{T}(X,Y,Z)$ read as

$$\tilde{T}(\vec{e}_a, \vec{e}_b, \partial_n) = 0,$$
$$\tilde{T}(\vec{e}_a, \partial_n, \vec{e}_b) = -g(\tilde{N}\vec{e}_a, \vec{e}_b),$$
$$\tilde{T}(\partial_n, \vec{e}_a, \vec{e}_b) = g(\tilde{N}\vec{e}_a, \vec{e}_b).$$

Theorem 3.1. *A quarter–symmetric connection is a connection of a sub–Riemannian quasi–static structure if and only if the identities $N = 2C + \psi$, $g(SX,Y) = g(X,SY)$ hold true.*

The proof of the theorem is reduced to checking the equivalence of the identity $g(SX,Y) = g(X,SY)$ to the identity $\varPhi(\vec{e}_a, \vec{e}_b, \partial_n) = 0$ as well as to the equivalence of the identities $\varPhi(\vec{e}_a, \partial_n, \vec{e}_b) = 0$ and $N = 2C + \psi$.

As an example we consider the case $\Phi(\vec{e}_a, \partial_n, \vec{e}_b) = 0$. Taking into consideration Proposition 2.3 and the expression for $\Phi(X,Y,Z)$, we have:

$$\Phi(\vec{e}_a, \partial_n, \vec{e}_b) = \nabla_a g_{nb} - \nabla_n g_{ab} - g_{cb}\tilde{N}^c_a = -S^c_a g_{cb} - \omega_{ba} - \partial_n g_{ab} + N^c_a g_{cb} + N^c_b g_{ca} - g_{cb}\tilde{N}^c_a = 0.$$

Taking into consideration the identities

$$\psi^b_a = g^{bc}\omega_{ac}, \qquad C_{ab} = \frac{1}{2}\partial_n g_{ab}, \qquad C^b_a = g^{bc}C_{ac},$$

we confirm that the theorem is true.

4. Non–holonomic Kenmotsu manifolds equipped with semi–metric quarter–symmetric connection

A normal almost contact metric manifold M is called a non–holonomic Kenmotsu manifold if the identity $d\Omega = 2\eta \wedge \Omega$ holds true. It is easy to show that the manifold M also satisfies the condition $L_{\vec{\xi}} g = 2(g - \eta \otimes \eta)$. A non–holonomic Kenmotsu manifold was introduced by one of the authors of the present work [1]. In contrast to the classical case of the Kenmotsu manifold [2], [21], the distribution on a non–holonomic Kenmotsu manifold is not supposed to possess involutive property.

The intrinsic geometry of a non–holonomic Kenmotsu manifold M possesses a series of wonderful properties [1]. Earlier it was established that the Schouten–Wagner tensor field vanishes: $P = L_{\vec{\xi}}\Gamma = 0$ [1]. The entries of the Schouten–Wagner field in the adapted coordinates are expressed by the identities $P^c_{ad} = \partial_n \Gamma^c_{ad}$.

We consider the case when the quarter–symmetric connection D_X is associated with the triple (∇, C, ψ). As it was shown in Proposition 2.4, in this case the connection D_X is metric one.

The non–zero coefficients G^k_{ij} of the connection D_X of a non–holonomic Kenmotsu manifold in the adapted coordinates read as

$$G^c_{ab} = \tilde{\Gamma}^c_{ab}, \qquad G^n_{ab} = \omega_{ba}, \qquad G^b_{na} = \delta^b_a, \qquad G^b_{an} = \psi^b_a.$$

Let us calculate the further needed entries of the curvature tensor K of the connection D_X. We have

$$K^d_{abc} = R^d_{abc} - \psi^d_b\omega_{ca} + \psi^d_a\omega_{cb} + 2\omega_{ab}\delta^d_c, \qquad K^n_{anb} = \omega_{ba}.$$

Let $k(X,Y)$ be the Ricci tensor associated with the tensor $K(X,Y)Z$. We have the identity

$$k_{ac} = r_{ac} + \psi^b_a\omega_{cb} + \omega_{ac},$$

where r_{ac} are the entries of the Schouten–Ricci tensor $r(X,Z) = \mathrm{tr}(Y \to R(X,Y)Z)$, $X, Y, Z \in \Gamma(D)$ [1].

Proposition 4.1. *For a non–holonomic Kenmotsu manifold of dimension $n = 2m + 1$ the following identity holds: $r_{[ac]} = 2m\omega_{ca}$.*

Proof. The proof is based on the following identity [1]:

$$\nabla_{[e}\nabla_{a]}g_{bc} = 2\omega_{ea}\partial_n g_{bc} - g_{dc}R^d_{eab} - g_{bd}R^d_{eac}.$$

In the case of the non–holonomic Kenmotus manifold this identity is rewritten as

$$0 = 4\omega_{ea}g_{bc} - g_{dc}R^d_{eab} - g_{bd}R^d_{eac}.$$

Applying necessary transformations and using the algebraic Bianchi identity for the Schouten curvature tensor, we arrive at the identity

$$2m\omega_{ca} = \frac{1}{2}(r_{ac} - r_{ca}).$$

The proof is complete. □

Theorem 4.1. *If a non–holonomic Kenmotsu manifold M is an Einstein manifold with respect to the quariter–symmetric connection D_X, then its dimension is equal to three.*

Proof. Let M be an Einstein manifold with respect to the quarter–symmetric connection D_X : $k_{ij} = \lambda g_{ij}$, $\lambda \in \mathbb{R}$. This implies that

$$k_{ac} = r_{ac} + 2mg_{ac} + 2\omega_{ac} = \lambda g_{ac}.$$

The latter identity yields: $r_{[ac]} = -2\omega_{ac} = 2\omega_{ca}$. Comparing the obtained identity with $r_{[ac]} = 2m\omega_{ca}$, we conclude that $2m = 2$ and $m = 1$. The proof is complete. □

Example 1. Non–holonomic Kenmotsu manifold of dimension 3. Let $M = \mathbb{R}^3$ and (∂_α), $\alpha = 1, 2, 3$, be the standard basis of the arithmetic space. On M we define an 1–form η letting $\eta = dx^3 + x^2dx^1$. Then we let

$$\vec{e}_1 = \partial_1 - x^2\partial_3, \qquad \vec{e}_2 = \partial_2, \qquad \vec{e}_3 = \vec{\xi} = \partial_3, \qquad D = \mathrm{Span}(\vec{e}_1, \vec{e}_2).$$

We define a metric tensor as

$$g(\vec{e}_1, \vec{e}_1) = g(\vec{e}_2, \vec{e}_2) = e^{2x^3}, \qquad g(\vec{e}_3, \vec{e}_3) = 1.$$

We straightforwardly confirm the validity of the identity $L_{\xi}g = 2(g - \eta \otimes \eta)$. We define the first structural endomorphism by the identities

$$\varphi(\vec{e}_1) = \vec{e}_2, \qquad \varphi(\vec{e}_2) = -\vec{e}_1, \qquad \varphi(\vec{e}_3) = \vec{0}.$$

By straightforward calculations we confirm that the non–zero entries of the intrinsic connection are the following ones: $\Gamma^1_{11} = \Gamma^2_{21} = -\Gamma^1_{22} = -x^2$. We then get

$$r_{12} = R^2_{122} = 1, \qquad r_{21} = R^1_{211} = -1.$$

Thus, $r_{12} - r_{21} = 2$. On the other hand, $4\omega_{12} = -2$. The identity

$$k_{12} = r_{12} + 2g_{12} + 2\omega_{12} = 1 + g_{12} - 1 = 2g_{12},$$

in particular implies that M is an Einstein manifold with respect to the quarter–symmetric connection D_X.

5. Almost quasi–Sasakian manifolds equipped with semi–metric quarter–symmetric connection

By an almost quasi–Sasakian manifold we mean an almost normal almost contact metric manifold with a closed fundamental form, which obeys the condition $d\eta(\vec{\xi}, \cdot\,) = 0$ [6], [17].

One of the authors of this paper called an almost contact metric manifold an almost normal if the identity

$$\tilde{N}_\varphi = N_\varphi + 2\varphi * d\eta \otimes \vec{\xi} = 0$$

holds, where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

For a normal almost contact metric space the condition $N_\varphi + 2d\eta \otimes \vec{\xi} = 0$ is satisfied. The expediency of introducing the concept of an almost normal almost contact metric manifold was realized after studying the so–called extended almost contact metric structures that naturally arise on distributions on sub–Riemannian manifolds of contact type [4], [6], [12], [18].

Let a quarter–symmetric connection D_X be associated with the triple (∇, C, ψ). Since for an almost quasi–Sasakian manifold the condition $C = 0$ holds [17], then non–zero coefficients G^k_{ij} of the connection D_X in the adapted coordinates become

$$G^c_{ab} = \tilde{\Gamma}^c_{ab}, \qquad G^n_{ab} = \omega_{ba}, \qquad G^b_{an} = \psi^b_a.$$

The components of the curvature tensor K of the connection D_X cast into the form

$$K^d_{abc} = R^d_{abc} - \psi^d_b\omega_{ca} + \psi^d_a\omega_{cb}, \qquad K^d_{abn} = \nabla_a\psi^d_b - \nabla_b\psi^d_a.$$

Let $k(X,Y)$ be the Ricci tensor associated with the tensor $K(X,Y)Z$. The identities

$$k_{ac} = r_{ac} + \psi^b_a\omega_{cb}, \qquad k_{an} = -\nabla_b\psi^b_a, \qquad k_{nn} = 0$$

hold, where r_{ac} are the entries of the Schouten–Ricci tensor $r(X,Z) = \mathrm{tr}(Y \to R(X,Y)Z)$, $X, Y, Z, \in \Gamma(D)$.

Now let $\nabla\psi = 0$. Then the following theorem turns out to be true.

Theorem 5.1. *Let M be an almost quasi–Sasakian manifold and let the structural endomorphism ψ be covariantly constant with respect to the intrinsic connection. Then the manifold M is an Einstein manifold with respect to the quarter–symmetric connection if and only if the identity*

$$r_{ac} + \psi^b_a\omega_{cb} = 0$$

holds true.

Example 2. Almost quasi–Sasakian Einstein manifold. As a simplest example of an almost quasi–Sasakian Einstein manifold we consider a cosymplectic manifold [21]. On the manifold $M = \mathbb{R}^5$ we introduce a cosymplectic structure by letting

1) $D = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \rangle$, where $\vec{e}_1 = \partial_1$, $\vec{e}_2 = \partial_2$, $\vec{e}_3 = \partial_3$, $\vec{e}_4 = \partial_4$, and $(\partial_1, \partial_2, \partial_3, \partial_4, \partial_5)$ is the natural basis in $\mathbb{R}^5$,

2) $\vec{\xi} = \partial_5$,

3) $\eta = dv$,

4) $\varphi\vec{e}_1 = \vec{e}_2$, $\varphi\vec{e}_2 = -\vec{e}_1$, $\varphi\vec{e}_3 = \vec{e}_4$, $\varphi\vec{e}_4 = -\vec{e}_3$, $\varphi\vec{\xi} = 0$,

5) in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{\xi})$ the metric tensor is defined by the identity $g = (dx)^2 + (dy)^2 + (dz)^2 + (du)^2 + (dv)^2$.

In the considered case the identity $r_{ab} = \omega_{da}\psi^d_b$ is naturally satisfied since both its left hand side and right hand side vanish.

6. Conclusion

In this paper we introduce a semi–metric quarter–symmetric connection on sub–Riemannian manifold of contact type by defining an intrinsic metric connection and two structural endomorphisms, which preserve the distribution on the sub–Riemannian manifold. The work consistently develops the idea of the fundamental role of intrinsic geometry in the context of the study of almost contact metric structures [12], [17]. Briefly speaking, intrinsic geometry is responsible for the parallel transport of the admissible vectors along the admissible curves. To the intrinsic geometry of almost contact metric manifolds we also include the endomorphisms studied in this article, which preserve the distributions on sub–Riemannian manifolds. It is impossible to describe in a limited work the variety of currently existing approaches allowing determine connections with torsion in almost contact metric spaces. This was partially done in work [5]. At the same time, Proposition 2.2 indicates the possibility of constructing connections with a torsion by introducing additional admissible tensor fields into the geometry of the studied manifolds. In our case, these are endomorphisms N, S.

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