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ASYMPTOTIC EXPANSION OF SOLUTION TO DIRICHLET PROBLEM IN PERFORATED DOMAIN: STRANGE TERM CASE

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Abstract. We consider an elliptic operator in a multi-dimensional space periodically perforated by closely spaced small cavities. The coefficients of the differential expression are varying and infinitely differentiable functions bounded uniformly with all their derivatives. For the coefficients at higher derivatives a uniform ellipticity condition is supposed. On the boundaries of the cavities we impose the Dirichlet condition. The sizes of the cavities and the distances between them are characterized by two small parameters. They are chosen to ensure the appearance of a strange term under the homogenization, which is an additional potential in the homogenized operator. The main result of the work is the scheme for constructing two-parametric asymptotics for the resolvent of the considered operator and its application for determining the leading terms in the asymptotics. The scheme is based on a combination of the multi-scaled method and the method of matching asymptotic expansions. The former is used to take into consideration the distribution of the cavities, while the latter takes into account the geometry of the cavities and the Dirichlet condition on its boundary.

Keywords: perforated domain, elliptic operator, asymptotic expansion, strange term.

Mathematics Subject Classification: 34B27, 35C20

1. Introduction

Elliptic problems in perforated domains are one of the classical models in the modern homogenization theory. The issues on convergence of solutions to such problems are actively studied, we only mention the classical monographs [4], [8]. Classical convergence results are formulated in terms of strong and weak convergence of solutions of perturbed problems to the homogenized ones for given right-hand sides of the considered equations. In recent years, an interest in these problems increased again in connection with proving the operator estimates for them, in which the L_2- or W_2^1-norm of the difference between solutions of the perturbed and homogenized problems is estimated in terms of the L_2-norm of the right-hand side multiplied by a small function, the form of which is determined by the geometry and perforation parameters. Estimates of this kind were established in a series of recent papers [2], [10]–[16], [22] for various perforation geometries.

Apart of the convergence issues, a separate interesting line of research is the construction of asymptotic expansions of solutions, including the case of complete asymptotic expansions. For the case of perforation along a given manifold, complete asymptotic expansions were constructed in very recent papers [2], [3]. In book [4, Ch. III, §6.5], as well as in the papers [17], [19]–[21], the case of strictly periodic perforation over the entire space was considered, with dimensions

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holes and the distance between them were proportional to the same small parameter. The construction of asymptotic expansions was based on the use of the multiscale method.

In this paper, we again consider the problem of constructing an asymptotic expansion for the resolvent of a general second order elliptic operator in a periodically perforated domain. However, we suppose that the sizes of the holes and the distances between them are described by two small positive parameters ε and η. The distances between the holes are described by the parameter ε, and the holes have a size $\varepsilon\eta$. The parameter η is considered to depend on ε and the condition $\varepsilon^{-2}\eta^{n-2}(\varepsilon) \to a$ with some constant a is imposed on it. This condition describes the critical hole size at which the occurrence of a strange term is guaranteed under the homogenization: this is the name of the additional potential arising in the homogenized operator. The condition on η is not strict in the sense that it does not fix the form of this function, but only its behavior as $\varepsilon \to +0$. Therefore, in fact, we are dealing with a problem described by two small parameters. The main result of the paper is a scheme for formal construction of the asymptotic expansion of the action of the resolvent of the operator under consideration on a function from the space $W_2^\infty(\mathbb{R}^n)$. The expansion is constructed with respect to two small parameters, ε and η, and is two-parametric. This scheme is used to determine the first few terms of the asymptotics. We also discuss issues related to the construction of a complete asymptotic expansion and the difficulties that arise along the way.

2. Formulation of problem and main results

Let $x = (x_1, \ldots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 3$, $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$ be some functions defined on $\mathbb{R}^n$ and satisfying the following conditions:

$$\begin{gathered} A_{ij}, A_j, A_0 \in C^\infty(\mathbb{R}^n), \qquad \frac{\partial^\alpha A_{ij}}{\partial x^\alpha}, \frac{\partial^\alpha A_j}{\partial x^\alpha}, \frac{\partial^\alpha A_0}{\partial x^\alpha} \in L_\infty(\mathbb{R}^n), \qquad \alpha \in \mathbb{Z}_+^n, \\ A_{ji} = A_{ij}, \qquad \sum_{i,j=1}^n A_{ij}(x)\xi_i\overline{\xi_j} \geqslant c_0 \sum_{j=1}^n |\xi_j|^2, \qquad x \in \mathbb{R}^n, \quad \xi_i \in \mathbb{C}, \end{gathered} \tag{2.1}$$

where c_0 is some positive constant independent of x and ξ_i. The functions A_{ij} are real, while the functions A_j, A_0 are complex-valued.

Let $\omega \subset \mathbb{R}^n$ be some bounded domain with an infinitely differentiable boundary. In the space $\mathbb{R}^n$ we make a fine periodic perforation as follows:

$$\Omega^\varepsilon := \mathbb{R}^n \setminus \overline{\theta^\varepsilon}, \qquad \theta^\varepsilon := \bigcup_{z \in \mathbb{Z}^n} (\varepsilon z + \varepsilon\eta\omega).$$

Here ε is a small positive parameter and $\eta = \eta(\varepsilon)$ is a positive function such that

$$\lim_{\varepsilon \to +0} \varepsilon^{-2}\eta^{n-2}(\varepsilon) = a, \tag{2.2}$$

where $a \geqslant 0$ is some fixed constant.

In a perforated domain Ω^ε we define an operator $\mathcal{H}^\varepsilon$ with the differential expression

$$\hat{\mathcal{H}} := -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} A_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{j=1}^n A_j \frac{\partial}{\partial x_j} + A_0$$

and the Dirichlet condition on $\partial\theta^\varepsilon$. Rigorously we define it as the m-sectorial operator in the space $L_2(\Omega^\varepsilon)$, which, by the first representation theorem [7, Ch. VI, Sect. 2.1], corresponds to a closed sectorial sesquilinear form

$$\mathfrak{h}^\varepsilon(u, v) := \sum_{i,j=1}^n \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega^\varepsilon)} + \sum_{j=1}^n \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega^\varepsilon)} + (A_0 u, v)_{L_2(\Omega^\varepsilon)}$$

in the space $L_2(\Omega^\varepsilon)$ on the domain $\mathfrak{D}(\mathfrak{h}^\varepsilon) := \mathring{W}_2^1(\Omega^\varepsilon)$, where $\mathring{W}_2^1(\Omega^\varepsilon)$ is the subspace of the Sobolev space $W_2^1(\Omega^\varepsilon)$ consisting of the functions with the zero trace on the boundary. By means of the standard smoothness improving theorems for elliptic boundary value problems, it is easy to confirm that the domain of the operator $\mathcal{H}^\varepsilon$ is of the form $\mathfrak{D}(\mathcal{H}^\varepsilon) := W_2^2(\Omega^\varepsilon)\cap\mathring{W}_2^1(\Omega^\varepsilon)$.

The main aim of the present work is to construct asymptotic expansion for the resolvent of the operator $\mathcal{H}^\varepsilon$ for small ε in the case when it acts on sufficiently smooth functions.

The convergence of the resolvents of the operators of form $\mathcal{H}^\varepsilon$ was studied in recent work [11] and the application of main theorems from [11] to our case gives the following result. Let $\mathcal{H}^0$ be one more m-sectorial operator but now in the space $L_2(\mathbb{R}^n)$ with the differential expression $\hat{\mathcal{H}}$. We again define this operator via the corresponding sectorial form by means of the first representation theorem; its domain is the space $W_2^2(\mathbb{R}^n)$.

We introduce the matrix

$$\mathrm{A}(x) := \begin{pmatrix} A_{11}(x) & \dots & A_{1n}(x) \\ \vdots & & \vdots \\ A_{n1}(x) & \dots & A_{nn}(x) \end{pmatrix}.$$

By $X = X(\zeta, x)$ we denote the solution to the boundary value problem

$$\begin{gathered} \operatorname{div}_\zeta \mathrm{A}(x)\nabla_\zeta X = 0 \quad \text{in} \quad \mathbb{R}^n\setminus\overline{\omega}, \qquad X = 0 \quad \text{on} \quad \partial\omega, \\ X(\zeta,x) = 1 + K(x)|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|^{-n+2} + O\big(|\zeta|^{-n+1}\big), \quad \xi\to\infty, \end{gathered} \tag{2.3}$$

where $K = K(x)$ is some function. It was shown in paper [11] that this function is uniformly bounded and non-positive. We denote:

$$V_0(x) := (2-n)\,\mathrm{mes}_{n-1}\,\mathbb{S}^{n-1}\sqrt{\det \mathrm{A}(x)}K(x),$$

where $\mathbb{S}^{n-1}$ is the unit sphere in the space $\mathbb{R}^n$, and mes_{n-1} is the measure on the manifolds in $\mathbb{R}^n$ of codimension one. Later in the work we show that the function V_0 is infinitely differentiable in $x\in\mathbb{R}^d$ and together with all its derivatives it belongs to the space $L_\infty(\mathbb{R}^d)$.

According to the main results of paper [11], there exists a fixed number λ_0 independent of $\varepsilon>0$ such that the half-plane $\operatorname{Re}\lambda<\lambda_0$ is a part of the resolvent set of the operators $\mathcal{H}^\varepsilon$ and $\mathcal{H}^0$ and for all $f\in L_2(\mathbb{R}^d)$ the estimate

$$\|(\mathcal{H}^\varepsilon-\lambda)^{-1}f-(\mathcal{H}^0+aV_0-\lambda)^{-1}f\|_{L_2(\Omega^\varepsilon)}\leqslant C\varepsilon\|f\|_{L_2(\Omega)} \tag{2.4}$$

holds true with a constant C independent of ε, η and f.

Our main result reads as follows.

Theorem 2.1. *Let $f\in W_2^\infty(\mathbb{R}^d)$. Then the leading terms of the asymptotic expansions of the action of the resolvent $u_\varepsilon=(\mathcal{H}^\varepsilon-\lambda)^{-1}f$ are of the form*

$$u_\varepsilon(x) = U_{ex}\left(\frac{x}{\varepsilon},x,\varepsilon,\eta,\mu\right)\big(1-\chi_\varepsilon(x)\big) + U_{in}\left(\frac{x}{\varepsilon\eta},x,\varepsilon,\eta,\mu\right)\chi_\varepsilon(x) + O(\varepsilon\eta^{1+\frac{2}{n}}) \tag{2.5}$$

in the norm of $W_2^1(\mathbb{R}^n)$ and

$$u_\varepsilon(x) = U_{ex}\left(\frac{x}{\varepsilon},x,\varepsilon,\eta,\mu\right)\big(1-\chi_\varepsilon(x)\big) + U_{in}\left(\frac{x}{\varepsilon\eta},x,\varepsilon,\eta,\mu\right)\chi_\varepsilon(x) + O(\varepsilon^2\eta^{1+\frac{2}{n}}) \tag{2.6}$$

in the norm of $L_2(\mathbb{R}^n)$*. Here we denote*

$$\begin{aligned}
U_{ex}(x,\xi,\eta,\mu) &:= u_0(x,\mu) + \varepsilon \sum_{j=0}^{n-1} \eta^j u_{1,j}(x,\mu) + \varepsilon^2 u_{2,0}(\xi,x,\mu) + \varepsilon^2\eta u_{2,1}(\xi,x,\mu),\\
U_{in}(\zeta,x,\varepsilon,\eta\mu) &:= v_{0,0}(\zeta^{(z)},x,\mu) + \varepsilon \sum_{j=0}^{n-1} \eta^j v_{1,j}(\zeta^{(z)},x,\mu)\\
&\quad + \varepsilon^2 v_{2,0}(\zeta^{(z)},x,\mu) + \varepsilon^2\eta v_{2,1}(\zeta^{(z)},x,\mu),\\
\chi_\varepsilon(x) &:= \sum_{z\in\mathbb{Z}^n} \chi(|x\varepsilon^{-1} - z|\eta^{-\frac{2}{n}}), \qquad \mu := \varepsilon^{-2}\eta^{n-2},
\end{aligned} \tag{2.7}$$

and $\chi = \chi(t)$ *is an infinitely differentiable cut-off function equalling to one as* $|t| < 1$ *and vanishing as* $|t| > 2$*. The function* $u_0(x,\mu) := (\mathcal{H}^0 + \mu V_0 - \lambda)^{-1} f$ *is an element of the space* $W_2^\infty(\mathbb{R}^n)$ *and is holomorphic in* μ *in the norm of this space. The function* v_0 *reads as* $v_0(\zeta,\mu) := u_0(x,\mu)X(\zeta,x)$*. Other functions in (2.7) are determined in Section 3.2.*

Let us briefly discuss the main results of the paper. The considered model is an elliptic operator in a periodically perforated domain, and the cavities have a critical size. Namely, the distance between the holes and their linear size are related by identity (2.2) and it leads to the appearance of a strange term: this name is used for the potential aV_0 arising in the homogenized operator, see (2.4). In this important case we construct an asymptotic expansion of the resolvent, namely, the first terms in the expansion. To be able to construct the asymptotics, we have to assume that the right side of f, on which the resolvent acts, is sufficiently smooth, namely, it is an element of the space $W_2^\infty(\mathbb{R}^n)$. In fact, our main result is a formal scheme for constructing an asymptotic expansion presented in Section 3.2. This scheme is based on a combination of the multiscale method [1] and the method of matching asymptotic expansions [6]. In this case, the multiscale method is used to take into account the periodic structure of the distribution of holes and it is used to construct the external expansion of U_{ex}, while the matching method gives the internal expansion of U_{in} and is used to take into account the geometry of the cavities and the boundary condition on their boundaries.

In the paper, this scheme is used to determine the first terms of the asymptotics, which are presented in Theorem 2.1. It can also be used to construct a complete asymptotic expansion under the assumptions of Theorem 2.1. At the same time, the question of determining the structure of this expansion turned out to be unexpectedly difficult. Namely, the first difficulty is connected with the presence of two small parameters ε and η, albeit connected by condition (2.2). It is clear that the asymptotics include power terms in ε and η and, most likely, it consists of terms of the form $\varepsilon^p\eta^q$ with $q = 0,\ldots,n-1$. Another point is related to the fact that in matching the outer and inner expansions one has to use both the fundamental solution of the Laplace operator in $\mathbb{R}^n$, see (3.3), and the solutions of chain of equations (3.4). This chain actually means that the fundamental solution is used as the right hand side in the Poisson equation, then the solution of such equation is again substituted into the right hand side of the new Poisson equation, and so on. As formulae (3.5) show, only power functions appear in odd dimensions, but logarithms also appear in even ones. Exactly these logarithms in matching in the formal construction of asymptotics give rise to the need to introduce additional terms of the form $\varepsilon^p\eta^q \ln^k \eta$ into the formal asymptotics. And a clarification of the dependence of the asymptotics on $\ln\eta$ in even dimensions is a separate non-trivial problem. One of the possible answers is a simple polynomial dependence on $\ln\eta$, that is, when for given p and q the degree of the logarithm changes from zero to some finite value depending on p and q . Another option, by analogy with asymptotic constructions for dimension two, is asymptotics with terms of the form $\varepsilon^p\eta^q u_{p,q}$, where the coefficients $u_{p,q}$ meromorphically depend on $\ln^{-1}\eta$. The third aspect is related to the presence of an additional parameter μ. As it turned out, even the first term of the outer expansion of u_0 nontrivially depends on μ, namely, it is holomorphic with respect to this parameter. Therefore, in the process of constructing a complete expansion, it is also necessary to track the dependence of μ on this parameter, and, most likely, the holomorphic dependence on the parameter μ will remain valid for other terms of the complete asymptotic expansion. In view of this, the problem of constructing a complete asymptotic expansion should be considered separately for even and odd dimensions, and it may well turn out that the case of dimension $n = 2$ is distinguished and requires a separate study. A separate difficulty, primarily of a technical nature, is to track the structure of the asymptotics at zero for the outer expansion functions and the asymptotics at infinity for the inner expansion functions. We postpone the study of these questions about the complete asymptotic expansion for future work, and in this article, as already mentioned above, we demonstrate the formal construction scheme itself.

3. Construction of asymptotics

For a given function $f \in W_2^\infty(\mathbb{R}^n)$, the action of the resolvent of the operator $\mathcal{H}^\varepsilon$ on this function gives a function $u_\varepsilon(x) := (\mathcal{H}^\varepsilon - \lambda)^{-1} f$, which solves the boundary value problem

$$(\mathcal{L} - \lambda) u^\varepsilon = f \quad \text{in} \quad \Omega^\varepsilon, \qquad u^\varepsilon = 0 \quad \text{on} \quad \partial\theta^\varepsilon. \tag{3.1}$$

In the present section we construct a formal asymptotic expansion for the solution to this problem under the assumptions of Theorem 2.1. At the same time, the choice of the number λ ensures the unique solvability of this problem owing to the results in work [11].

The scheme of constructing the asymptotics is based on a combination of the method of matching asymptotic expansions [6] and the multiscaled method [1]. First we prove two auxiliary lemmata, which will be used in the formal construction of the asymptotics.

3.1. Auxiliary lemmata. Here we provide auxiliary lemmata, which will be used for studying the problems for the coefficients in the formal asymptotics expansions, which will be constructed in the next section. In fact, we consider two model auxiliary problems. The first is posed in the periodicity cell $\square := (-\frac{1}{2}, \frac{1}{2})^n$:

$$-\operatorname{div}_\xi \mathrm{A}(x) \nabla_\xi u = h \quad \text{in} \quad \square \setminus \{0\} \tag{3.2}$$

with periodic boundary conditions on $\partial\square$. At a point $\xi = 0$ the function h has a singularity, which will be described later and this generates a singularity of the solution to this problem.

We denote:

$$E_0(t) := \frac{1}{(2-n) t^{n-2} \operatorname{mes}_{n-1} \mathbb{S}^{n-1}}. \tag{3.3}$$

By $E_j = E_j(t)$, $t \geqslant 1$, we denote the sequence of solutions of a recurrent system of equations

$$\frac{1}{t^{n-1}} \frac{d}{dt} t^{n-1} \frac{dE_j}{dt} = E_{j-1}. \tag{3.4}$$

This sequence can be found explicitly:

$$E_j(t) = c_j t^{-n+2+2j}$$

for odd n and

$$\begin{aligned} &E_j(t) = c_j t^{-n+2+2j}, \quad j < \frac{n}{2} - 1, \qquad E_{\frac{n}{2}-1}(t) = c_{\frac{n}{2}-1} \ln t, \\ &E_j(t) = t^{-n+2+2j} c_j(\ln t), \quad j \geqslant \frac{n}{2}, \end{aligned} \tag{3.5}$$

for even n. Here c_j are some non-zero constants for all j as n is odd and $j \leqslant \frac{n}{2} - 1$ as n is even. For $j \geqslant \frac{n}{2}$ and even n the symbols c_j denote some polynomials of the first degree.

Let us describe the smoothness of the function h. We suppose that this is an infinitely differentiable in $\overline{\square}\setminus\{0\}$ function satisfying periodic boundary conditions together with all its derivatives and having the following asymptotic expansion at zero:

$$h(\xi,x)=F_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)+F_1(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)+O\big(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|^{-n+M+3}\big),\qquad \xi\to 0, \tag{3.6}$$

$$\begin{aligned}&F_0(\varsigma,x)=\sum_{j=0}^{m}\mathcal{L}_p(x)E_j(|\varsigma|),\qquad \mathcal{L}_p(x):=\sum_{\substack{\gamma\in\mathbb{Z}_+^n\\ |\gamma|\leqslant M_j}}\alpha_{j,\gamma}(x)\frac{\partial^\gamma}{\partial\varsigma^\gamma},\\ &M:=\min\{2j-M_j,\ j=0,\dots,m\},\end{aligned} \tag{3.7}$$

where M_j are the order of the differential expressions $\mathcal{L}_p(x)$, the coefficients $\alpha_{j,\gamma}(x)$ of which belong to the space $W_2^\infty(\mathbb{R}^n)$, while m is some given natural number; we additionally assume that $M\geqslant n-2$. The function $F_1=F_1(\varsigma,x)$ is a polynomial in ς of degree at most $2M-n+2$ with the coefficients depending on x and belonging to the space $W_2^\infty(\mathbb{R}^n)$. We suppose that asymptotics (3.6) is infinitely differentiable in ξ and x.

We treat the function $h(\xi,x)-F_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)$ as a mapping of the space $\mathbb{R}^n$ into $W_2^{M-n+3}(\square)$ acting by the rule

$$x\mapsto h(\xi,x)-F_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x) \tag{3.8}$$

and suppose that it belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $W_2^{M-n+3}(\square)$-valued functions defined on $\mathbb{R}^n$.

Lemma 3.1. *Problem (3.2) has a unique solution infinitely differentiable in $\xi\in(\overline{\square}\setminus\{0\})\times\mathbb{R}^n$ with the asymptotics*

$$\begin{aligned}u(\xi,x)=&U_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)+U_1(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)\\ &+b(x)G(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|)+O\big(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|^{M-n+5}\big),\qquad \xi\to 0,\\ U_0(\varsigma,x):=&\sum_{j=1}^{m+1}\mathcal{L}_p(x)E_j(|\varsigma|),\end{aligned} \tag{3.9}$$

where $U_1=U_1(\varsigma,x)$ is a polynomial in ς of degree at most $M-n+4$ with the coefficients depending on x and belonging to the space $W_2^\infty(\mathbb{R}^n)$ and satisfying the conditions

$$-\Delta_\varsigma U_1=F_1. \tag{3.10}$$

The function $b(x)$ reads as

$$b(x)=\lim_{r\to 0}\left(\frac{1}{\sqrt{\det\mathrm{A}(x)}}\int\limits_{\square\setminus\{\xi:\,|\mathrm{A}^{-\frac{1}{2}}(x)\xi|<r\}}f(\xi,x)\,d\xi+\int\limits_{\{\varsigma:\,|\varsigma|=r\}}\frac{\partial U_0}{\partial|\varsigma|}(\varsigma,x)\,ds\right), \tag{3.11}$$

where the limit is well-defined and the belonging $b\in W_2^\infty(\mathbb{R}^n)$ holds. The function

$$u(\xi,x)-U_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)-b(x)E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|)$$

considered as a mapping of the space $\mathbb{R}^n$ into $W_2^{M-2}(\square)$ acting by the rule

$$x\mapsto u(\xi,x)-U_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x)-b(x)E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|) \tag{3.12}$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $W_2^{M-n+5}(\square)$-valued functions defined on $\mathbb{R}^n$. The general solution of problem (3.2) differs from the described one by an arbitrary function depending only on x.

Proof. Since F_1 is a polynomial in the variable ς, then the polynomial U_1 satisfying conditions (3.10) is constructed elementary. Let $\chi_1 = \chi_1(\xi)$ be an infinitely differentiable cut-off function equalling to one as $|\xi| < \frac{1}{5}$ and vanishing as $|\xi| > \frac{2}{5}$. We seek a solution to problem (3.2) as

$$\begin{aligned}&u(\xi,x) = \big(\tilde{U}(\mathrm{A}^{-\frac{1}{2}}(x)\xi,x) + b(x)E_0(|\varsigma|)\big)\chi_1(\mathrm{A}^{-\frac{1}{2}}(x)\xi) + \tilde{u}(\xi,x),\\ &\tilde{U}(\varsigma,x) := U_0(\varsigma,x) + U_1(\varsigma,x),\end{aligned} \tag{3.13}$$

where $b(x)$ is some function. Since by construction the function $U(x,\varsigma)$ is infinitely differentiable in $(\xi,x) \in (\overline{\square}\setminus\{0\})\times\mathbb{R}^n$, for the function $\tilde{u}$ we get the boundary value problem

$$-\operatorname{div}_\xi \mathrm{A}\nabla_\xi \tilde{u} = \tilde{h} \quad \text{in} \quad \square, \qquad \tilde{h} := h - \chi_1 F + 2\nabla_\xi\chi_1\cdot \mathrm{A}\nabla_\xi \tilde{U} + \tilde{U}\operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1, \tag{3.14}$$

with periodic boundary conditions and a right hand side belonging at least to the space $W_2^{M-n+3}(\square)$ by asymptotics (3.6). The solvability condition of such problem is standard:

$$\begin{aligned}0 = \int\limits_{\square} \tilde{h}\,d\xi &= \lim_{r\to 0} \int\limits_{\square\setminus\{\xi:\,|\mathrm{A}^{-\frac{1}{2}}(x)\xi|<r\}} \big(h + \operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1\tilde{U} + b(x)\operatorname{div}_\xi \mathrm{A}\nabla_\xi E_0\chi_1\big)\,d\xi\\
&= \lim_{r\to 0}\left(\int\limits_{\square\setminus\{\xi:\,|\mathrm{A}^{-\frac{1}{2}}(x)\xi|<r\}} h\,d\xi + \sqrt{\det \mathrm{A}} \int\limits_{\left\{\varsigma:\, r<|\varsigma|<\frac{2}{5}\right\}} \Delta_\varsigma\big(\tilde{U}(x,\varsigma) + b(x)E_0(|\varsigma|)\big)\,d\varsigma\right)\\
&= \lim_{r\to 0}\left(\int\limits_{\square\setminus\{\xi:\,|\mathrm{A}^{-\frac{1}{2}}(x)\xi|<r\}} h\,d\xi - \sqrt{\det \mathrm{A}} \int\limits_{\{\varsigma:\,|\varsigma|=r\}} \frac{\partial U_0}{\partial|\varsigma|}(x,\varsigma)\,ds\right)\\
&\quad - b(x)\sqrt{\det \mathrm{A}(x)}\lim_{r\to 0}\int\limits_{\{\varsigma:\,|\varsigma|=r\}} \frac{\partial G}{\partial|\varsigma|}(\varsigma)\,ds\\
&= \lim_{r\to 0}\left(\int\limits_{\square\setminus\{\xi:\,|\mathrm{A}^{-\frac{1}{2}}(x)\xi|<r\}} h\,d\xi + \sqrt{\det \mathrm{A}} \int\limits_{\{\varsigma:\,|\varsigma|=r\}} \frac{\partial U_0}{\partial|\varsigma|}(x,\varsigma)\,ds\right) - b(x)\sqrt{\det \mathrm{A}(x)},\end{aligned}$$

which implies formula (3.11). In the same way we obtain one more formula b:

$$\begin{aligned}0 &= \int\limits_{\square}\tilde{h}\,d\xi\\
&= \int\limits_{\square}\big(h - \chi_1 F + 2\nabla_\xi\chi_1\cdot\mathrm{A}\nabla_\xi\tilde{U} + \tilde{U}\operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1\big)\,d\xi + b(x)\sqrt{\det \mathrm{A}} \int\limits_{\left\{\varsigma:\, r<|\varsigma|<\frac{2}{5}\right\}} \Delta_\varsigma E_0(|\varsigma|)\,d\varsigma\\
&= \int\limits_{\square}\big(h - \chi_1 F + 2\nabla_\xi\chi_1\cdot\mathrm{A}\nabla_\xi\tilde{U} + \tilde{U}\operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1\big)\,d\xi - b(x)\sqrt{\det \mathrm{A}(x)},\end{aligned}$$

and this yields

$$b(x) = \frac{1}{\sqrt{\det \mathrm{A}(x)}}\int\limits_{\square}\big(h - \chi_1 F + 2\nabla_\xi\chi_1\cdot\mathrm{A}\nabla_\xi\tilde{U} + \tilde{U}\operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1\big)\,d\xi.$$

By this formula, an explicit form of the function F_1 and an assumed smoothness of the function h as mapping (3.8) we immediately obtain that the function b belongs to the space $W_2^\infty(\mathbb{R}^n)$.

For the functions $u \in W_2^1(\square)$ with the zero mean over $\square$ an obvious estimate

$$\frac{(\mathrm{A}(x)\nabla_\xi u, \nabla_\xi u)_{L_2(\square)}}{\|u\|^2_{L_2(\square)}} \geqslant c_0\frac{\|\nabla_\xi u\|^2_{L_2(\square)}}{\|u\|^2_{L_2(\square)}} \geqslant c_1$$

holds, where c_1 is some constant independent of x and u. This estimate ensures the unique solvability of problem (3.14) for all $\tilde{f} \in L_2(\square)$ with the zero means in the subspace of the functions in $W_2^2(\square)$ satisfying periodic boundary conditions and having a zero mean. At the same time, the operator mapping $\tilde{f}$ into the mentioned solution is bounded uniformly in $x \in \mathbb{R}^n$. Standard smoothness improving theorems for solutions to elliptic boundary value problems then immediately ensure that this operator is bounded uniformly in $x \in \mathbb{R}^n$ as acting into the space $W_2^{M-n+5}(\square)$. This fact allows us to differentiate problem (3.14) in x obtaining in this way similar problems for the derivatives of the solution in x and estimating then them in the norms of the space $W_2^{M-n+5}(\square)$. Returning then back to the function u, we immediately obtain its required smoothness as of mapping (3.12). Since the general solution to homogeneous problem (3.2) (with $f = 0$) is constant in the variable ξ, the statement on the general solution of the inhomogeneous problem is obvious. The proof is complete. $\square$

The second model problem is outer in the domain $\mathbb{R}^n \setminus \overline{\omega}$:

$$-\operatorname{div}_\zeta \mathrm{A}(x)\nabla_\zeta u = g \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\omega}, \qquad u = 0 \quad \text{on} \quad \partial\omega. \tag{3.15}$$

Here $g = g(\zeta, x)$ is an infinitely differentiable in $(\zeta, x) \in (\mathbb{R}^n \setminus \omega) \times \mathbb{R}^n$ function with the following asymptotics at infinity:

$$g(\zeta, x) = F_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + F_1(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + O(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|^{M-n+1}),$$

where the functions F_0, F_1 and the number M are same as in (3.7), while the number M is assumed to satisfy the inequality $M \leqslant -2$. This asymptotics is supposed to be infinitely differentiable in $(\zeta, x) \in (\mathbb{R}^n \setminus \omega) \times \mathbb{R}^n$.

For given numbers $k \in \mathbb{Z}$ and $p \in \mathbb{Z}_+$ by $\mathfrak{C}^{p,k}$ we denote the subspaces of the functions in $C^\infty(\mathbb{R}^n \setminus)$, for which the following norms are finite:

$$\|u\|_{\mathfrak{C}^{p,k}} := \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| \leqslant p}} \sup_{\mathbb{R}^n \setminus \omega} (|\zeta| + 1)^{k+p} \left| \frac{\partial^\alpha u}{\partial \zeta^\alpha}(\zeta) \right|.$$

Let $\chi_2 = \chi_2(\zeta)$ be an infinitely differentiable cut-off function vanishing on some fixed ball containing the domain ω and the origin and equalling to one outside some bigger ball. For the function g we additionally suppose that the mapping of the space $\mathbb{R}^n$ into $\mathfrak{C}^{p,n-M-1+p}$ acting by rule

$$x \mapsto g(\zeta, x) - \Big(F_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + F_1(\mathrm{A}^{-\frac{1}{2}}\zeta, x) \Big) \chi_2(\zeta)$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $\mathfrak{C}^{p,n-M-1+p}$-valued functions defined on $\mathbb{R}^n$ for all $p \in \mathbb{Z}_+$.

Lemma 3.2. *Problem (3.15) has a unique solution infinitely differentiable in $(\zeta, x) \in (\mathbb{R}^n \setminus \omega) \times \mathbb{R}^n$ with the asymptotics*

$$\begin{aligned} u(\zeta, x) =& U_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + U_1(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) \\ &+ U_2(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|) + O\big(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|^{M-n+3}\big), \qquad \xi \to 0, \end{aligned}$$

where U_0, U_1 are same as in (3.9), while the function $U_2(\varsigma, x)$ reads as

$$U_2(\varsigma, x) = \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| \leqslant |M|-1}} b_\alpha(x) \frac{\partial^\alpha}{\partial \varsigma^\alpha} E_0(|\varsigma|), \tag{3.16}$$

where b_α are some functions in $W_2^\infty(\mathbb{R}^n)$. The function

$$u(\zeta, x) - \Big(U_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + U_1(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + U_2(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) \Big) \chi_1(\zeta)$$

considered as a mapping of the space $\mathbb{R}^n$ *into* $\mathfrak{C}^{p,n-M-3+p}(\mathbb{R}^n\setminus\omega)$ *acting by the rule*

$$x\mapsto u(\zeta,x)-\Big(U_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)+U_1(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)+U_2(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)\Big)\chi_1(\zeta)$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ *consisting of* $\mathfrak{C}^{p,n-M-3+p}(\mathbb{R}^n\setminus\omega)$*-valued functions defined on* $\mathbb{R}^n$ *for all* $p\in\mathbb{Z}_+$.

Proof. We seek a solution to problem (3.15) as

$$u(\zeta,x)=\tilde{u}(\zeta,x)+\tilde{U}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)\chi_2(\zeta),$$

where $\tilde{U}$ is from (3.13). Then for the function $\tilde{u}$ we obtain the problem

$$-\operatorname{div}_\zeta \mathrm{A}(x)\nabla_\zeta\tilde{u}=\tilde{g}\quad\text{in}\quad\mathbb{R}^n\setminus\overline{\tilde{\omega}},\qquad \hat{u}=0\quad\text{on}\quad\partial\omega, \tag{3.17}$$

where we have denoted

$$\begin{aligned}\tilde{g}(\zeta,x):=&g-\Big(F_0(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)+F_1(\mathrm{A}^{-\frac{1}{2}}\zeta,x)\Big)\chi_2(\zeta)\\ &+2\nabla_\zeta\chi_2(\zeta)\cdot\mathrm{A}(x)\nabla_\zeta\tilde{U}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta)+\tilde{U}(\mathrm{A}^{-\frac{1}{2}}\zeta,x)\operatorname{div}_\xi \mathrm{A}\nabla_\xi\chi_1(\zeta).\end{aligned}$$

The base of the following proof is the usage of the Kelvin transform. Namely, let ζ_0 be some internal point of the set ω. We introduce the Kelvin transform as follows:

$$\hat{u}(\hat{\zeta}):=|\hat{\zeta}|^{-n+2}\tilde{u}\big(\zeta_0+\hat{\zeta}|\hat{\zeta}|^{-2}\big),\qquad \hat{\zeta}:=|\mathrm{A}^{-\frac{1}{2}}(x)(\zeta-\zeta_0)|^{-2}\mathrm{A}^{-\frac{1}{2}}(x)(\zeta-\zeta_0).$$

Under such change, the domain ω is mapped into some unbounded domain $\hat{\omega}$ depending on x and containing the vicinity of the infinity, while problem (3.15) is transformed into a boundary value problem in a bounded domain $\mathbb{R}^n\setminus\hat{\omega}$:

$$-\Delta_{\hat{\zeta}}\hat{u}=\hat{g}\quad\text{in}\quad\mathbb{R}^n\setminus\overline{\tilde{\omega}},\qquad\hat{u}=0\quad\text{on}\quad\partial\omega,\qquad \hat{g}(\hat{\zeta}):=|\hat{\zeta}|^{-n-2}\tilde{g}\big(\zeta_0+\hat{\zeta}|\hat{\zeta}|^{-2}\big). \tag{3.18}$$

The above conditions for the function g imply immediately that the function g is infinitely differentiable everywhere in $\mathbb{R}^n\setminus\overline{\tilde{\omega}}$ except for the origin, and at the origin it has the smoothness at least $C^{|M|-2+\vartheta}$ with an arbitrary $\vartheta\in(0,1)0$.

Although the domain $\tilde{\omega}$ depends on the variable x, this dependence is smooth and regular. Namely, this domain is obtained under the change of the variable $\zeta\mapsto|\zeta|^{-2}\zeta$ from the domain generated from ω under the linear change $\zeta\to\mathrm{A}^{-\frac{1}{2}}(x)(\zeta-\zeta_0)$. Conditions (2.1) ensure the smoothness, the uniform positivity and the boundedness of the matrix A. This is why the curvatures of the domain $\tilde{\omega}$ are uniformly bounded and the dependence of this boundary on x is also smooth. As a result this gives an opportunity to reproduce the proof of the standard Schauder estimates for problem (3.18) tracking the dependence on the parameter x. As a result we obtain uniform estimates of the following form:

$$\begin{aligned}&\|\hat{u}\|_{C^{|M|}(\mathbb{R}^n\setminus\hat{\omega})}\leqslant C\|\hat{g}\|_{C^{|M|-2+\vartheta}(\mathbb{R}^n\setminus\hat{\omega})}\leqslant C\|\tilde{g}\|_{\mathfrak{C}^{|M|-1,n+|M|+1}},\\ &\|\hat{u}\|_{C^k(\Omega)}\leqslant C\big(\|\hat{g}\|_{C^{k+2+\vartheta}(\hat{\Omega})}+\|\hat{g}\|_{C^\vartheta(\mathbb{R}^n\setminus\hat{\omega})}\big)\leqslant C\Big(\|\tilde{g}\|_{C^{k+3}(\overline{\tilde{\Omega}})}+\|\tilde{g}\|_{\mathfrak{C}^{1,n+3}}\Big),\end{aligned}$$

where C are some constants independent of x, $\hat{g}$ and $\tilde{g}$, while $\Omega\subset\hat{\Omega}\subset\mathbb{R}^n\setminus\hat{\omega}$ are some subdomains in $\mathbb{R}^n\setminus\hat{\omega}$, the closures of which can contain the boundary of the set $\hat{\omega}$ but at the same time they are separated from the origin by a positive distance. The symbol $\tilde{\Omega}$ denotes the image of the domain $\hat{\Omega}$ under the above Kelvin transform; the domain $\tilde{\Omega}$ is bounded.

Since $\hat{u}$ belongs at least to the space $C^{|M|}(\mathbb{R}^n\setminus\hat{\omega})$, its asymptotics at the zero is given by the Taylor series with the remained of order $O(|\hat{\zeta}|^{|M|})$. While returning back to the function $\tilde{u}$, this asymptotics is transformed into the function U_2 defined by formula (3.16). Taking this fact into consideration and returning back to the function $\tilde{u}$, we see immediately that it is infinitely differentiable in ξ on each compact set in $\mathbb{R}^n\setminus\omega$, which can contain entire boundary $\partial\omega$ or a part of it and C^k-norms of the function $\tilde{u}$ on this compact set are estimated by the norm $\|\tilde{g}\|_{\mathfrak{C}^{1,n+3}}$ and C^{k+3}-norms of the function $\tilde{g}$ on a bigger compact set. This allows us to differentiate problem (3.17) in ξ infinitely many times obtaining similar problems for the derivatives of the function $\tilde{u}$ with inhomogeneous Dirichlet conditions. Applying then the Kelvin transform and the aborementioned Schauder estimates for these problems, we can successively estimate $\mathfrak{C}^{p,n-M-1+p}(\mathbb{R}^n \setminus \omega)$-norms for the function $\tilde{u} - U_2(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x)\chi_1(\zeta)$. These apriori estimates then allow us to differentiate problem (3.17) in x and to obtain similar estimates for the derivatives of the function $\tilde{u}$ in x. This completes the proof. □

Remark 3.1. *We note that the only condition for the choice of the polynomial U_2 in the proven lemma is equation (3.10). It fixes the choice of this polynomial up to an arbitrary harmonic polynomial. Lemma 3.2 holds for each possible choice of the polynomial U_2.*

3.2. Formal construction. We construct a formal asymptotic expansion for the solution of problem (3.1) as a combination of the internal and external expansions. The external expansion is used for approximating the solution outside small neighbourhoods of the cavities from the set θ^ε, the internal expansion is employed in small neighbourhoods of these cavities. In the intermediate zone near each cavity these expansions are matched on the base of the method of matching asymptotic expansions.

Applying the multiscale method, we construct the external expansion as

$$u_{ex}^\varepsilon(x) = u_0(x,\mu) + \varepsilon\sum_{j=0}^{n-1}\eta^j u_{1,j}(x,\mu) + \varepsilon^2 u_{2,0}(\xi,x,\mu) + \varepsilon^2\eta u_{2,1}(\xi,x,\mu) + \dots. \tag{3.19}$$

The variable $\xi = \frac{x}{\varepsilon}$ is used to take into consideration the microstructure of the cavities, the variable x is regarded as a slow variable. This is why we seek the functions $u_{k,p}$ as 1-periodic in each of the variables ξ_j. The dependence on the slow variable x should be so that to ensure the belonging of these functions to the space W_2^2 and in fact, this is a condition on the behavior at infinity. At the same time, the coefficients of the external expansion have increasing singularities at the points $z \in \mathbb{Z}^n$ and this is why we seek them in the space W_2^2 on $\mathbb{R}^n$ except for small neighbourhoods of the points including the cavities in θ^ε.

We substitute expansion(3.19) into problem (3.1), take into consideration the presence of the fast variable $\xi = \frac{x}{\varepsilon}$ in the functions u_j, collect the coefficients at the like powers of ε and η and replace the cavities θ^ε by the points they shrink to. Then we obtain the equations

$$-\operatorname{div}_\xi \mathrm{A}(x)\nabla_\xi u_{2,0} = f_{2,0}, \quad -\operatorname{div}_\xi \mathrm{A}(x)\nabla_\xi u_{2,1} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{Z}^n, \tag{3.20}$$

where the function $f_{2,0} = f_{2,0}(x)$ reads as

$$f_{2,0}(x) := \operatorname{div}_x \mathrm{A}(x)\nabla_x u_0(x) - \sum_{j=1}^{n} A_j(x)\frac{\partial u_0}{\partial x_j}(x) + (\lambda - A_0(x))u_0(x) + f(x). \tag{3.21}$$

By convergence (2.4) we can state that the identity

$$u_0(\,\cdot\,,a) := (\mathcal{H}^0 + aV_0 - \lambda)^{-1} f$$

should hold. In view of the assumed smoothness of the function f and the coefficients A_{ij}, A_j, A_0 and V_0, according to smoothness improving theorems for elliptic boundary value problems we easily obtain that $u_0(\,\cdot\,,0) \in W_2^\infty(\mathbb{R}^n)$. Therefore, by the standard embedding theorems, the function $u_0(\,\cdot\,,a)$ is infinitely differentiable and uniformly bounded with all its derivatives on $\mathbb{R}^n$. In what follows, once we obtain an equation for the function u_0, we shall show that the mentioned smoothness holds true also for μ from a small neighbourhood of the point a.

To determine the functions $u_{2,0}$ and $u_{2,1}$ we should complement equations (3.20) by conditions describing their behavior at integer points in $\mathbb{Z}^n$. This will be done during the matching with the internal expansion. This internal expansion in the vicinity of each integer point $z \in \mathbb{Z}^n$ is constructed in the form

$$\begin{aligned} u^{\varepsilon}_{in}(x) =& v_{0,0}(\zeta^{(z)}, x, \mu) + \varepsilon \sum_{j=0}^{n-1} \eta^j v_{1,j}(\zeta^{(z)}, x, \mu) \\ &+ \varepsilon^2 v_{2,0}(\zeta^{(z)}, x, \mu) + \varepsilon^2 \eta v_{2,1}(\zeta^{(z)}, x, \mu) + \ldots, \end{aligned} \tag{3.22}$$

where $v_{k,j}$ are some functions, while $\zeta^{(z)} := (\xi - \zeta)\eta^{-1}$ is one more rescaled variable. In the vicinity of each point $z \in \mathbb{Z}^n$ we pass to the variables $\zeta = \zeta^{(z)}$ in the derivatives and we substitute the internal expansion into boundary value problem (3.1). In the coefficients of the equation and in the right hand side we do not pass to the variable ζ. Then we obtain outer boundary value problems for $v_{k,j}$:

$$-\operatorname{div}_\zeta \mathrm{A}(x) \nabla_\zeta v_{k,j} = F_{k,j} \quad \text{in} \quad \mathbb{R}^n \setminus \omega, \qquad v_k = 0 \quad \text{on} \quad \partial\omega, \tag{3.23}$$

where the right hand sides $F_k = F_k(x, \zeta, \mu)$ are given by the formulae:

$$\begin{aligned} &F_{0,0}(\zeta, x, \mu) := 0, \qquad F_{1,j}(\zeta, x, \mu) := 0, \qquad j \neq 1, \\ &F_{2,0}(\zeta, x, \mu) := 0, \qquad F_{2,1}(\zeta, x, \mu) := 0, \\ &F_{1,1}(\zeta, x, \mu) := \mathcal{L}_1 v_{0,0}, \qquad \mathcal{L}_1 := \big(\operatorname{div}_x \mathrm{A}(x) \nabla_\zeta + \operatorname{div}_\zeta \mathrm{A}(x) \nabla_x\big) - \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial \zeta_j}. \end{aligned} \tag{3.24}$$

We should complement the obtained equation by conditions describing the behavior of the functions v_k at infinity and this will be done as a result of the matching the external and internal expansions.

While matching, we need to determine the behavior of the coefficients of the external expansion in the vicinity of the points, to which the cavities shrink, and of the coefficients in the internal expansion at infinity. This behaviour will be studied as $\xi \to z$, $z \in \mathbb{Z}^n$ for the functions in the external expansion and as $\zeta \to \infty$ for the functions in the internal expansion. The variable x in these functions plays the role of a parameter and there will be no asymptotic formulae with respect to this variable.

In view of the said above and by the method of matching asymptotic expansions, we immediately obtain a condition determining the behavior of the function v_0 at infinity:

$$v_{0,0}(\zeta, x, \mu) = u_0(x, \mu) + \ldots, \qquad \zeta \to \infty.$$

In view of homogeneous equation (3.23), (3.24) for this function, we immediately conclude that it reads as

$$v_0(\zeta, x, \mu) = u_0(x, \mu) X(\zeta, x), \tag{3.25}$$

where X is a solution of problem (2.3). The properties of the function X we shall need in what follows are described in the following lemma.

Lemma 3.3. *The function X is infinitely differentiable in $(\zeta, x) \in (\mathbb{R}^n \setminus \omega) \times \mathbb{R}^n$. Its asymptotic expansion as $\zeta \to \infty$ reads as*

$$X(\zeta, x) = 1 + \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{Z}^n_+ \\ |\alpha| = k}} K_\alpha(x) \frac{\partial^\alpha}{\partial \zeta^\alpha} E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|), \qquad n \geqslant 3, \tag{3.26}$$

where $K_\alpha = K_\alpha(x)$ are some functions belonging to the space $W_2^\infty(\mathbb{R}^n)$. The mapping of the space $\mathbb{R}^n$ into $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n \setminus \omega)$ acting by the rule

$$x \mapsto X(\zeta, x) - \left(1 + \sum_{k=0}^{m} \sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=k}} K_\alpha(x) \frac{\partial^\alpha}{\partial\zeta^\alpha} E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|)\right) \chi_1(\zeta),$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n \setminus \omega)$-valued funcitons defined on $\mathbb{R}^n$ for all $p, m \in \mathbb{Z}_+$.

The statement of this lemma is a striaghtforward corollary of Lemma 3.2 and Remark 3.1: it is sufficient to choose the polynomial U_2 as $U_2 = 1$.

We rewrite asymptotics (3.26) of the function X in the variable $\xi = \zeta\eta + z$ and we substitute it into formula (3.25). Then we compare external and internal expansions (3.19), (3.22) and by the method of matching asymptotic expansion we immediately conclude that the functions of the external expansion should have the following behavior at the points $z \in \mathbb{Z}^n$:

$$u_{2,0}(\xi, x, \mu) = \mu u_0(x, \mu) K_0(x) E_0\big(|\mathrm{A}^{-\frac{1}{2}}(x)(\xi - z)|\big) + o(|\xi - z|^{-n+2}), \qquad \xi \to z, \quad (3.27)$$

$$u_{2,1}(\xi, x, \mu) = \mu u_0(x, \mu) \sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=1}} K_\alpha(x) \frac{\partial^\alpha}{\partial\xi^\alpha} E_0(\mathrm{A}^{-\frac{1}{2}}(x)(\xi - z)) + o(|\xi - z|^{-n+1}), \quad \xi \to z. \quad (3.28)$$

The existence of solution to problem (3.20) with asymptotics (3.27) is ensured by Lemma 3.1. Here as the function $b(x)$ the function $\mu K_0(x)$ serves and identity (3.11) should be treated as the solvability condition. Namely, in this case the function U_0 in (3.9) is absent, that is, $U_0 = 0$, while the function $f_{2,0}$ is independent of ξ. This is why identity (3.11) is rewritten to the form

$$\mu\sqrt{\det \mathrm{A}} K_0 u_0 = f_{2,0},$$

and in view of definition (3.21) this gives the equation for u_0:

$$(\mathcal{H}^0 + \mu V_0 - \lambda) u_0 = f.$$

For μ in a small neighbourhood of a, the potential μV_0 is an analytic perturbation of the potential aV_0, and this is why the resolvent $(\mathcal{H}^0 + \mu V_0 - \lambda)^{-1}$ is holomorphic in μ in a small neighbourhood of the point a and an obvious formula

$$u_0 = \big(\mathcal{I} + (\mu - a)(\mathcal{H}^0 + aV_0 - \lambda)^{-1} V_0\big)^{-1} (\mathcal{H}^0 + aV_0 - \lambda)^{-1} f$$

holds true. By this formula and standard smoothness improving theorems for elliptic boundary value problems we easily find that the function u_0 is holomorphic in μ in the vicinity of the point a in the sense of the norm of the space $W_2^p(\mathbb{R}^n)$ for all $p > 0$.

The mentioned choice of the function u_0 ensures the solvability of problem (3.20), (3.27). According to Lemma 3.1, the solution $u_{2,0}$ is represented as

$$u_{2,0}(\xi, x, \mu) = \mu u_0(x, \mu) K_0(x) \tilde{u}_{2,0}(\xi, x),$$

where $\tilde{u}_{2,0}$ is a periodic solution to problem

$$-\operatorname{div}_\xi \mathrm{A}(x) \nabla_\xi \tilde{u}_{2,0} = \sqrt{\det \mathrm{A}} \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{Z}^n,$$

$$\tilde{u}_{2,0}(\xi, x, \mu) = E_0\big(|\mathrm{A}^{-\frac{1}{2}}(x)(\xi - z)|\big) + \ldots, \quad \xi \to z.$$

Applying Lemma 3.1 with an arbitrarily large M, we can specify the asymptotics of the function $\tilde{u}_{2,0}$ at zero:

$$\tilde{u}_{2,0}(\xi, x) = E_0\big(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|\big) + \frac{\sqrt{\det \mathrm{A}(x)}}{2n} |\mathrm{A}^{-\frac{1}{2}}(x)\xi|^2 + \sum_{p=0}^{\infty} U_{2,0,p}(\mathrm{A}^{-\frac{1}{2}}(x)\xi, x), \qquad \xi \to 0, \quad (3.29)$$

where $U_{2,0,p}(\varsigma, x)$ are homogeneous harmonic polynomials of degree p with infinitely differentiable in $x \in \mathbb{R}^d$ coefficients belonging to the space $W_2^\infty(\mathbb{R}^d)$. At the same time, the function $U_{2,0,0}(\varsigma, x) = U_{2,0,0}(x)$ can be chosen arbitrarily, we apriori assume that it belongs to the space $W_2^\infty(\mathbb{R}^n)$. The mapping

$$x \mapsto \tilde{u}_{2,0}(\xi, x) - E_0\big(|\mathrm{A}^{-\frac{1}{2}}(x)\xi|\big)$$

of the space $\mathbb{R}^n$ into $W_2^q(\square)$ is an element of the space $W_2^\infty(\mathbb{R}^n)$ for each $q > 0$.

In the same way we study the solvability of problem (3.20), (3.28) for the function $u_{2,1}$. Namely, there exists a solution of this problem of form

$$u_{2,1}(\xi, x, \mu) = \mu u_0(x, \mu)\tilde{u}_{2,1}(\xi, x, \mu),$$

where $\tilde{u}_{2,1}$ is a periodic solution to equation (3.20) with the asymptotics at zero

$$\tilde{u}_{2,1}(\xi, x, \mu) = \sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=1}} K_\alpha(x)\frac{\partial^\alpha}{\partial\xi^\alpha}E_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi) + \sum_{p=0}^{\infty} U_{2,1,p}(\mathrm{A}^{-\frac{1}{2}}(x)\xi, x), \qquad \xi \to 0, \tag{3.30}$$

where $U_{2,1,p}(\varsigma, x)$ are homogeneous harmonic polynomials of degree p with infinitely differentiable in $x \in \mathbb{R}^d$ coefficients belonging to the space $W_2^\infty(\mathbb{R}^d)$. The function $U_{2,1,0}(\varsigma, x) = U_{2,1,0}(x)$can be chosen arbitrarily, we apriori assume that it belongs to the space $W_2^\infty(\mathbb{R}^n)$. The mapping

$$x \mapsto \tilde{u}_{2,1}(\xi, x) - \sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=1}} K_\alpha(x)\frac{\partial^\alpha}{\partial\xi^\alpha}E_0(\mathrm{A}^{-\frac{1}{2}}(x)\xi)$$

of the space $\mathbb{R}^n$ into $W_2^q(\square)$ is an element of the space $W_2^\infty(\mathbb{R}^n)$ for each $q > 0$.

Now we rewrite asymptotics (3.29), (3.30) in the variables ζ and we match with the internal expansion (3.22). Then we obtain that the coefficients of the internal expansion should behave as follows at infinity:

$$\begin{aligned}
v_{1,j}(\zeta, x, \mu) =& u_{1,j}(x, \mu) + \ldots, \\
v_{2,0}(\zeta, x, \mu) =& \mu K_0(x)u_0(x, \mu)U_{2,0,0}(x) + \ldots, \\
v_{2,1}(\zeta, x, \mu) =& \mu u_0(x, \mu)\big(U_{2,0,1}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x) + U_{2,1,0}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta, x)\big) + \ldots,
\end{aligned}$$

as $\zeta \to \infty$. The existence of solutions to problem (3.23) with the aforementioned asymptotics is ensured by Lemma 3.2. Namely, the functions $v_{1,j}$ and $v_{2,0}$ can be found explicitly:

$$v_{1,j}(\zeta, x, \mu) = u_{1,j}(x, \mu)X(\xi, x), \qquad j \neq 1, \qquad v_{2,0} = \mu K_0(x)u_0(x, \mu)U_{2,0,0}(x)X(\xi, x).$$

The presence of the right hand side in the equation for the function $v_{1,1}$ prevents finding this function explicitly. However, the application of Lemma 3.2 ensures the existence of a partial solution of form $u_0(x, \mu)\tilde{v}_{1,1}(\xi, x, \mu)$, where the function $\tilde{v}_{1,1}$ solves problem for $v_{1,1}$ but with $F_{1,1}$ replaced by $\mathcal{L}_1 X$. The function $\tilde{v}_{1,1}$ is infinitely differentiable in $(\xi, x) \in (\mathbb{R}^n \setminus \omega) \times \mathbb{R}^n$ and has the following behavior at infinity:

$$\tilde{v}_{1,1}(\zeta, x, \mu) = \sum_{k=0}^{\infty}\sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=k}} K_\alpha(x)\mathcal{L}_1\frac{\partial^\alpha}{\partial\zeta^\alpha}E_1(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|) + \sum_{k=1}^{\infty}\sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=k}} K_{1,1,\alpha}(x)\frac{\partial^\alpha}{\partial\zeta^\alpha}E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|),$$

where $K_{1,1,\alpha}$ are some functions in $W_2^\infty(\mathbb{R}^n)$. The mapping of the space $\mathbb{R}^n$ into $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n\setminus\omega)$ acting by the rule

$$x \mapsto \tilde{v}_{1,1}(\zeta,x) - \chi_1(\zeta)\sum_{k=0}^{m+1}\sum_{\substack{\alpha\in\mathbb{Z}_+^n\\|\alpha|=k}} K_\alpha(x)\mathcal{L}_1\frac{\partial^\alpha}{\partial\zeta^\alpha}E_1(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|)$$
$$- \chi_1(\zeta)\sum_{k=1}^{m}\sum_{\substack{\alpha\in\mathbb{Z}_+^n\\|\alpha|=k}} K_{1,1,\alpha}(x)\frac{\partial^\alpha}{\partial\zeta^\alpha}E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|),$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n\setminus\omega)$-valued function defined on $\mathbb{R}^n$ for all $p,m\in\mathbb{Z}_+$. A solution of problem (3.23), (3.24) is as follows:

$$v_{1,1}(\zeta,x,\mu) = u_0(x,\mu)\tilde{v}_{1,1}(\zeta,x) + u_{1,1}(x,\mu)X(\zeta,x).$$

The problem for the function $v_{2,1}$ is also solvable and its solution reads as $\mu u_0(x,\mu)\tilde{v}_{2,1}(\zeta,x)$, where the function $\tilde{v}_{2,1}$ solves the same problem but has the following asymptotic behavior at infinity:

$$\tilde{v}_{2,1}(\zeta,x) = U_{2,0,1}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x) + U_{2,1,0}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x) + \sum_{k=1}^{\infty}\sum_{\substack{\alpha\in\mathbb{Z}_+^n\\|\alpha|=k}} K_{2,1,\alpha}(x)\frac{\partial^\alpha}{\partial\zeta^\alpha}E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|),$$

where $K_{2,1,\alpha}$ are some functions in $W_2^\infty(\mathbb{R}^n)$. The mapping of the space $\mathbb{R}^n$ into $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n\setminus\omega)$ acting by the rule

$$x \mapsto \tilde{v}_{1,1}(\zeta,x) - \chi_1(\zeta)\Bigg(U_{2,0,1}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x) + U_{2,1,0}(\mathrm{A}^{-\frac{1}{2}}(x)\zeta,x)$$
$$+ \sum_{k=1}^{m}\sum_{\substack{\alpha\in\mathbb{Z}_+^n\\|\alpha|=k}} K_{2,1,\alpha}(x)\frac{\partial^\alpha}{\partial\zeta^\alpha}E_0(|\mathrm{A}^{-\frac{1}{2}}(x)\zeta|)\Bigg)$$

belongs to the space $W_2^\infty(\mathbb{R}^n)$ consisting of $\mathfrak{C}^{p,n+m-2+p}(\mathbb{R}^n\setminus\omega)$-valued functions defined on $\mathbb{R}^n$ for all $p,m\in\mathbb{Z}_+$.

The above described formal scheme allows one to construct further terms in the internal and external expansions. To determine still arbitrary functions $u_{1,j}$, one should construct the terms of order $O(\varepsilon^3)$ in the external expansion. By applying then Lemma 3.1 to this functions, namely, identity (3.11), one can determine the function $u_{1,j}$. A similar situation holds also further: to determine completely a coefficient at $\varepsilon^p\eta^q$, one has to construct the terms up to $\varepsilon^{p+2}\eta^q$ and to write then identity (3.11). The coefficients of the external and internal expansions turn out to be infinitely differentiable in (ξ,x) and (ζ,x) functions; in addition, the analyticity in μ is expected to hold. In odd dimensions the internal and external expansions are power in ε and η. In even dimensions there arises an additional dependence on $\ln\eta$. This is related with the emergence of $\ln t$ in the functions E_j, see (3.5). The study of the dependence of the coefficients in the external and internal expansion on $\ln\eta$ is an independent problem.

3.3. Justification of asymptotics. A rigorous justification of the asymptotics and establishing of the error terms here is made in a standard way. One should construct sufficiently many terms in the external and internal expansions so that they give a sufficiently small error being substituted into the equation for the resolvent. At that the issue on the estimates for the error terms is elementary solved on the base of the known apriori estimates for the resolvent.

Namely, let $Q_{ex}(\xi,x,\varepsilon,\mu)$ and $Q_{in}(\xi,x,\varepsilon,\mu)$ be additional terms in the external and internal expansions. Then we can construct sufficiently many additional terms so that the functions

$$\begin{aligned}U_\varepsilon(x) &:= U_{ex}\left(\frac{x}{\varepsilon},x,\varepsilon,\eta,\mu\right)\chi_\varepsilon(x)+U_{in}\left(\frac{x}{\varepsilon\eta},x,\varepsilon,\eta,\mu\right)\big(1-\chi_\varepsilon(x)\big),\\ Q_\varepsilon(x) &:= Q_{ex}\left(\frac{x}{\varepsilon},x,\varepsilon,\eta,\mu\right)\chi_\varepsilon(x)+Q_{in}\left(\frac{x}{\varepsilon\eta},x,\varepsilon,\eta,\mu\right)\big(1-\chi_\varepsilon(x)\big)\end{aligned}$$

will satisfy the equation and estimates

$$\begin{aligned}&(\mathcal{H}^\varepsilon-\lambda)(U_\varepsilon+Q_\varepsilon)=f+h_\varepsilon, && \|h_\varepsilon\|_{L_2(\Omega^\varepsilon)}=O(\varepsilon^2),\\ &\|Q_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}=O(\varepsilon\eta^{1+\frac{2}{n}}), && \|Q_\varepsilon\|_{L_2(\Omega^\varepsilon)}=O(\varepsilon^2\eta^{1+\frac{2}{n}}).\end{aligned}$$

This implies immediately that

$$\|(\mathcal{H}^\varepsilon-\lambda)^{-1}f-U_\varepsilon-Q_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}\leqslant C\|h_\varepsilon\|_{L_2(\Omega^\varepsilon)},$$

where a constant C is independent of ε. This is why

$$\|(\mathcal{H}^\varepsilon-\lambda)^{-1}f-U_\varepsilon\|_{W_2^1(\Omega^\varepsilon)}=O(\varepsilon\eta^{1+\frac{2}{n}}),\qquad \|(\mathcal{H}^\varepsilon-\lambda)^{-1}f-U_\varepsilon\|_{L_2(\Omega^\varepsilon)}=O(\varepsilon^2\eta^{1+\frac{2}{n}}),$$

and this gives required estimates for the errors in (2.5), (2.6).

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