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PROBLEM ON STRING SYSTEM VIBRATIONS ON STAR-SHAPED GRAPH WITH NONLINEAR CONDITION AT NODE

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Abstract. We consider a system of n strings being in the equilibrium position along a geometrical star-graph. We suppose that the edges of the graph have the same lengths and the graph is oriented to the node. We study the case when the initial velocity of each string is zero. The initial shape of each string is defined by means of given functions on the edges. We assume that at the boundary vertices the strings are fixed. We study the oscillatory process for the case, when the node point of the string system is located inside the motion limiter. At the same time we suppose that the limiter can move in the direction perpendicular to the graph plane. While the limiter does not touch the node point of the string system, the transmission condition holds (the Kirchoff condition). Once the limiter touches the node, their joint motion begins and an additional restriction for the sign of the sum of derivatives at the node appears. Thus, at the node, a hysteresis type condition is satisfied.

In the work we obtain a representation for the solution and prove its existence. For a particular case we consider a case on periodic oscillations of the node point of the string system. We solve a problem on boundary control of the oscillatory process under the assumption that the oscillation time does not exceed the length of the strings.

Keywords: wave equation, oscillatory process, hysteresis, geometric graph.

Mathematics Subject Classification: 35L05, 35L20, 35R02

1. Introduction

Differential equations on spatial networks (geometric graphs), which attracted the attention of mathematicians several decades ago, are relevant in many areas of technology and natural science. They arise when describing phenomena in continuous systems with a network-like structure (electrical, hydraulic, acoustic networks, heat pipes, waveguides, neural and computing systems, elastic lattice structures, electronic systems, etc.). An active mathematical interest in studying such problems led to the appearance of numerous publications; we mention works [1]-[3], [7]-[15], [17]-[21]. However, in all these works problems with linear boundary conditions were considered. In papers [2], [3] the study of problems on deformations of string systems on graphs with various nonlinear conditions was initiated. However, oscillatory processes for such systems were not thoroughly studied.

In the present paper we obtain a representation for the solution to an initial boundary value problem describing the oscillations of a string system located along a geometric star graph with

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a hysteresis type condition at the node. This condition arises due to a limiter for the oscillatory process installed at the node. In its turn, the limiter can move in a direction perpendicular to the graph plane so that its movement is described by the mapping $C(t) = [-h, h] + \xi(t)$, $t \geqslant 0$. In what follows we use the terminology from [9].

We are going to describe the formulation of the problem. Let points O, A_1, A_2, $\ldots$, A_n belong to the horizontal plane π. We consider a mechanical system consisting of n strings, the equilibria of which coincide with the segments OA_1, OA_2, $\ldots$, OA_n. The end-points of the strings are tied at the point O. A geometric star graph Γ consists of the edges (intervals) OA_1, OA_2, $\ldots$, OA_n, the node O and boundary vertices A_1, A_2, $\ldots$, A_n. We suppose that while oscillating, the strings deviate from their equillibria in the direction perpendicular to the plane π and we consider the case of small oscillations.

Let the edges of the graph have the same lengths and the graph is oriented to the node. The introduced parametrization associated the node with the point $x = l$, while the boundary vertices are associated with $x = 0$. By $u(x, t)$ we denote a function defined on the graph, which describes the deviation of the string system from the equilibrium at the point x and at the time t. The restriction of $u(x, t)$ to the edges is denoted by $u^i(x, t)$, $i = 1, 2, \ldots, n$. Thus, each function $u^i(x, t)$ determines the shape of ith string. At the points $x = 0$ and $x = l$ the functions $u^i(x, t)$ are defined by the corresponding boundary values. The tie condition for the strings at the node means that $u(l, t) = u^1(l, t) = u^i(l, t) = \ldots = u^n(l, t)$. We suppose that the initial shape of the strings is described by the functions $\varphi^i(x)$ $(i = 1, 2, \ldots, n)$. We consider the case, when the initial velocity for all strings is zero. We assume that the end-points of the strings are fixed at the boundary vertices, which means the validity of the conditions $u^i(0, t) = 0$, $(i = 1, 2, \ldots, n)$.

We suppose that in the oscillatory process the node point of the string system $u(l, t)$ is located inside the limiter, that is, the condition $u(l, t) \in C(t)$ is satisfied. As $u(l, t)$ is an internal point of $C(t)$, the transmission condition (the Kirchoff condition) is satisfied:

$$\sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l, t) = 0.$$

The derivatives at the node for each function $u^i(x, t)$ are treated as corresponding one-sided derivatives. If the node point of the string system touches the boundary points of the limiter, then during some time one of the following conditions hold:

$$u(l, t) = \xi(t) + h, \quad \text{at the same time} \quad \sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l, t) \leqslant 0,$$

or

$$u(l, t) = \xi(t) - h, \quad \text{at the same time} \quad \sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l, t) \geqslant 0.$$

The condition for the sign of the sum of the derivatives at the node describes the influence of the support reaction force from the limiter, which blocks the movement of the node. Thus, we should have

$$-\sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l, t) \in N_{C(t)}(u(l, t)),$$

where the set $N_{C(t)}(u(l, t))$ stands for an outward normal cone to $C(t)$ at the point $u(l, t) \in C(t)$, which is defined as

$$N_{C(t)}(u(l, t)) = \{\xi \in R^1 : \xi \cdot (c - u(l, t)) \leqslant 0 \quad \forall c \in C(t)\}.$$

We observe that if $u(l,t)$ is an internal point of $C(t)$, then $N_{C(t)}(u(l,t)) = 0$. If $u(l,t) = \xi(t)+h$, then $C(t) = [0,+\infty)$. As $u(l,t) = \xi(t) - h$, then $C(t) = (-\infty, 0]$.

Thus, the mathematical model of the problem reads as

$$\begin{cases} \dfrac{\partial^2 u^i}{\partial x^2} = \dfrac{\partial^2 u^i}{\partial t^2}, \quad 0 < x < l, \quad t > 0 \quad (i = 1, 2, \dots, n), \\ u^i(x,0) = \varphi^i(x), \\ \dfrac{\partial u^i}{\partial t}(x,0) = 0, \\ u^1(l,t) = u^2(l,t) = \dots = u^n(l,t) = u(l,t), \\ -\displaystyle\sum_{i=1}^{n} \dfrac{\partial u^i}{\partial x}(l,t) \in N_{C(t)}(u(l,t)), \\ u^i(0,t) = 0 \quad (i = 1, 2, \dots, n), \\ u(l,t) \in C(t). \end{cases} \tag{1.1}$$

Hereafter we suppose that the conditions hold:

$$\varphi(l) = \varphi^1(l) = \varphi^2(l) = \dots = \varphi^n(l), \qquad \varphi(l) \in C(0),$$
$$\varphi^1(0) = \varphi^2(0) = \dots = \varphi^n(0) = 0.$$

In the present work we obtain an analogue of the D'Alembert formula for the solution of problem (1.1).

2. Preliminaries

In this section we provide some notions and definitions, which will be needed in what follows.

Let H be a Hilbert space. The scalar product in H is denoted by $\langle \cdot, \cdot \rangle$. For a closed convex set $C \subset H$ and $x \in C$ the set

$$N_C(x) = \{\xi \in H : \langle \xi, c - x \rangle \leqslant 0 \quad \forall c \in C\}$$

denotes an outward normal cone to C at the point x. We note that we always have $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in \operatorname{int} C$, where $\operatorname{int} C$ is the set of interior points of C; we suppose that $\operatorname{int} C \neq \emptyset$. The latter relation shows that the outward normal cone is non-trivial only as $x \in \partial C$, where ∂C is the boundary of the set C.

The Hausdorff distance $d_H(C_1, C_2)$ between closed sets C_1 and C_2 is defined by the formula

$$d_H(C_1, C_2) = \max\{\sup_{x \in C_2} \operatorname{dist}(x, C_1), \sup_{x \in C_1} \operatorname{dist}(x, C_2)\},$$

where

$$\operatorname{dist}(x, C) = \inf\{\|x - c\|, c \in C\}.$$

We consider a so-called sweeping process [16]:

$$-u'(t) \in N_{C(t)}(u(t)), \quad t \in [0,T], \tag{2.1}$$
$$u(0) = u_0 \in C(0). \tag{2.2}$$

A function $u : [0,T] \to H$ is called a solution of sweeping process (2.1), (2.2) if

(a) $u(0) = u_0$;

(b) $u(t) \in C(t)$ for all $t \in [0,T]$;

(c) u is differentiable for almost all $t \in [0,T]$;

(d) $-u'(t) \in N_{C(t)}(u(t))$ for almost all $t \in [0,T]$.

We shall need the following theorems from [16].

Theorem 2.1. *Assume that map $t \to C(t)$ satisfies the Lipschitz condition in the sense of the Hausdorff distance, that is,*

$$d_H(C(t), C(s)) \leqslant L|t-s|,$$

and $C(t) \subset H$ is nonempty, closed and convex for every $t \in [0,T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0,T] \to H$ of problem (2.1), (2.2), which is Lipschitz continuous with constant L. In particular, $|u'(t)| \leqslant L$ for almost every $t \in [0,T]$.

Theorem 2.2. *The solution of (2.1), (2.2) is unique in the class of absolutely continuous functions.*

In what follows we apply the classes of functions introduced by V.A. Ilin in [5]. By Q_T we denote a rectangle

$$Q_T = [0 \leqslant x \leqslant l] \times [0 \leqslant t \leqslant T].$$

We shall say that the function $u(x,t)$ belongs to the class $\widehat{W}_2^1(Q_T)$ if $u(x,t)$ is continuous in Q_T and possesses both generalized partial derivatives $u_x(x,t)$ and $u_t(x,t)$ in this rectangle and each of these derivatives belongs to the class $L_2(Q_T)$ and to the class $L_2[0 \leqslant x \leqslant l]$ for each fixed t in the segment $[0,T]$ and to the class $L_2[0 \leqslant t \leqslant T]$ for each fixed x in the segment $[0,l]$.

We shall say that $\Phi(x,t)$ belongs to the class $\widehat{W}_2^2(Q_T)$ if the function $\Phi(x,t)$ and its first partial derivatives are continuous in Q_T and if $\Phi(x,t)$ possesses in this rectangle all generalized derivatives of second order, each of which belongs to the class $L_2(Q_T)$ and to the class $L_2[0 \leqslant x \leqslant l]$ for each fixed t in the segment $[0,T]$ and to the class $L_2[0 \leqslant t \leqslant T]$ for each fixed x in the segment $[0,l]$.

3. Problem on graph with a nonlinear condition at node

A solution to problem (1.1) is a function $u(x,t)$ such that

1) the restrictions of $u(x,t)$ on the edges coincide with $u^i(x,t)$ $(i = 1,2,\dots,n)$ and $u^i(x,t) \in \widehat{W}_2^1(Q_T)$ for all $T > 0$;
2) for $t \geqslant 0$ the conditions
$$u^1(l,t) = u^2(l,t) = \ldots = u^n(l,t) = u(l,t), \qquad u(l,t) \in C(t), \qquad u^i(0,t) = 0$$
hold;
3) for almost all $t \geqslant 0$ the condition $-\sum\limits_{i=1}^{n} \dfrac{\partial u^i}{\partial x}(l,t) \in N_{C(t)}(u(l,t))$ holds;
4) the conditions $u^i(x,0) = \varphi^i(x)$ holds for all $x \in [0,l]$, and the conditions $\dfrac{\partial u^i}{\partial t}(x,0) = 0$ hold for almost all $x \in [0,l]$, $i = 1,2,\dots,n$;
5) for each $T > 0$ the integral identity
$$\begin{aligned}\sum_{i=1}^{n} \int_0^l \int_0^T u^i(x,t)\Big[\frac{\partial^2 \Psi^i}{\partial t^2}(x,t) - \frac{\partial^2 \Psi^i}{\partial x^2}(x,t)\Big]\,dx\,dt + \sum_{i=1}^{n} \int_0^l \frac{\partial \Psi^i}{\partial t}(x,0)\varphi^i(x)\,dx \\ + \sum_{i=1}^{n} \int_0^T \Big(u^i(l,t)\frac{\partial \Psi^i}{\partial x}(l,t) - \Psi^i(l,t)\frac{\partial u^i}{\partial x}(l,t)\Big)\,dt = 0\end{aligned} \tag{3.1}$$
holds, where arbitrary functions $\Psi^i \in \widehat{W}_2^2(Q_T)$, $(i = 1,2,\dots,n)$, are such that
$$\Psi^i(0,t) = 0, \quad \Psi^i(x,T) = 0, \quad \frac{\partial \Psi^i}{\partial t}(x,T) = 0, \quad \Psi^1(l,t) = \Psi^2(l,t) = \ldots = \Psi^n(l,t).$$

We consider functions Φ^i of the following form:

– if $x \in [0, l]$, then $\Phi^i(x) = \varphi^i(x)$;

– if $x \in [(m+1)l, (m+2)l]$ and m is an even number, then

$$\Phi^i(x) = 2 \cdot \sum_{k=0}^{\frac{m}{2}} g_{2k}(x - (m+1-2k)l) - \varphi^i((m+2)l - x);$$

– if $x \in [(m+1)l, (m+2)l]$ and m is an odd number, then

$$\Phi^i(x) = 2 \cdot \sum_{k=1}^{\frac{m+1}{2}} g_{2k-1}(x - (m+2-2k)l) + \varphi^i(x - (m+1)l);$$

$$\Phi^i(-x) = -\Phi^i(x).$$

Here the functions $g_0(t)$ and $g_1(t)$ are solutions of problems

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(l-t), \quad t \in [0, l], \\ g_0(0) = \varphi(l), \end{cases}$$

$$\begin{cases} -g_1'(t) \in N_{C(t)}(g_1(t)) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(t-l), \quad t \in [l, 2l], \\ g_1(l) = g_0(l). \end{cases}$$

The functions $g_m(t)$, where $t \in [ml, (m+1)l]$, with even numbers $m \geqslant 2$ are solutions to the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=0}^{\frac{m-2}{2}} g_{2k}'(t - ml + 2kl) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(ml + l - t), \\ g_m(ml) = g_{m-1}(ml), \end{cases}$$

while for odd $m \geqslant 3$ and $t \in [ml, (m+1)l]$ they solve the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=1}^{\frac{m-1}{2}} g_{2k-1}'(t - l - ml + 2kl) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(t - ml), \\ g_m(ml) = g_{m-1}(ml). \end{cases}$$

Theorem 3.1. *Let the functions $\xi(t)$ and $\varphi^i(x)$ satisfy the Lipschitz condition on their domains. Then the solution to problem (1.1) can be represented as*

$$u^i(x,t) = \frac{\Phi^i(x-t) + \Phi^i(x+t)}{2}, \tag{3.2}$$

where $i = 1, 2, \ldots, n$.

Proof. We first formally suppose that the solution to problem (1.1) is of form (3.2). Then $u^i(x, 0) = \Phi^i(x) = \varphi^i(x)$, where $x \in [0, l]$. It follows from the condition $u^i(0, t) = 0$ that the functions $\Phi^i(x)$ should be defined for $x < 0$ in the odd way. Since

$$u_x^i(x,t) = \frac{\Phi^{i'}(x-t) + \Phi^{i'}(x+t)}{2}, \qquad u_t^i(x,t) = \frac{-\Phi^{i'}(x-t) + \Phi^{i'}(x+t)}{2},$$

then

$$-u_t^i(l,t) = -u_x^i(l,t) + \Phi^{i'}(l-t)$$

and therefore,

$$-\frac{1}{n}\sum_{i=1}^{n} u_t^i(l,t) = -\frac{1}{n}\sum_{i=1}^{n} u_x^i(l,t) + \frac{1}{n}\sum_{i=1}^{n} \Phi^{i'}(l-t).$$

We denote

$$g(t) = \frac{1}{n}\sum_{i=1}^{n} u^i(l,t) = u(l,t).$$

We note that since

$$-\sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l,t) \in N_{C(t)}(u(l,t))$$

then

$$-\frac{1}{n}\sum_{i=1}^{n} \frac{\partial u^i}{\partial x}(l,t) \in N_{C(t)}(u(l,t)),$$

and hence,

$$-g'(t) \in N_{C(t)}(g(t)) + \frac{1}{n}\sum_{i=1}^{n} \Phi^{i'}(l-t).$$

We consider the case $0 \leqslant t \leqslant l$. Then

$$\frac{1}{n}\sum_{i=1}^{n} \Phi^{i'}(l-t) = \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(l-t).$$

We introduce a function $g_0(t)$, which is equal to $g(t)$ as $0 \leqslant t \leqslant l$. We then see that $g_0(t)$ is a solution to problem

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(l-t), \quad t \in [0,l], \\ g_0(0) = \varphi(l). \end{cases} \tag{3.3}$$

Let us show that this problem possesses a unique solution, which is defined for all $t \in [0,l]$.

We consider a function

$$w(t) = g_0(t) + \int_0^t \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(l-s)\,ds$$

and a set

$$D(t) = C(t) + \int_0^t \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(l-s)\,ds.$$

Since the functions $\xi(t)$ and $\varphi^i(x)$ satisfy the Lipschitz condition, then the mapping $D(t)$ also satisfies the Lipschitz condition in the sense of the Hausdorff distance. We observe that $N_{C(t)}(g_0(t)) = N_{D(t)}(w(t))$. Thus, we obtain the problem

$$-\frac{d}{dt}w(t) \in N_{D(t)}(w(t)), \qquad w(0) = \varphi(l) \in D(0), \quad t \in [0,l].$$

According to Theorems 2.1 and 2.2, this problem possesses a unique solution $w(t)$ defined on the entire segment $[0,l]$. The function $w(t)$ satisfies the Lipschitz condition and its derivative is almost everywhere bounded. Then problem (3.3) possesses a unique solution $g_0(t)$, where $g_0(t) \in C(t)$ and $g_0(t)$ also satisfies the Lipschitz condition. Since

$$\Phi^i(l-t) + \Phi^i(l+t) = 2g_0(t),$$

we obtain

$$\Phi^i(x) = 2g_0(x-l) - \varphi^i(2l-x),$$

where $x \in [l, 2l]$. We note that each function $\Phi^i(x)$ satisfies the Lipschitz condition on the segment $[l, 2l]$ and its derivative is bounded almost everywhere. Thus, $\Phi^i \in W_2^1[l, 2l]$.

We are going to show that $\Phi^i(l-0) = \Phi^i(l+0)$. We have

$$\Phi^i(l-0) = \varphi(l), \quad \text{and} \quad \Phi^i(l+0) = 2g_0(0) - \varphi^i(l) = 2\varphi(l) - \varphi(l) = \varphi(l).$$

We consider the case $t \in [l, 2l]$ and on this segment we define a function $g_1(t) = g(t)$. We consider a problem

$$\begin{cases} -\dfrac{d}{dt} g_1(t) \in N_{C(t)}(g_1(t)) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \Phi^{i'}(l-t), \quad t \in [l, 2l], \\ g_1(l) = g_0(l). \end{cases}$$

We note that for all $i = 1, 2, \ldots, n$ we have $\Phi^i(l-t) = -\varphi^i(t-l)$. We then obtain the problem

$$\begin{cases} -g_1'(t) \in N_{C(t)}(g_1(t)) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(t-l), \quad t \in [l, 2l], \\ g_1(l) = g_0(l). \end{cases}$$

Similarly to (3.3) we prove that the latter problem possesses a unique solution $g_1(t)$, where $g_1(t) \in C(t)$ and $g_1(t)$ satisfies the Lipschitz condition. Thus, we can determine $\Phi^i(x)$ for $x \in [2l, 3l]$ as

$$\Phi^i(x) = 2g_1(x-l) + \varphi^i(x-2l).$$

We observe that $\Phi^i \in W_2^1[2l, 3l]$.

Let us show that $\Phi^i(2l-0) = \Phi^i(2l+0)$. We have

$$\Phi^i(2l-0) = 2g_0(l) - \varphi^i(0) = 2g_0(l)$$

and

$$\Phi^i(2l+0) = 2g_1(l) + \varphi^i(0) = 2g_0(l).$$

In the same way we consider the case $t \in [2l, 3l]$. We define the function $g_2(t) = g(t)$, where $t \in [2l, 3l]$. Then $g_2(t)$ is a solution to the problem

$$\begin{cases} -\dfrac{d}{dt} g_2(t) \in N_{C(t)}(g_2(t)) + 2g_0'(t-2l) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(3l-t), \quad t \in [2l, 3l], \\ g_2(2l) = g_1(2l). \end{cases}$$

Now we can determine each function $\Phi^i(x)$ on the segment $x \in [3l, 4l]$ as

$$\Phi^i(x) = 2g_2(x-l) + 2g_0(x-3l) - \varphi^i(4l-x).$$

We consider the case $t \in [3l, 4l]$. Having determined $g_3(t) = g(t)$, we see that $g_3(t)$ is a solution of the problem

$$\begin{cases} -\dfrac{d}{dt} g_3(t) \in N_{C(t)}(g_3(t)) + 2g_1'(t-2l) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \varphi^{i'}(t-3l), \quad t \in [3l, 4l], \\ g_3(3l) = g_2(3l), \end{cases}$$

and for $x \in [4l, 5l]$ we determine the functions

$$\Phi^i(x) = 2g_3(x-l) + 2g_1(x-3l) + \varphi^i(x-4l).$$

Let us show that as $x \in [(m+1)l, (m+2)l]$, for even m we have

$$\Phi^i(x) = 2 \cdot \sum_{k=0}^{\frac{m}{2}} g_{2k}(x - (m+1-2k)l) - \varphi^i((m+2)l - x);$$

while for odd m we have

$$\Phi^i(x) = 2 \cdot \sum_{k=1}^{\frac{m+1}{2}} g_{2k-1}(x - (m+2-2k)l) + \varphi^i(x - (m+1)l).$$

In their turn, the functions $g_m(t)$, where $t \in [ml, (m+1)l]$, are solution of the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=0}^{\frac{m-2}{2}} g_{2k}'(t - ml + 2kl) + \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(ml + l - t), \\ g_m(ml) = g_{m-1}(ml), \end{cases}$$

for even $m \geqslant 2$, while for odd $m \geqslant 3$ they solve the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=1}^{\frac{m-1}{2}} g_{2k-1}'(t - l - ml + 2kl) + \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(t - ml), \\ g_m(ml) = g_{m-1}(ml). \end{cases}$$

For $m = 2, 3$ the statement is proved. Suppose that it is true for $m \leqslant M$ and let us show that then it holds for $m = M + 1$.

We consider the case of even M and we are going to show that

$$\Phi^i(x) = 2\sum_{k=1}^{\frac{M+2}{2}} g_{2k-1}(x - (M+3-2k)l) + \varphi^i(x - (M+2)l),$$

where $x \in [(M+2)l, (M+3)l]$. Having determined $g(t) = g_{M+1}(t)$ for $t \in [(M+1)l, (M+2)l]$, we obtain

$$-g_{M+1}'(t) \in N_{C(t)}(g_{M+1}(t)) + \frac{1}{n}\sum_{i=1}^{n} \Phi^{i'}(l - t).$$

Since

$$\Phi^{i'}(l - t) = 2 \cdot \sum_{k=1}^{\frac{M}{2}} g_{2k-1}'(t - l - (M+1-2k)l) + \varphi^{i'}(t - l - Ml),$$

then

$$-g_{M+1}'(t) \in N_{C(t)}(g_{M+1}(t)) + 2 \cdot \sum_{k=1}^{\frac{M}{2}} g_{2k-1}'(t - 2l - Ml + 2kl) + \frac{1}{n}\sum_{i=1}^{n} \varphi^{i'}(t - l - Ml).$$

We note that

$$g_{M+1}((M+1)l) = \frac{\Phi((2+M)l) - \Phi(Ml)}{2}.$$

Since

$$\Phi((M+2)l) = 2\sum_{k=0}^{\frac{M}{2}} g_{2k}(l + 2kl) \quad \text{and} \quad \Phi(Ml) = 2\sum_{k=0}^{\frac{M-2}{2}} g_{2k}(l + 2kl),$$

then

$$g_{M+1}((M+1)l) = g_M((M+1)l).$$

The problem

$$\begin{cases} -g'_{M+1}(t) \in N_{C(t)}(g_{M+1}(t)) + 2 \cdot \sum\limits_{k=1}^{\frac{M}{2}} g'_{2k-1}(t - 2l - Ml + 2kl) + \dfrac{1}{n} \sum\limits_{i=1}^{n} \varphi^{i'}(t - l - Ml), \\ g_{M+1}((M+1)l) = g_M((M+1)l) \end{cases}$$

possesses a unique solution $g_{M+1}(t)$ defined on the segment $[(M+1)l, (M+2)l]$. Then

$$g_{M+1}(t) = \frac{\Phi^i(l-t) + \Phi^i(l+t)}{2}.$$

Hence,

$$\Phi^i(x) = 2g_{M+1}(x - l) - \Phi^i(2l - x),$$

where $x \in [(M+2)l, (M+3)l]$. Since

$$\Phi^i(2l - x) = -2 \sum_{k=1}^{\frac{M}{2}} g_{2k-1}(x - 3l - Ml + 2kl) - \varphi^i(x - 2l - Ml),$$

then

$$\begin{aligned} \Phi^i(x) &= 2g_{M+1}(x - l) + 2 \sum_{k=1}^{\frac{M}{2}} g_{2k-1}(x - 3l - Ml + 2kl) + \varphi^i(x - 2l - Ml) \\ &= 2 \sum_{k=1}^{\frac{M+2}{2}} g_{2k-1}(x - 3l - Ml + 2kl) + \varphi^i(x - 2l - Ml), \end{aligned}$$

and this is what we needed. Other cases can be treated in the same way.

We thus have obtained a representation for the functions $\Phi^i(x)$ $(i = 1, 2, \ldots, n)$. Let us show that the functions $u^i(x,t)$ defined by identity (3.2) are solutions to problem (1.1). We observe that $u^i \in \widehat{W}_2^1(Q_T)$ for all T since the functions $\Phi_i(x)$ are continuous on the entire axis and $\Phi^i \in W_2^1[ml, (m+1)l]$ for $m = 0, 1, 2, \ldots$, while for $x < 0$ the functions $\Phi^i(x)$ are defined in the odd way.

Since $u(l,t) = g(t)$, where $g(t) = g_m(t)$ as $t \in [ml, (m+1)l]$, $g_m(ml) = g_{m-1}(ml)$ and $g_m(t) \in C(t)$, then $u(l,t) \in C(t)$ for all $t \geqslant 0$. We note that the conditions

$$u^i(0,t) = 0, \qquad u^1(l,t) = u^2(l,t) = \ldots = u^n(l,t) = g(t)$$

are satisfied for all $t \geqslant 0$; the condition

$$\frac{\partial u^i}{\partial t}(x, 0) = 0$$

holds for almost each $x \in [0, l]$ and the condition

$$u^i(x, 0) = \varphi^i(x)$$

holds for all $x \in [0, l]$.

Since

$$\begin{gathered} -u_x^i(l,t) = -\frac{1}{2}(\Phi^{i'}(l-t) + \Phi^{i'}(l+t)), \\ \Phi^{i'}(l+t) = 2g'(t) + \Phi^{i'}(l-t) \end{gathered}$$

almost everywhere, then

$$-\frac{1}{n} \sum_{i=1}^{n} u_x^i(l,t) = -\frac{1}{n} \sum_{i=1}^{n} \Phi^{i'}(l-t) - g'(t).$$

Since

$$-g'(t) \in N_{C(t)}(g(t)) + \frac{1}{n}\sum_{i=1}^{n} \Phi^{i'}(l-t),$$

then

$$-\frac{1}{n}\sum_{i=1}^{n} u_x^i(l,t) \in N_{C(t)}(g(t)) = N_{C(t)}(u(l,t)),$$

and hence,

$$-\sum_{i=1}^{n} u_x^i(l,t) \in N_{C(t)}(u(l,t))$$

almost everywhere.

Now we are going to show that the integral identity holds true. Integral identity (3.1) can be represented as

$$\sum_{i=1}^{n}\int_0^l \left(\int_0^T u^i(x,t)\Psi_{tt}^i(x,t)\,dt\right)dx - \sum_{i=1}^{n}\int_0^T \left(\int_0^l u^i(x,t)\Psi_{xx}^i(x,t)\,dx\right)dt$$

$$+\sum_{i=1}^{n}\int_0^l \Psi_t^i(x,0)\varphi^i(x)\,dx - \sum_{i=1}^{n}\int_0^T \Psi^i(l,t)u_x^i(l,t)\,dt + \sum_{i=1}^{n}\int_0^T \Psi_x^i(l,t)u^i(l,t)\,dt$$

$$=\sum_{i=1}^{n}\int_0^l \left(u^i(x,T)\Psi_t^i(x,T) - u^i(x,0)\Psi_t^i(x,0)\right)dx - \sum_{i=1}^{n}\int_0^l\int_0^T u_t^i(x,t)\Psi_t^i(x,t)\,dt\,dx$$

$$-\sum_{i=1}^{n}\int_0^T \left(\Psi_x^i(l,t)u^i(l,t) - \Psi_x^i(0,t)u^i(0,t)\right)dt + \sum_{i=1}^{n}\int_0^T\int_0^l u_x^i(x,t)\Psi_x^i(x,t)\,dx\,dt$$

$$+\sum_{i=1}^{n}\int_0^l \Psi_t^i(x,0)\varphi^i(x)\,dx - \sum_{i=1}^{n}\int_0^T \Psi^i(l,t)u_x^i(l,t)\,dt + \sum_{i=1}^{n}\int_0^T \Psi_x^i(l,t)u^i(l,t)\,dt$$

$$=\sum_{i=1}^{n}\int_0^T\int_0^l u_x^i(x,t)\Psi_x^i(x,t)\,dx\,dt - \sum_{i=1}^{n}\int_0^l\int_0^T u_t^i(x,t)\Psi_t^i(x,t)\,dx\,dt - \sum_{i=1}^{n}\int_0^T \Psi^i(l,t)u_x^i(l,t)\,dt$$

$$=\frac{1}{2}\sum_{i=1}^{n}\int_0^T\int_0^l \left(\Phi^{i'}(x-t) + \Phi^{i'}(x+t)\right)\Psi_x^i(x,t)\,dx\,dt$$

$$-\frac{1}{2}\sum_{i=1}^{n}\int_0^l\int_0^T \left(\Phi^{i'}(x+t) - \Phi^{i'}(x-t)\right)\Psi_t^i(x,t)\,dt\,dx - \sum_{i=1}^{n}\int_0^T \Psi^i(l,t)u_x^i(l,t)\,dt$$

$$=\frac{1}{2}\sum_{i=1}^{n}\int_0^l \Psi_x^i(x,T)(-\Phi^i(x-T) + \Phi^i(x+T))\,dx - \frac{1}{2}\sum_{i=1}^{n}\int_0^l \Psi_x^i(x,0)(-\Phi^i(x) + \Phi^i(x))\,dx$$

$$-\frac{1}{2}\sum_{i=1}^{n}\int_0^T\int_0^l \Psi_{xt}^i(x,t)(\Phi^i(x+t) - \Phi^i(x-t))\,dx\,dt$$

$$-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi_t^i(l,t)(\Phi^i(l+t)-\Phi^i(l-t))\,dt+\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi_t^i(0,t)(\Phi^i(t)-\Phi^i(-t))\,dt$$

$$+\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{l}\int\limits_{0}^{T}\Psi_{tx}^i(x,t)(\Phi^i(x+t)-\Phi^i(x-t))\,dx\,dt-\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi(l,t)u_x^i(l,t)\,dt$$

$$=-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi_t^i(l,t)(\Phi^i(l+t)-\Phi^i(l-t))\,dt-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}(\Phi^{i'}(l-t)+\Phi^{i'}(l+t))\Psi^i(l,t)\,dt$$

$$=-\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi_t^i(l,t)(\Phi^i(l+t)-\Phi^i(l-t))\,dt+\frac{1}{2}\sum_{i=1}^{n}\int\limits_{0}^{T}\Psi_t^i(l,t)(\Phi^i(l+t)-\Phi^i(l-t))\,dt=0.$$

The proof is complete. □

Remark 3.1. *We observe that problem (1.1) has a unique solution.*

Proof. Suppose that the functions $u^i(x,t)$ form a solution to problem (1.1). Then the function

$$\widetilde{u}(x,t)=\frac{1}{n}\sum_{i=1}^{n}u^i(x,t)$$

is a solution of the problem

$$\begin{cases}\dfrac{\partial^2\widetilde{u}}{\partial x^2}=\dfrac{\partial^2\widetilde{u}}{\partial t^2}, \quad 0<x<l, \quad t>0,\\ \widetilde{u}(x,0)=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\varphi^i(x),\\ \dfrac{\partial\widetilde{u}}{\partial t}(x,0)=0,\\ \dfrac{\partial\widetilde{u}}{\partial x}(l,t)\in N_{C(t)}(\widetilde{u}(l,t)),\\ \widetilde{u}(l,t)\in C(t),\\ \widetilde{u}(0,t)=0.\end{cases}\tag{3.4}$$

This problem possesses a unique solution. Indeed, if $\varphi(l)\in(-h+\xi(0),h+\xi(0))$, then for all $t\in[0,t_1]$ the function $\widetilde{u}(x,t)$ is a solution of the problem

$$\begin{cases}\dfrac{\partial^2\widetilde{u}}{\partial x^2}=\dfrac{\partial^2\widetilde{u}}{\partial t^2}, \quad 0<x<l, \quad 0<t<t_1,\\ \widetilde{u}(x,0)=\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}\varphi^i(x),\\ \dfrac{\partial\widetilde{u}}{\partial t}(x,0)=0,\\ \widetilde{u}(0,t)=0,\\ \widetilde{u}'_x(l,t)=0.\end{cases}$$

As it is known [4], the latter problem possesses a unique solution $\widetilde{u}(x,t)$. At time t_1 either the condition $\widetilde{u}(l,t_1)=-h+\xi(t)$ or $\widetilde{u}(l,t_1)=h+\xi(t)$ is satisfied and for $t\in[t_1,t_2]$ the function

$\widetilde{u}(x,t)$ is a solution of one of problems

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}^*}{\partial x^2} = \dfrac{\partial^2 \widetilde{u}^*}{\partial t^2}, \quad 0 < x < l, \quad t_1 < t < t_2, \\ \widetilde{u}^*(x,t_1) = \widetilde{u}(x,t_1), \\ \dfrac{\partial \widetilde{u}^*}{\partial t}(x,t_1) = \widetilde{u}_t'(x,t_1), \\ \widetilde{u}(0,t) = 0, \\ \widetilde{u}(l,t) = \pm h + \xi(t). \end{cases}$$

Such problems also have a unique solution on $[t_1, t_2]$ [4]. Continue similar arguing, we obtain that the initial problem can have only a unique solution.

We introduce functions $\omega^i(x,t) = u^i(x,t) - \widetilde{u}(x,t)$, $(i = 1, 2, \dots, n)$. We note that $\omega^i(x,t)$, $(i = 1, 2, \dots, n)$ are a solution to the problem

$$\begin{cases} \dfrac{\partial^2 \omega^i}{\partial x^2} = \dfrac{\partial^2 \omega^i}{\partial t^2}, \quad 0 < x < l, \quad t > 0 \quad (i = 1, 2, \dots, n), \\ \omega^i(x,0) = \varphi^i(x) - \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \varphi^j(x), \\ \dfrac{\partial \omega^i}{\partial t}(x,0) = 0, \\ \omega^i(0,t) = 0, \\ \omega^i(l,t) = 0. \end{cases}$$

According to [4], for each $i = 1, 2, \dots, n$ the functions $\omega^i(x,t)$ are defined uniquely. Hence, the functions $u^i(x,t)$ are also defined uniquely and this completes the proof. □

We consider an example of solving a problem of form (1.1). Namely, we consider a problem

$$\begin{cases} \dfrac{\partial^2 u^i}{\partial x^2} = \dfrac{\partial^2 u^i}{\partial t^2}, \quad 0 < x < l, \quad t > 0 \quad (i = 1, 2, \dots, n), \\ u^i(x,0) = 0, \\ \dfrac{\partial u^i}{\partial t}(x,0) = 0, \\ u^1(l,t) = u^2(l,t) = \ldots = u^n(l,t) = u(l,t), \\ -\displaystyle\sum_{i=1}^{n} \dfrac{\partial u^i}{\partial x}(l,t) \in N_{C(t)}(u(l,t)), \\ u^i(0,t) = 0 \quad (i = 1, 2, \dots, n), \\ u(l,t) \in C(t), \end{cases}$$

where $C(t) = [-h, h] + \xi(t)$ and $\xi(t)$ is defined as l-periodic function of form

$$\xi(t) = \begin{cases} \dfrac{8h}{l} t & t \in \left[0, \dfrac{l}{4}\right], \\ -\dfrac{8h}{l}\left(t - \dfrac{l}{2}\right), & t \in \left[\dfrac{l}{4}, \dfrac{3l}{4}\right], \\ \dfrac{8h}{l}(t - l), & t \in \left[\dfrac{3l}{4}, l\right]. \end{cases}$$

We note that the function $\xi(t)$ satisfies the Lipschitz condition with the constant $L = \dfrac{8h}{l}$. As it was proved above, such problem possesses a unique solution, where

$$u^i(x,t) = \frac{\Phi^i(x-t) + \Phi^i(x+t)}{2}, \quad i = 1, 2, \dots, n.$$

Each of the functions $\Phi^i(x)$ can be represented as follows:

1) If $x \in [0, l]$, then $\Phi^i(x) = 0$;

2) If $x \in [(m+1)l, (m+2)l]$ and m is even, then

$$\Phi^i(x) = 2 \cdot \sum_{k=0}^{\frac{m}{2}} g_{2k}(x - (m+1-2k)l);$$

3) If m is odd, then

$$\Phi^i(x) = 2 \cdot \sum_{k=1}^{\frac{m+1}{2}} g_{2k-1}(x - (m+2-2k)l);$$

$$\Phi^i(-x) = -\Phi^i(x).$$

Here the functions $g_0(t)$ and $g_1(t)$ are solutions of problems

$$\begin{cases} -g_0'(t) \in N_{C(t)}(g_0(t)), \quad t \in [0, l], \\ g_0(0) = 0, \end{cases} \tag{3.5}$$

$$\begin{cases} -g_1'(t) \in N_{C(t)}(g_1(t)), \quad t \in [l, 2l], \\ g_1(l) = g_0(l). \end{cases} \tag{3.6}$$

The functions $g_m(t)$ for even numbers $m \geqslant 2$ are solutions of the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=0}^{\frac{m-2}{2}} g_{2k}'(t - ml + 2kl), \quad t \in [ml, (m+1)l], \\ g_m(ml) = g_{m-1}(ml), \end{cases}$$

while for odd $m \geqslant 3$ they solve the problems

$$\begin{cases} -g_m'(t) \in N_{C(t)}(g_m(t)) + 2\displaystyle\sum_{k=1}^{\frac{m-1}{2}} g_{2k-1}'(t - l - ml + 2kl), \quad t \in [ml, (m+1)l], \\ g_m(ml) = g_{m-1}(ml). \end{cases}$$

We consider problem (3.5). Having solved it, we obtain

$$g_0(t) = \begin{cases} 0, & t \in \left[0, \dfrac{l}{8}\right], \\ \xi(t) - h, & t \in \left[\dfrac{l}{8}, \dfrac{l}{4}\right], \\ h, & t \in \left[\dfrac{l}{4}, \dfrac{l}{2}\right], \\ \xi(t) + h, & t \in \left[\dfrac{l}{2}, \dfrac{3l}{4}\right], \\ -h, & t \in \left[\dfrac{3l}{4}, l\right]. \end{cases}$$

We consider problem (3.6). Its solution reads as

$$g_1(t) = \begin{cases} \xi(t) - h, & t \in \left[l, \frac{5l}{4}\right], \\ h, & t \in \left[\frac{5l}{4}, \frac{3l}{2}\right], \\ \xi(t) + h, & t \in \left[\frac{3l}{2}, \frac{7l}{4}\right], \\ -h, & t \in \left[\frac{7l}{4}, 2l\right]. \end{cases}$$

Let us show that for all $m \in \mathbb{N}$ we have

$$g_m(t) = \begin{cases} \xi(t) - h, & t \in \left[ml, \frac{l(4m+1)}{4}\right], \\ h, & t \in \left[\frac{l(4m+1)}{4}, \frac{l(2m+1)}{2}\right], \\ \xi(t) + h, & t \in \left[\frac{l(2m+1)}{2}, \frac{l(4m+3)}{4}\right], \\ -h, & t \in \left[\frac{l(4m+3)}{4}, l(m+1)\right]. \end{cases}$$

For $m = 1$ the statement is true. Suppose that it is true for $m \leqslant N$ and let us show that in this case it is true for $m = N+1$. We consider the case, when $N = 2M$, and we are going to prove that

$$g_{2M+1}(t) = \begin{cases} \xi(t) - h, & t \in \left[2Ml + l, 2Ml + \frac{5l}{4}\right], \\ h, & t \in \left[2Ml + \frac{5l}{4}, 2Ml + \frac{3l}{2}\right], \\ \xi(t) + h, & t \in \left[2Ml + \frac{3l}{2}, 2Ml + \frac{7l}{4}\right], \\ -h, & t \in \left[2Ml + \frac{7l}{4}, 2Ml + 2l\right]. \end{cases} \tag{3.7}$$

We have

$$\begin{cases} -g'_{2M+1}(t) \in N_{C(t)}(g_{2M+1}(t)) + 2\sum_{k=1}^{M} g'_{2k-1}(t - 2l - 2Ml + 2kl), \\ g_{2M+1}((2M+1)l) = -h, \quad t \in [(2M+1)l, (2M+2)l]. \end{cases}$$

We denote

$$v(t) = g_{2M+1}(t) + 2\int\limits_{(2M+1)l}^{t} \sum_{k=1}^{M} g'_{2k-1}(s - 2l - 2Ml + 2kl)\, ds.$$

We have

$$\begin{aligned} v(t) &= g_{2M+1}(t) + 2\sum_{k=1}^{M}(g_{2k-1}(t - 2l - 2Ml + 2kl) - g_{2k-1}((2k-1)l) \\ &= g_{2M+1}(t) + 2\sum_{k=1}^{M} g_{2k-1}(t - 2l - 2Ml + 2kl) + 2Mh. \end{aligned}$$

We denote

$$\tilde{\xi}(t) = \xi(t) + 2Mh + 2\sum_{k=1}^{M} g_{2k-1}(t - 2l - 2Ml + 2kl),$$
$$D(t) = [-h, h] + \tilde{\xi}(t), t \in [(2M+1)l, (2M+2)l].$$

In view of the induction assumption and the representation for the function $\xi(t)$ we get

$$\tilde{\xi}(t) = \begin{cases} (1+2M)\xi(t), & t \in \left[2Ml + l, 2Ml + \frac{5l}{4}\right], \\ \xi(t) + 4Mh, & t \in \left[2Ml + \frac{5l}{4}, 2Ml + \frac{3l}{2}\right], \\ \xi(t) + 4Mh + 2M\xi(t), & t \in \left[2Ml + \frac{3l}{2}, 2Ml + \frac{7l}{4}\right], \\ \xi(t), & t \in \left[2Ml + \frac{7l}{4}, 2Ml + 2l\right]. \end{cases}$$

Since $v(t)$ is a solution of problem

$$\begin{cases} -v'(t) \in N_{D(t)}(v(t)), & t \in [(2M+1)l, (2M+2)l], \\ \quad v((2M+1)l) = -h, \end{cases}$$

then

$$v(t) = \begin{cases} -h + \xi(t) + 2M\xi(t), & t \in \left[2Ml + l, 2Ml + \frac{5l}{4}\right], \\ \quad h + 4Mh, & t \in \left[2Ml + \frac{5l}{4}, 2Ml + \frac{3l}{2}\right], \\ \quad \xi(t) + 4Mh + 2M\xi(t) + h, & t \in \left[2Ml + \frac{3l}{2}, 2Ml + \frac{7l}{4}\right], \\ -h, & t \in \left[2Ml + \frac{7l}{4}, 2Ml + 2l\right]. \end{cases}$$

Therefore, $g_{2M+1}(t) = v(t) - \tilde{\xi}(t) + \xi(t)$ and we obtain (3.7) for the function $g_{2M+1}(t)$. Other cases can be considered in the same way.

We define a function $g(t)$ coinciding with the function $g_m(t)$ on each segment $t \in [ml, (m+1)l]$, where $m = 0, 1, 2, \ldots$ Since $u(l, t) = g(t)$, the node of the string system periodically oscillates with the period l from the time $t = \frac{l}{4}$. At the same time, the touching of the limiter occurs at times $\frac{l}{8}$, $\frac{nl}{2}$, where $n \in \mathbb{N}$.

4. Boundary control problem

Problems on boundary control of oscillating processes on a segment in the case of linear boundary conditions were studied, for instance, in works by V.A. Ilin and E.I. Moiseev [4]–[6]. We consider a problem of such kind for the star graph for the case of a nonlinear condition at the node:

$$
\begin{cases}
\dfrac{\partial^2 u^i}{\partial x^2} = \dfrac{\partial^2 u^i}{\partial t^2}, \quad 0 < x < l, \quad 0 < t < T \quad (i = 1, 2, \ldots, n), \\
u^i(x, 0) = \varphi^i(x), \\
\dfrac{\partial u^i}{\partial t}(x, 0) = 0, \\
-\displaystyle\sum_{i=1}^{n} \dfrac{\partial u^i}{\partial x}(l, t) \in N_{C(t)}(u(l, t)), \\
u(l, t) = u^1(l, t) = u^2(l, t) = \ldots = u^n(l, t), \\
u(l, t) \in C(t), \\
u^i(0, t) = \mu^i(t).
\end{cases}
\tag{4.1}
$$

We need to find functions $\mu^i(t) \in W_2^1[0, T]$ such that

$$
u^i(x, T) = \varphi *^i (x), \qquad (u^i)'_t(x, T) = \psi *^i (x),
$$

where $\varphi*^i \in W_2^1[0, l]$, $\psi*^i \in L^2[0, l]$ are given functions. Suppose that the functions $\xi(t)$ and $\varphi^i(x)$ satisfy the Lipschitz condition on their domains.

A solution of problem (4.1) is a function $u(x, t)$ such that

1) the restrictions of $u(x, t)$ to the edges coincide with $u^i(x, t)$, $(i = 1, 2, \ldots, n)$, and $u^i(x, t) \in \widehat{W}_2^1(Q_T)$;
2) for $0 \leqslant t \leqslant T$ the conditions

$$
u^1(l, t) = u^2(l, t) = \ldots = u^n(l, t) = u(l, t), \qquad u(l, t) \in C(t), \qquad u^i(0, t) = \mu^i(t),
$$

hold;
3) for almost all $0 \leqslant t \leqslant T$ the condition $-\sum\limits_{i=1}^{n} \dfrac{\partial u^i}{\partial x}(l, t) \in N_{C(t)}(u(l, t))$ holds;
4) the conditions $u^i(x, 0) = \varphi^i(x)$ hold for all $x \in [0, l]$, while the conditions $\dfrac{\partial u^i}{\partial t}(x, 0) = 0$ hold for almost all $x \in [0, l]$, $i = 1, 2, \ldots, n$;
5) The integral identity

$$
\sum_{i=1}^{n} \int_0^l \int_0^T u^i(x, t)\Big[\frac{\partial^2 \Psi^i}{\partial t^2}(x, t) - \frac{\partial^2 \Psi^i}{\partial x^2}(x, t)\Big]\, dx\, dt + \sum_{i=1}^{n} \int_0^l \frac{\partial \Psi^i}{\partial t}(x, 0)\varphi^i(x)\, dx
$$
$$
+ \sum_{i=1}^{n} \int_0^T \Big(u^i(l, t)\frac{\partial \Psi^i}{\partial x}(l, t) - \Psi^i(l, t)\frac{\partial u^i}{\partial x}(l, t)\Big)\, dt - \sum_{i=1}^{n} \int_0^T \frac{\partial \Psi^i}{\partial x}(0, t)\mu^i(t)\, dt = 0
$$

hold, where arbitrary functions $\Psi^i \in \widehat{W}_2^2(Q_T)$, $(i = 1, 2, \ldots, n)$, are such that

$$
\Psi^i(0, t) = 0, \qquad \Psi^i(x, T) = 0, \qquad \frac{\partial \Psi^i}{\partial t}(x, T) = 0, \qquad \Psi^1(l, t) = \Psi^2(l, t) = \ldots = \Psi^n(l, t).
$$

We consider the case $T < l$.

Theorem 4.1. *For $T < l$ a solution to problem (4.1) is uniquely defined. The functions $\mu^i(t)$ should read as*

$$
\mu^i(t) = \frac{1}{2}(\varphi^i(t) - \widehat{\psi*^i}(T - t) + \varphi *^i (T - t)).
$$

At the same time, for all $i = 1, 2, \dots, n$ the initial and final data should be related by the identities

$$\begin{aligned}
&\widehat{\psi *^i}(x) - \varphi *^i (x) + \varphi^i(x-T) \equiv 0, && T \leqslant x \leqslant l,\\
&\widehat{\psi *^i}(x) + \varphi *^i (x) - \varphi^i(x+T) \equiv 0, && 0 \leqslant x \leqslant l-T,\\
&\widehat{\psi *^i}(x) + \varphi *^i (x) - 2g_0(T+x-l) + \varphi^i(2l-x-T) \equiv 0, && l-T \leqslant x \leqslant l.
\end{aligned}$$

Here for each $i = 1, 2, \dots, n$ by $\widehat{\psi *^i}$ we denote the primitive for the function $\psi *^i$, which obeys the identity

$$\widehat{\psi *^i}(x_0^i) - \varphi *^i (x_0^i) + \varphi^i(x_0^i - T) = 0,$$

$x_0^i \in [T, l]$ are fixed, $g_0(t)$ is a solution of problem (3.3).

Proof. We introduce the functions

$$\underline{\mu}^i(t) = \begin{cases} \mu^i(t), & t \geqslant 0,\\ 0, & t < 0, \end{cases} \qquad i = 1, 2, \dots, n,$$

We denote by $v^i(x,t)$ a solution to problem (1.1), $(i = 1, 2, \dots, n)$. Similarly to Theorem 3.1, by straightforward checking of the conditions we confirm that

$$u^i(x,t) = \underline{\mu}^i(t-x) + v^i(x,t)$$

is a solution of problem (4.1), $(i = 1, 2, \dots, n)$. Thus,

$$u^i(x,t) = \underline{\mu}^i(t-x) + \frac{\Phi^i(x-t) + \Phi^i(x+t)}{2}.$$

Then

$$\underline{\mu}^i(T-x) + \frac{\Phi^i(x-T) + \Phi^i(x+T)}{2} = \varphi *^i (x),$$

and therefore,

$$-\underline{\mu}^{i\prime}(T-x) + \frac{\Phi^{i\prime}(x+T) + \Phi^{i\prime}(x-T)}{2} = \varphi *^{i\prime} (x). \tag{4.2}$$

On the other hand,

$$\underline{\mu}^{i\prime}(T-x) + \frac{\Phi^{i\prime}(x+T) - \Phi^{i\prime}(x-T)}{2} = \psi *^i (x). \tag{4.3}$$

Deducting identity (4.2) from (4.3), we obtain

$$2\underline{\mu}^{i\prime}(T-x) - \Phi^{i\prime}(x-T) = \psi *^i (x) - \varphi *^{i\prime} (x). \tag{4.4}$$

We proceed to the case $T \leqslant x \leqslant l$. Using representations for the functions $\underline{\mu}^i$ and Φ^i, we obtain

$$\widehat{\psi *^i}(x) - \varphi *^i (x) + \varphi^i(x-T) \equiv 0, \quad T \leqslant x \leqslant l,$$

where we choose the primitive $\widehat{\psi *^i}(x)$ of the function $\psi *^i (x)$ so that it to satisfy the identity

$$\widehat{\psi *^i}(x_0^i) - \varphi *^i (x_0^i) + \varphi^i(x_0^i - T) = 0,$$

where $x_0^i \in [T, l]$ is fixed for each $i = 1, 2, \dots, n$.

We consider identity (4.4) for $0 \leqslant x \leqslant T$. We obtain

$$2\mu^i(T-x) = \varphi^i(T-x) - \widehat{\psi *^i}(x) + \varphi *^i (x),$$

which for all $0 \leqslant t \leqslant T$ implies

$$\mu^i(t) = \frac{1}{2}(\varphi^i(t) - \widehat{\psi *^i}(T-t) + \varphi *^i (T-t)).$$

Summing (4.2) and (4.3), we find

$$\Phi^{i'}(x+T) = \psi *^{i} (x) + \varphi *^{i'} (x). \tag{4.5}$$

Let us consider the case $l - T \leqslant x \leqslant l$. Using the representations for the functions Φ^i obtained in Theorem 3.1, we get

$$\widehat{\psi *^{i}}(x) + \varphi *^{i} (x) - 2g_0(T + x - l) + \varphi^{i}(2l - x - T) \equiv 0.$$

If $0 \leqslant x \leqslant l - T$, then

$$\widehat{\psi *^{i}}(x) + \varphi *^{i} (x) - \varphi^{i}(x + T) \equiv 0.$$

The proof is complete. □

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