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CONVERGENCE OF SERIES OF EXPONENTIAL MONOMIALS

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Abstract. In the paper we study the convergence of series of exponential monomials, special cases of which are the series of exponentials, Dirichlet series and power series. We provide a description of the space of coefficients of series of exponential monomials converging in a given convex domain in the complex plane is described. Under a single natural restriction on the degrees of monomials, we provide a complete analogue of the Abel theorem for such series, which, in particular, implies results on the continued convergence of series of exponential monomials. We also obtain a complete analogue of the Cauchy-Hadamard theorem, in which we give a formula allowing to recover the convergence domain of these series by their coefficients. The obtained results include, as special cases, all previously known results related with the Abel and Cauchy-Hadamard theorems for exponential series, Dirichlet series and power series.

Keywords: exponential monomial, convex domain, Abel theorem, Cauchy-Hadamard theorem.

Mathematics Subject Classification: 30D10

1. Introduction

In the work we study the convergence of the series of exponential monomials, that is, of the series of form

$$\sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n e^{\lambda_k z}. \tag{1.1}$$

We study the problem on describing the space of the coefficients of converging series (1.1), the nature of their convergence, describe their convergence domain and study the question on continuing the convergence of series (1.1).

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and their multiplicities $n_k \in \mathbb{N}$. We suppose that $|\lambda_k|$ are non-decreasing and $|\lambda_k| \to \infty$, $k \to \infty$. We let

$$m(\Lambda) = \varlimsup_{k\to\infty} \frac{n_k}{|\lambda_k|}, \quad \sigma(\Lambda) = \varlimsup_{j\to\infty} \frac{\ln j}{|\xi_j|}, \tag{1.2}$$

where $\{\xi_j\}$ is a non-increasing by the absolute value sequence formed by the points λ_k and each point λ_k appears in it exactly n_k times.

The direction related with the series of exponential monomials and their particular case, the series of exponentials, that is, series (1.1), where $n_k = 1$, $k \geqslant 1$, the Dirichlet series ($n_k = 1$ and $\lambda_k > 0$) and the Taylor series has a rich history. It goes back to works by Taylor, Cauchy, Hadamard, Abel and Dirichlet. The aforementioned problems for such series were studied in works E. Khille, G.L. Lunts, A.F. Leontiev and other mathematicians.

For series (1.1), as in the theory of exponential series, in particular, of power series and the Dirichlet series, the most important problems are on describing the classes of the convergence domains including the problem on continuing the convergence and on describing the nature

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of the convergence of the series, as well as the recovering of the convergence domain by the coefficients of the series. In the theory of power series the first two problems are solved by means of the Abel theorem, while the latter problem is solved by means of the Cauchy-Hadamard problem. For the Dirichlet series there is an analogue of the Abel theorem [1, Ch. II, Lm. 1.1], which states that the convergence of the Dirichlet series at a single point z_0 implies its convergence in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < \operatorname{Re} z_0\}$. If $\sigma(\Lambda) = 0$, then this convergence is absolute and uniform in each half plane $\{z \in \mathbb{C} : \operatorname{Re} z < \operatorname{Re} z_0 - \varepsilon\}$ [1, Ch. 2, Thm. 1.1]. There is also a complete analogue of the Cauchy-Hadamard theorem for the Dirichlet series, in which under the condition $\sigma(\Lambda) = 0$, the convergence abscissa is calculated [1, Ch. 2, Thm. 1.2].

In the case of the exponential series a complete analogue of the Abel theorem is absent. There is a result [3], [1, Ch. 2, Thm. 2.1] stating that the set of the points of the absolute convergence of the series is convex. And on the compact subsets of the interior of this set the seris converges uniformly [1, Ch. 2, Thm. 2.2]. If the condition $\sigma(\Lambda) = 0$ is satisfied, then [1, Ch. 2, Thm. 2.3] the simple and the absolute convergence of the exponential series in the convex domain are equivalent. Moreover, for the exponential series, an analogue of the Cauchy-Hadamard theorem is known [3]–[5], [2, Thm. 3.1.3]. In the case of general series of form (1.1) we can only mention the result from work [6]. Here it was proved that the convergence domain of series (1.1) is convex if $m(\Lambda) = 0$.

In work [7] under the conditions $\sigma(\Lambda) = m(\Lambda) = 0$, a complete analogue of the Abel theorem is given for (1.1), in particular, for the exponential series. It was shown that the convergence domain of series (1.1) is a convex domain of a special form. It was proved that the pointwise convergence of series (1.1) in this domain is equivalent to its absolute continuity, uniform continuity on the compact sets and even a convergence in a stronger topology. Also an analogue of Cauchy-Hadamard theorem was provided, which contained all above results as particular cases.

A disadvantage of work [7] is the condition $m(\Lambda) = 0$, which is well appropriate for the case of a bounded convergence domain of series (1.1). In the case of an unbounded domain, the condition $m(\Lambda) = 0$ becomes too restrictive. The aim of the present work is to obtain the results similar to ones in work [7] under a weaker, in the case of an unbounded domain, condition on the multiplicities n_k of the points λ_k.

2. Space of coefficients of converging series

By the symbols $B(z, r)$ and $S(z, r)$ we denote respectively an open ball and a circumference of the radius $r > 0$ centered at the point $z \in \mathbb{C}$. Let $M \subset \mathbb{C}$ and $\overline{M}$ be the closure of the set M. By

$$H(\varphi, M) = \sup_{z \in M} \operatorname{Re}(ze^{-i\varphi}), \quad \varphi \in \mathbb{R},$$

we denote the support function of M and

$$J(M) = \{e^{i\varphi} \in S(0,1) : h(\varphi, M) = +\infty\}.$$

We note that the support function H_M is always lower semi-continuous and is continuous inside the interval, on which it is bounded. In particular, if M is bounded set ([8]), then $H(\varphi, M)$ is a continuous function. If the set $\overline{J(M)} \setminus J(M)$ is non-empty, then it consists of one or two points.

If D is a bounded convex domain, then $J(D) = \emptyset$. In the case of an unbounded convex domain there can be the following situations:

1) $J(D) = S(0,1)$, that is, $D = \mathbb{C}$,
2) D is the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}\}$,
3) D is the strip $\{z \in \mathbb{C} : b < \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}, e^{i\varphi+\pi}\}$,

4) in other cases $J(D)$ is an arc on the unit circumference, which corresponds to an angle at least π.

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ and $D \subset\in \mathbb{C}$ be a convex domain. We describe the set of the sequences of the coefficients $\{d_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$, for which series (1.1) converges in the domain D. By $\mathcal{K}(D) = \{K_p\}_{p=1}^{\infty}$ we denote the sequence of the convex compact sets in the domain D, which strictly exhausts it, that is,

$$K_p \subset \operatorname{int} K_{p+1}, \quad p \geqslant 1, \qquad D = \bigcup_{p=1}^{\infty} K_p. \tag{2.1}$$

Here the symbol int stands for the interior of the set. By the embedding in (2.1) and the definition of the support function, for each $p \geqslant 1$ there exists $\alpha_p > 0$ such that

$$H(\varphi, K_p)(\varphi) + \alpha_p \leqslant H(\varphi, K_{p+1}), \quad \varphi \in [0, 2\pi]. \tag{2.2}$$

We let

$$Q_p(\Lambda) = \{d = \{d_{k,n}\} : \|d\|_p = \sup_{k,n} |d_{k,n}| p^n \exp(r_k H(-\varphi_k, K_p)) < \infty\}, \quad \lambda_k = r_k e^{i\varphi_k},$$

$$Q_{p,0}(\Lambda) = \{d = \{d_{k,n}\} : \|d\|_{p,0} = \sup_{k,n} |d_{k,n}| \exp(r_k H(-\varphi_k, K_p)) < \infty\},$$

$$Q(D, \Lambda) = \bigcap_{p=1}^{\infty} Q_p(\Lambda), \quad Q_0(D, \Lambda) = \bigcap_{p=1}^{\infty} Q_{p,0}(\Lambda).$$

The following inequalities are obvious:

$$\|d\|_{p,0} \leqslant \|d\|_p, \quad p \geqslant 1, \quad \forall d = \{d_{k,n}\}. \tag{2.3}$$

By (2.2), for all $d \in Q(D, \Lambda)$ and $d_0 \in Q_0(D, \Lambda)$ we also have the inequalities

$$\|d\|_1 \leqslant \|d\|_2 \leqslant \ldots \leqslant \|d\|_p \leqslant \ldots, \qquad \|d_0\|_{1,0} \leqslant \|d_0\|_{2,0} \leqslant \ldots \leqslant \|d_0\|_{p,0} \leqslant \ldots. \tag{2.4}$$

Let $\overline{\lambda}$ be the complex conjugate number of λ. By the symbol $\Theta(\Lambda)$ we denote the set of the limits of all converging sequences of form $\left\{\overline{\lambda}_{k_j}/|\lambda_{k_j}|\right\}_{j=1}^{\infty}$. It is obvious that $\Theta(\Lambda)$ is a closed subset of the circumference $S(0, 1)$. We let

$$m(\Lambda, \mu) = \sup \varlimsup_{j\to\infty} \frac{n_{k_j}}{\lambda_{k_j}},$$

where the supremum is taken over all subsequences $\{\lambda_{k_j}\}$ such that $\overline{\lambda_{k_j}}/|\lambda_{k_j}| \to \mu$. If $\mu \notin \Theta(\Lambda)$, then we obviously have the identity $m(\Lambda, \mu) = 0$.

Lemma 2.1. *Let $\Lambda = \{\lambda_k, n_k\}$ and D be a convex domain. Assume that*

$$J(D) \cap \Theta(\Lambda) = \overline{J(D)} \cap \Theta(\Lambda), \quad m(\Lambda) < \infty, \quad m(\Lambda, \mu) = 0, \quad \mu \in \Theta(\Lambda) \setminus \overline{J(D)}. \tag{2.5}$$

Then the following statements hold:

1) for each $p \geqslant 1$ there exist $C > 0$ and $m \geqslant 1$ such that

$$p^{n_k} \exp(r_k H(-\varphi_k, K_p)) \leqslant C \exp(r_k H(-\varphi_k, K_m)), \qquad k \geqslant 1;$$

2) for each $p \geqslant 1$ there exist $C > 0$ and $m \geqslant 1$ such that

$$\|d\|_p \leqslant C \|d\|_{m,0}, \quad \forall d = \{d_{k,n}\};$$

3) topological vector spaces $Q(D, \Lambda)$ and $Q_0(D, \Lambda)$ coincide.

Proof. Suppose that Statement 1 is wrong. Then for some index $p \geqslant 1$ there exists a subsequence $\{\lambda_{k(m)}\}$ such that

$$p^{n_{k(m)}} \exp(r_{k(m)} H(-\varphi_{k(m)}, K_p)) \geqslant m \exp(r_{k(m)} H(-\varphi_{k(m)}, K_m)), \quad m \geqslant 1. \tag{2.6}$$

We choose a subsequence $\{\lambda_{k(m(j))}\}$ so that $\overline{\lambda_{k(m(j))}}/|\lambda_{k(m(j))}|$ converges to some point $\mu = e^{-i\varphi_0} \in \Theta(\Lambda)$. First let $\mu \notin J(D)$. Then, by the first identity in (2.5), $\mu \in \Theta(\Lambda) \setminus \overline{J(D)}$. By the second identity in (2.5) we have: $m(\Lambda, \mu) = 0$. This is why there exists j_0 such that

$$p^{n_{k(m(j))}} = \exp(n_{k(m(j))} \ln p) \leqslant \exp(\alpha_p r_{k(m(j))}), \quad j \geqslant j_0.$$

In view of (2.2) we get:

$$p^{n_{k(m(j))}} \exp(r_{k(m(j))} H(-\varphi_{k(m(j))}, K_p)) \leqslant \exp(r_{k(m(j))} H(-\varphi_{k(m(j))}, K_{p+1})), \quad j \geqslant j_0.$$

This contradicts (2.6).

Now let $\mu \in J(D)$. By the inequality in (2.5) for some $b > 0$ we have $n_k \leqslant b r_k$, $k \geqslant 1$. Then

$$p^{n_{k(m(j))}} \exp(r_{k(m(j))} H(-\varphi_{k(m(j))}, K_p)) \leqslant \exp(b r_{k(m(j))} \ln p + b_p r_{k(m(j))}), \quad j \geqslant 1, \qquad (2.7)$$

where

$$b_p = \max_{z \in K_p} |z|.$$

Since the function $H(\varphi, D)$ is lower semicontinuous, there exists $\delta > 0$ such that

$$H(\varphi, D) > b \ln p + b_p, \quad \varphi \in [\varphi_0 - \delta, \varphi_0 + \delta].$$

Then by the identity in (2.1) for some index l we have:

$$H(\varphi, K_l) \geqslant b \ln p + b_p, \quad \varphi \in [\varphi_0 - \delta, \varphi_0 + \delta].$$

This implies:

$$H(-\varphi_{k(m(j))}, K_l) \geqslant b \ln p + b_p, \quad j \geqslant j_1.$$

Together with (2.7) this contradicts (2.6).

Thus, Statement 1 is true. It implies Statement 2. The latter, in view of (2.3), gives Statement 3. The proof is complete. □

Remark 2.1. *Assume that the first identity in (2.5) is wrong and $m(\Lambda, \mu) > 0$, where μ is a point in the set $\overline{J(D)} \cap (\Lambda)$, which does not belong to the set $J(D) \cap \Theta(\Lambda)$. Then all three statements in Lemma 2.1 are wrong. As an example we consider the domain $D = \{z : \operatorname{Re} z < 0\}$ and the sequence $\Lambda = \{2^k, 2^k\}$. We have*

$$J(D) = S(0,1) \setminus \{1\}, \qquad \Theta(\Lambda) = \{1\}, \qquad m(\Lambda) = m(\Lambda, 1) = 1.$$

Moreover, $0 > H(0, K_p) \to 0$, $p \to \infty$. This implies that Statement 1 of Lemma 2.1 fails. Let $d = \{1, 1, \ldots\}$. Then $d \in Q_0(D, \Lambda)$ and for sufficiently large indices p the identity $\|d\|_p = +\infty$ holds, that is, $d \in Q(D, \Lambda)$.

Remark 2.2. *The first identity in (2.5) is employed in the proof of Statement 1 only to ensure the identity $m(\Lambda, \mu) = 0$, $\mu \notin J(D)$. If we impose the condition $m(\Lambda, \mu) = 0$, $\mu \in \Theta(\Lambda) \setminus J(D)$, then all three Statements of Lemma 2.1 are true independent of the first condition in (2.5). Thus, we obtain the following lemma.*

Lemma 2.2. *Let $\Lambda = \{\lambda_k, n_k\}$ and D be a convex domain. Assume that*

$$m(\Lambda) < \infty, \quad m(\Lambda, \mu) = 0, \quad \mu \in \Theta(\Lambda) \setminus \overline{J(D)}.$$

Then the following statements hold:

1) for each $p \geqslant 1$ there exist $C > 0$ and $m \geqslant 1$ such that

$$p^{n_k} \exp(r_k H(-\varphi_k, K_p)) \leqslant C \exp(r_k H(-\varphi_k, K_m)), \quad k \geqslant 1;$$

2) for each $p \geqslant 1$ there exist $C > 0$ and $m \geqslant 1$ such that

$$\|d\|_p \leqslant C \|d\|_{m,0}, \quad \forall d = \{d_{k,n}\};$$

3) topological vector spaces $Q(D, \Lambda)$ and $Q_0(D, \Lambda)$ coincide.

Let us show that the space $Q(D,\Lambda)$ coincides with the space of the coefficients of converging in the domain D series (1.1).

First of all we formulate two auxiliary statements proved in work [7].

Lemma 2.3. *Let $\Lambda = \{\lambda_k, n_k\}$. The series*

$$\sum_{k=1}^{\infty} n_k e^{-\varepsilon|\lambda_k|}$$

converges for each $\varepsilon > 0$ if and only if $\sigma(\Lambda) = 0$.

Let $E \subset \mathbb{C}$, Θ be a closed subset of the unit circumference $S(0,1)$. A Θ-convex hull of E is the set

$$E(\Theta) = \{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < H(\varphi, E), e^{i\varphi} \in \Theta\}.$$

We observe that $\operatorname{int} E \subset E(\Theta)$. Indeed, let $z \in \operatorname{int} E$. Then the definition of the support function implies the inequalities $\operatorname{Re}(ze^{-i\varphi}) < H_E(\varphi)$, $e^{i\varphi} \in \Theta$. This means that $z \in E(\Theta)$. If $\Theta = S(0,1)$, then the Θ-convex hull of the set coincides with its usual convex hull, more precisely, with the interior of this convex hull, and thus, is a convex domain. The latter holds also in the general situation.

Lemma 2.4. *Let $E \subset \mathbb{C}$, Θ be a closed subset of the circumference $S(0,1)$. Then $E(\Theta)$ is a convex domain.*

Lemma 2.5. *Let $\Lambda = \{\lambda_k, n_k\}$ and $E \subset \mathbb{C}$. Assume that*

1) the general term of the series (1.1) is bounded on the set E, that is,

$$|d_{k,n} z^n e^{\lambda_k z}| \leqslant A(z) < +\infty, \quad k \geqslant 1, \quad n = \overline{0, n_k - 1}, \quad z \in E;$$

2) $m(\Lambda) < \infty$ and $m(\Lambda,\mu) = 0$, $\mu \in \Theta(\Lambda) \setminus \overline{J(E)}$;

3) if $\mu = e^{-i\varphi_0} \in \Theta(\Lambda) \cap \left(\overline{J(E)} \setminus J(E)\right)$ and $m(\Lambda,\mu) > 0$, then there exists $b(\varphi_0) \in \mathbb{R}$ such that the set $B(\varphi_0) = \{z \in E : \operatorname{Re}(ze^{i\varphi_0}) \geqslant b(\varphi_0)\}$ is unbounded;

4) if 0 is an isolated domain in the set E, then the sequence $\{d_{k,n}\}_{k=1,n=0}^{\infty, m_k - 1}$ is bounded.

Then $d = \{d_{k,n}\} \in Q(D,\Lambda)$, where $D = E(\Theta(\Lambda))$.

Proof. Suppose that $d \notin Q(D,\Lambda)$. Then $d \notin Q_p$ for some $p \geqslant 1$, that is, there exists a sequence $\{d_{k(j),n(j)}\}$ such that

$$|d_{k(j),n(j)}| p^{n(j)} \exp(r_{k(j)} H(-\varphi_{k(j)}, K_p)) \to +\infty, \quad j \to \infty. \tag{2.8}$$

Passing to a subsequence once again, we can suppose that $\left\{\overline{\lambda_{k(j)}}/|\lambda_{k(j)}|\right\}$ converges to $\mu = e^{-i\varphi_0} \in \Theta(\Lambda)$. First let

$$\lim_{j\to\infty} \frac{n(j)}{|\lambda_{k(j)}|} = 0. \tag{2.9}$$

By the definition of the quantity $m(\Lambda,\mu)$, for each $\varepsilon > 0$ there exists j_0 such that

$$n_{k(j)} \leqslant \varepsilon r_{k(j)}, \quad j \geqslant j_0. \tag{2.10}$$

Hence, by (2.2) and (2.8) we obtain:

$$|d_{k(j),n(j)}| \exp(r_{k(j)} H(-\varphi_{k(j)}, K_{p+1})) \to +\infty, \quad j \to \infty. \tag{2.11}$$

Since K_{p+2} is a compact set in the domain $D = E(\Theta(\Lambda))$, it follows from the definitions of the set $E(\Theta(\Lambda))$ and the support function that for some $z_0 \in E$ the estimate holds: $\operatorname{Re}(z_0 e^{i\varphi_0}) > H(-\varphi_0, K_{p+2})$. Then in view of (2.2) and the continuity of the support function there exists $\delta > 0$ such that

$$\operatorname{Re}(ze^{i\varphi}) > H(-\phi, K_{p+2}) \geqslant H(-\varphi, K_{p+1}) + \alpha_{p+1}, \quad z \in B(z_0,\delta), \quad e^{i\varphi} \in B(e^{i\varphi_0},\delta). \tag{2.12}$$

We choose an index $j_1 \geqslant j_0$ so that

$$\lambda_{k(j)}/|\lambda_{k(j)}| = e^{i\varphi_{k(j)}} \in B(e^{i\varphi_0}, \delta), \quad j \geqslant j_1. \tag{2.13}$$

Two cases are possible.

1. The set $B(z_0, \delta) \cap E$ contains the point $z_1 \neq 0$.
2. The point $z_0 = 0$ is an isolated one in the set E.

We consider the first case. By (2.10)

$$|z_1|^{n(j)} \geqslant \exp(-\alpha_{p+1} r_{k(j)}), \quad j \geqslant j_2 \geqslant j_1.$$

By (2.12), (2.13) and (2.11) this yields:

$$|d_{k(j),n(j)}(z_1)^{n(j)} e^{\lambda_{k(j)} z_1}| \geqslant |d_{k(j),n(j)}| \exp(r_{k(j)} H(-\varphi_{k(j)}, K_{p+1})) \to +\infty, \quad j \to \infty.$$

This contradicts Condition 1).

If $z_0 = 0$ is an isolated point in E, by (2.11)–(2.13) we find:

$$|d_{k(j),n(j)}| = |d_{k(j),n(j)} e^{\lambda_{k(j)} z_0}| \geqslant |d_{k(j),n(j)}| \exp H(-\varphi_{k(j)}, K_{p+1}) \to +\infty, \quad j \to \infty.$$

This contradicts Condition 4).

Now suppose that (2.9) is wrong. Passing to a subsequence, we can suppose that

$$\lim_{j\to\infty} \frac{n(j)}{|\lambda_{k(j)}|} > 0. \tag{2.14}$$

According to Condition 2), two cases are possible.

1. $\mu \in J(E)$.
2. $\mu \in \Theta(\Lambda) \cap \left(\overline{J(E)} \setminus J(E)\right)$.

By (2.7) and (2.8) we obtain:

$$|d_{k(j),n(j)}| \exp(b r_{k(j)} \ln p + b_p r_{k(j)}) \to +\infty, \quad j \to \infty. \tag{2.15}$$

We consider the first case. Since $\mu \in J(E)$, it follows from the definitions of the set $E(\Theta(\Lambda))$ and the support function that for some $z_0 \in E$ the estimate holds:

$$\mathrm{Re}(z_0 e^{i\varphi_0}) > b \ln p + b_p.$$

We can suppose that $|z_0| \geqslant 1$. We choose $\delta > 0$ such that

$$\mathrm{Re}(z_0 e^{i\varphi}) > b \ln p + b_p, \quad e^{i\varphi} \in B(e^{i\varphi_0}, \delta).$$

Then in view of (2.15) we have:

$$|d_{k(j),n(j)}(z_0)^{n(j)} e^{\lambda_{k(j)} z_0}| \geqslant |d_{k(j),n(j)} e^{\lambda_{k(j)} z_0}| \to +\infty, \quad j \to \infty.$$

This contradicts Condition 1).

Finally, let $\mu \in \Theta(\Lambda) \cap \left(\overline{J(E)} \setminus J(E)\right)$. By Condition 3) of the lemma, the set $B(\varphi_0)$ is unbounded. Hence, in view of (2.14), there exists $z_0 \in E$ such that

$$|(z_0)^{n(j)}| \geqslant \exp((b \ln p + b_p - b(\varphi_0)) r_{k(j)}), \quad j \geqslant j_3.$$

By (2.15) this implies:

$$|d_{k(j),n(j)}(z_0)^{n(j)} e^{\lambda_{k(j)} z_0}| \to +\infty, \quad j \to \infty.$$

This contradicts Condition 1). Thus, $d \in Q(D, \Lambda)$. The proof is complete. □

Remark 2.3. *Condition 4) of Lemma 2.5 is essential. As an example we consider the series*

$$\sum_{k=1}^{\infty} e^{2k} z e^{kz}.$$

Here $\Theta(\Lambda) = \{1\}$. Let $E = \{-2, 0\}$. Then $E(\Theta(\Lambda))$ coincides with the half-plane $\operatorname{Re} z < 0$, while the general term of the series is bounded on E. But this series does not converge on this half-plane since it diverges on the circumference $S(0,1)$. It converges in the half-plane $\operatorname{Re} z < -2$, which coincides with the set $E'(\Theta(\Lambda))$, where $E' = \{-2\}$. In this case Condition 4) of Lemma 2.1 fails, while others are satisfied and the statement of the lemma becomes wrong.

Lemma 2.6. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$ and $\sigma(\Lambda) = 0$. Then for each $p \geqslant 1$ there exist $C_p > 0$ and an index $m(p)$ such that*

$$\sum_{k=1,n=0}^{\infty, m_k-1} |d_{k,n}| \sup_{z\in K_p} |z^n e^{z\lambda_k}| \leqslant C_p \|d\|_{m(p)}, \quad d = \{d_{k,n}\} \in Q(D,\Lambda). \tag{2.16}$$

Proof. We fix $p \geqslant 1$. We choose an index $m(p) > p$ such that

$$m(p) \geqslant b_p = \max_{z\in K_p} |z|.$$

Then by (2.2) and the definition of the support function we obtain:

$$\begin{aligned}\sum_{k=1,n=0}^{\infty, m_k-1} |d_{k,n}| \sup_{z\in K_p} |z^n e^{z\lambda_k}| \leqslant& \sum_{k=1,n=0}^{\infty, m_k-1} |d_{k,n}| (m(p))^n \exp(r_k H(-\varphi_k, K_p)) \\ \leqslant& \|d\|_{m(p)} \sum_{k=1,n=0}^{\infty, m_k-1} \exp(r_k(H(-\varphi_k, K_p) - H(-\varphi_k, K_m(p)))) \\ \leqslant& \|d\|_{m(p)} \sum_{k=1,n=0}^{\infty, m_k-1} \exp(-r_k \alpha_p).\end{aligned}$$

By Lemma 2.3 this implies (2.16). The proof is complete. □

Theorem 2.1. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Assume that $\sigma(\Lambda) = 0$, $m(\Lambda) < \infty$ and $m(\Lambda, \mu) = 0$, $\mu \in \Theta(\Lambda) \setminus \overline{J(D)}$. Then the following statements are equivalent:*
1) Series (1.1) converges in the domain D.
2) The belonging $d = \{d_{k,n}\} \in Q(D,\Lambda)$ holds.

Proof. Let Statement 1) be satisfied. We let $E = D$. By the convergence of series (1.1), in the domain D Condition 1) of Lemma 2.5 is satisfied. According to the assumptions of the theorem, also Condition 2) of Lemma 2.5 is obeyed. Since $E = D$ is a convex domain, it has no isolated points. This is why Condition 4) of Lemma 2.5 holds trivially. We are going to show that Condition 3) of Lemma 2.5 holds as well.

Let $\mu = e^{-i\varphi_0} \in \overline{J(D)} \setminus J(D)$ and $b(\varphi_0) < H(-\varphi_0, D)$. According to the definition of the support function we find a point $z_0 \in D$ such that

$$\operatorname{Re}(z_0 e^{i\varphi_0}) > b(\varphi_0).$$

Since $\mu \in \overline{J(D)} \setminus J(D)$, then μ is a bounded point of an open arc $\gamma \subset S(0,1)$ of the opening π, which is completely contained in the set $J(D)$. Let, for the sake of definiteness,

$$\gamma = \{e^{-i\varphi} : \varphi \in (\varphi_0, \varphi_0 + \pi)\}.$$

We consider the ray $z_t = z_0 + te^{-i(\varphi_0 + \pi/2)}$, $t > 0$. We have:

$$\operatorname{Re}(z_t e^{i\varphi_0}) = \operatorname{Re}(z_0 e^{i\varphi_0}) + \operatorname{Re}(te^{-i\pi/2}) = \operatorname{Re}(z_0 e^{i\varphi_0}) > b(\varphi_0). \tag{2.17}$$

Since $\gamma \subset J(D)$, then

$$\operatorname{Re}(z_t e^{i\varphi}) < H(-\varphi, D) = +\infty, \quad \varphi \in (\varphi_0, \varphi_0 + \pi).$$

Moreover, since $z_0 \in D$, then

$$\operatorname{Re}(z_t e^{i\varphi}) = \operatorname{Re}(z_0 e^{i\varphi}) + t \operatorname{Re}(e^{-i(\varphi_0+\pi/2-\varphi)}) \leqslant \operatorname{Re}(z_0 e^{i\varphi}) < H(-\varphi, D), \quad \varphi \in [\varphi_0 - \pi, \varphi_0].$$

It follows from latter identities that

$$\operatorname{Re}(z_t e^{i\varphi}) < H(-\varphi, D), \quad t > 0, \quad \varphi \in [\varphi_0 - \pi, \varphi_0 + \pi],$$

that is, $z_t \in D$, $t > 0$. Together with (2.17) this means that the set $B(\varphi_0)$ is unbounded.

Thus, all assumptions of Lemma 2.5 are satisfied. Then according to this lemma, $d = \{d_{k,n}\} \in Q(D(\Theta(\Lambda)), \Lambda)$. Since D is a convex domain, then

$$D = \{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < H(\varphi, D), \varphi \in [0, 2\pi]\}.$$

This is why $D \subset D(\Theta(\Lambda))$. Now by the definition of $Q(D, \Lambda)$ we easily obtain the embedding

$$Q(D(\Theta(\Lambda))) \subset Q(D, \Lambda).$$

Therefore, $d \in Q(D, \Lambda)$.

Now let Statement 2) be true. Then by Lemma 2.6 series (1.1) converges on each compact set in the domain D, and hence, on the entire domain D. The proof is complete. □

Remark 2.4. *According to Theorem 2.7 and Lemma 2.6, under the assumptions of this theorem, the pointwise convergence of series (1.1) in the domain D is equivalent to its absolute and uniform convergence on the compact sets in this domain.*

3. Analogue of Abel and Cauchy-Hadamard theorems for series of exponential monomials

The following result is an analogue of the Abel theorem for series (1.1).

Theorem 3.1. *Let $\Lambda = \{\lambda_k, n_k\}$ and $E \subset \mathbb{C}$. Assume that*

1) the general term of series (1.1) is bounded on the set E, that is,

$$|d_{k,n} z^n e^{\lambda_k z}| \leqslant A(z) < +\infty, \quad k \geqslant 1, \quad n = \overline{0, n_k - 1}, \quad z \in E;$$

2) $\sigma(\Lambda) = 0$, $m(\Lambda) < \infty$ and $m(\Lambda, \mu) = 0$, $\mu \in \Theta(\Lambda) \setminus \overline{J(E)}$;

3) if $\mu = e^{-i\varphi_0} \in \Theta(\Lambda) \cap \left(\overline{J(E)} \setminus J(E)\right)$ and $m(\Lambda, \mu) > 0$, then there exists $b(\varphi_0) \in \mathbb{R}$ such that the set $B(\varphi_0) = \{z \in E : \operatorname{Re}(ze^{i\varphi_0}) \geqslant b(\varphi_0)\}$ is unbounded;

4) if 0 is an isolated point in the set E, then the sequence $\{d_{k,n}\}_{k=1,n=0}^{\infty,m_k-1}$ is bounded.

Then for each $p \geqslant 1$ there exist $C_p > 0$ and an index $m(p)$ such that

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \sup_{z \in K_p} |z^n e^{z\lambda_k}| \leqslant C_p \|d\|_{m(p)}, \quad d = \{d_{k,n}\} \in Q(D, \Lambda),$$

where $D = E(\Theta(\Lambda))$, $\{K_p\} = \mathcal{K}(D)$. In particular, series (1.1) converges absolutely and uniformly on each compact set in the domain D.

Proof. Let the assumptions of the theorem hold. Then by Lemma 2.5, $d \in Q(D, \Lambda)$. Therefore, by Lemma 2.6, inequality (2.16) holds. In particular, this means that series (1.1) converges absolutely and uniformly on each compact set in the domain D. The proof is complete. □

Remark 3.1. *Let E be the convergence domain of series (1.1). Theorem 3.1 implies that under its assumptions the interior of the set E is always convex and is even a $\Theta(\Lambda)$-convex domain, that is, is a domain of the form*

$$\{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < H(\varphi), e^{i\varphi} \in \Theta(\Lambda)\},$$

where $H(\varphi)$ is a lower semi-continuous function.

Remark 3.2. *If we omit the assumption $\sigma(\Lambda)=0$ from Theorem 3.1, its statement becomes wrong. To support this, we consider a series from book* [1]*:*

$$\sum_{k=1}^{\infty}(-1)^k e^{\lambda_k z}, \quad \lambda_k=\ln\ln k, \quad k\geqslant 1. \tag{3.1}$$

It converges as a sign-alternating series for all $z=x<0$ and diverges at the point $z=0$. All assumptions of Theorem 3.1 except for the identity $\sigma(\Lambda)=0$ are satisfied. In our case $\sigma(\Lambda)=+\infty$. Had the statement of Theorem 3.1 been true, then series (3.1) would have converged in the half-plane $\{z\in\mathbb{C}:\operatorname{Re}z<0\}$. However, for each $x>0$ and all sufficiently large indices k we have:

$$|e^{\lambda_k x}|=\frac{1}{(\ln k)^x}>\frac{1}{k},$$

that is, series (3.1) converges absolutely on the negative real semi-axis.

Remark 3.3. *The condition $m(\Lambda,\mu)=0$, $\mu\in\Theta(\Lambda)\setminus\overline{J(E)}$ is also essential. We consider the series*

$$\sum_{k=1}^{\infty}e^{2k}z^{2k-1}e^{kz}.$$

It is easy to show that this series converges in some domain lying in the half-plane $\operatorname{Re}z<-a$, where $a>1$ is chosen by the condition $a>2(2\ln a+1)$, and in the circle $B(0,r)$, where $r\in(0,1)$ is such that $-2^{-1}2\ln r>3$. At the same time it obviously diverges on the circumference $S(0,1)$. Thus, the interior of the convergence set of this series is not a convex domain and even is not a domain since its not connected.

Theorem 3.1 is an analogue of the Abel theorem for power series which are a particular case of the exponential series. However, if we reformulate Theorem 3.1 for this particular case, we obtain a weaker statement that the Abel theorem. This is explained by the fact that the circles, on which the power series is to converge absolutely and uniformly, are mapped into unbounded domains under the mentioned transformation. At the same time, in Theorem 3.1, the uniform convergence is guaranteed only on the compact subsets. Complicating essentially the proof of this theorem, we can show that series (1.1) still converges uniformly in some cases on unbounded sets. However, these sets not always contain the images of circles under the change of the variable that transforms a power series into an exponential series. We consider the series

$$\sum(e^{kz}+ze^{kz}). \tag{3.2}$$

Let $E=\{0\}$. We have $\Theta(\Lambda)=\{1\}$. By Theorem 3.1, series (3.2) converges in the domain $E(\Theta(\Lambda))=\{z\in\mathbb{C}:\operatorname{Re}z<0\}$ and uniformly on its compact subsets. One can show that series (3.2) converges uniformly on some unbounded set, for instance, on the angles of opening strictly less than π with the vertices on the negative real semi-axis. But it does not converge uniformly in any half-plane of form $\Pi(a)=\{z:\operatorname{Re}z<-a\}$, $a>0$. Now we consider the series

$$\sum e^{kz}. \tag{3.3}$$

It is obtained from the power series $\sum w^k$ by means of the transformation $w=e^z$. The latter converges in the ball $B(0,1)$, while by the Abel theorem it converges in each ball of a smaller radius. Under the mentioned transform, these balls are mapped onto the half-planes $\Pi(a)$. Therefore, series (3.2) converges uniformly in each of these half-planes. Such difference in the set of uniform convergence of series (3.2) and (3.3) is related with the presence of the factors z in series (3.2). Keeping the same factors, it is impossible to prove theorem of like Theorem 3.1 so that its particular case would the Abel theorem for power series. However, this situation can be fixed by omitting the factors z^n in series (1.1), that is, considering only «pure» exponential series; this is confirmed by the following result from work [7].

Let $E \subset \mathbb{C}$ and Θ be a closed subset in $S(0,1)$. We let

$$E(\Theta,\varepsilon) = \{z \in \mathbb{C} : \mathrm{Re}(ze^{-i\varphi}) < H(\varphi, E), e^{i\varphi} \in \Theta\}, \quad \varepsilon > 0.$$

We note that in the case when Θ lies in some angle with the vertex at the origin of an opening at most π, the set $E(\Theta,\varepsilon)$ is unbounded for a sufficiently small $\varepsilon \geqslant 0$.

Theorem 3.2. *Let $\Lambda = \{\lambda_k, 1\}$, $\sigma(\Lambda) = 0$, $E \subset \mathbb{C}$ and Θ be a closed subset in $S(0,1)$ such that*

$$\overline{\lambda_k}/|\lambda_k| \in \Theta, \quad k \geqslant k_0.$$

Assume that general term of series (1.1) is uniformly bounded on the set E, that is,

$$|d_k e^{\lambda_k z}| \leqslant A, \quad k \geqslant 1, \quad z \in E. \tag{3.4}$$

Then for each $\varepsilon > 0$ there exists $c(\varepsilon,\Lambda) > 0$ such that

$$\sum_{k=k_0}^{\infty} |d_k e^{\lambda_k z}| \leqslant Ac(\varepsilon,\Lambda), \quad z \in E(\Theta,\varepsilon).$$

In particular, series (1.1) converges absolutely and uniformly on the set $E(\Theta,\varepsilon)$.

Remark 3.4. *We consider a series of exponentials*

$$\sum d_k e^{kz}, \tag{3.5}$$

into which the power series $\sum d_k w^k$ is transformed under the change $w = e^z$. In this case $\sigma(\Lambda) = 0$ and for each $k \geqslant 1$ the belonging holds: $\lambda_k/|\lambda_k| \in \Theta = \{1\}$. Let $E = \{z_0\}$ and (3.4) be true. Then by Theorem 3.2 series (3.5) converges absolutely and uniformly on the set $E(\Theta,\varepsilon) = \{z : \mathrm{Re}\, z < \mathrm{Re}\, z_0 - \varepsilon\}$, $\varepsilon > 0$. This gives the Abel theorem for the power series.

The Cauchy-Hadamard theorem provides a formula for calculating the convergence radius of a power series. For exponential series, an analogue of the circle is the half-plane, it is the image of the circle under the change $w = e^z$. An analogue of the radius of the circle is the distance from the origin to the half-plane. If $\Theta(\Lambda)$ consists of two points, then the corresponding $\Theta(\Lambda)$-convex convergence domain of series (1.1) is the intersection of two half-planes. This domain has already two «convergence radii», which are the distance from the origin to two straight lines being the boundaries of these half-planes. If $\Theta(\Lambda)$ is an infinite set, then there are infinitely many corresponding «convergence radii» of series (1.1). It should be noted that some of the distance are to be taken with the minus sign. Such situation arises in the case, when the convergence domain does not contain the origin.

We consider the series

$$\sum 2^k e^{kz}.$$

Applying the Abel theorem to the associated power series, we establish easily that its convergence domain is the half-plane $\{z : \mathrm{Re}\, z < \ln \frac{1}{2}\}$. To avoid ambiguities, here as the «convergence radius» the quantity $-\ln 2$ should be treated, which is equal to the distance from the origin to the straight line bounding the half-plane taken with minus sign and not the distance itself. Let us clarify the said statement. We consider one more series

$$\sum 2^{-k} e^{kz}.$$

The convergence domain of this series is the half-plane $\{z : \mathrm{Re}\, z < \ln 2\}$. Here the «convergence radius» is equal to $\ln 2$, that is, to the distance from the origin to the straight line bounding the half-plane.

Let $e^{i\varphi} \in \Theta(\Lambda)$. For the sequence of the coefficients d of series (1.1) we let

$$h_0(d,\varphi) = \inf_{\lambda_{k(j)}} \varliminf_{j\to\infty} \min_{0\leqslant n\leqslant n_{k(j)}-1} \frac{-\ln|d_{k(j),n}|}{r_{k(j)}},$$

$$h(d,\varphi) = \inf_{\{\lambda_{k(j)}\}} \inf_{p\in\mathbb{N}} \varliminf_{j\to\infty} \min_{0\leqslant n\leqslant n_{k(j)}-1} \frac{-(\ln|d_{k(j),n}| + n\ln p)}{r_{k(j)}},$$

where the infimum is taken over all subsequences $\{\lambda_{k(j)}\}$ such that $\overline{\lambda_{k(j)}}/|\lambda_{k(j)}| \to e^{i\varphi}$, $j\to\infty$. It follows from the definition of the function $h(d,\varphi)$ that it is lower semi-continuous . Then, as in Lemma 2.4 proven in work [7], one can show that the set

$$D(d,\Lambda) = \{z : \mathrm{Re}(ze^{-i\varphi}) < h_0(d,\varphi), e^{i\varphi} \in \Theta(\Lambda)\}$$

is a $\Theta(\Lambda)$-convex domain. The definitions of the quantities $h_0(d,\varphi)$ and $h(d,\varphi)$ imply that in the cases when $m(\Lambda,e^{i\varphi}) = 0$ or $m(\Lambda,e^{i\varphi}) < \infty$ and $h_0(d,\varphi) = +\infty$ the identity $h_0(d,\varphi) = h(d,\varphi)$ holds.

Theorem 3.3. *Let $\Lambda = \{\lambda_k, n_k\}$. Suppose that $D(d,\Lambda) \neq \emptyset$,*

$$\sigma(\Lambda) = 0, \quad m(\Lambda) < \infty, \quad m(\Lambda,\mu) = 0, \quad \mu \in \Theta(\Lambda) \setminus \overline{J(D(d,\Lambda))}, \tag{3.6}$$

$$h(d,\varphi) > -\infty, \quad e^{i\varphi} \in \Theta(\Lambda) \cap \left(\overline{J(D(d,\Lambda))} \setminus J(D(d,\Lambda))\right). \tag{3.7}$$

Then series (1.1) converges at each point in the domain $D(d,\Lambda)$ and diverges at each point of its exterior $\mathbb{C} \setminus \overline{D(d,\Lambda)}$ except for the origin if the series $\sum d_{k,0}$ diverges.

Proof. Let $\{K_p\} = \mathcal{K}(D)$, where $D = D(d,\Lambda)$. We are going to show that $d \in Q(D,\Lambda)$. Suppose that this is wrong. Then for $p \geqslant 1$ there exists a subsequence $\{d_{k(j),n(j)}\}$ such that (2.8) holds an $\{\overline{\lambda_{k(j)}}/|\lambda_{k(j)}|\}$ converges to $\mu = e^{-i\varphi_0} \in \Theta(\Lambda)$. First let (2.9) be true. Then relation (2.11) holds.

Since K_{p+1} is a compact set in the domain D, it follows from its definition that

$$H(-\varphi_0, K_{p+1}) \leqslant h_0(d,-\varphi_0) - 2\varepsilon$$

for some $\varepsilon > 0$. Then according to the definition of the quantity $h_0(d,-\varphi_0)$ there exists an index $j_1 \geqslant j_0$ such that

$$\frac{-\ln|d_{k(j),n(j)}|}{r_{k(j)}} \geqslant H(-\varphi_0, K_{p+1}) + \varepsilon, \quad j \geqslant j_1.$$

By the continuity of the support function of a compact set we can suppose that

$$H(-\varphi_0, K_{p+1}) + \varepsilon \geqslant H(-\varphi_{k(j)}, K_{p+1}), \quad j \geqslant j_1.$$

Then

$$\frac{-\ln|d_{k(j),n(j)}|}{r_{k(j)}} \geqslant H(-\varphi_{k(j)}, K_{p+1}), \quad j \geqslant j_1.$$

This yields:

$$|d_{k(j),n(j)}| \leqslant \exp(-r_{k(j)} H(-\varphi_{k(j)}, K_{p+1}), \quad j \geqslant j_1,$$

which contradicts (2.11).

Now suppose that (2.9) is wrong. Passing to a subsequence, we can suppose that (2.14) holds. In accordance with (3.6), two cases are possible.

1. $\mu \in J(D)$.
2. $\mu \in \Theta(\Lambda) \cap \left(\overline{J(D)} \setminus J(D)\right)$.

As in Lemma 2.5, convergence (2.15) holds true. Let us consider the first case. Since $\mu \in J(D)$, then $h_0(d, -\varphi_0) = +\infty$. This is why there exists an index j_2 such that

$$\frac{-\ln |d_{k(j),n(j)}|}{r_{k(j)}} \geqslant b \ln p + b_p, \quad j \geqslant j_2.$$

This gives:

$$|d_{k(j),n(j)}| \leqslant \exp(-b r_{k(j)} \ln p - b_p r_{k(j)}), \quad j \geqslant j_2,$$

which contradicts (2.15).

Finally, let $\mu \in \Theta(\Lambda) \cap \left(\overline{J(D)} \setminus J(D)\right)$. It follows from (3.7) and (2.14) that

$$\frac{-\ln |d_{k(j),n(j)}|}{r_{k(j)}} \to +\infty, \quad j \to \infty.$$

This contradicts (2.15).

Thus, $d \in Q(D, \Lambda)$. Then by Theorem 2.7, series (1.1) converges in the domain D.

Now let $z \in \mathbb{C} \setminus \overline{D}$. If $z = 0$ and series $\sum d_{k,0}$ converges, then series (1.1) converges at the point $z = 0$. Let $z \neq 0$. By the definition of the domain D there exists $e^{i\varphi} \in \Theta(\Lambda)$ such that

$$\operatorname{Re}(z e^{-i\varphi}) > h_0(d, \varphi). \tag{3.8}$$

According to the definition of the quantity $h_0(d, \varphi)$, there exists a subsequence

$$\overline{\lambda_{k(j)}}/|\lambda_{k(j)}| \to e^{-i\varphi},$$

for which

$$h_0(d, -\varphi) = \lim_{j\to\infty} \frac{-\ln |d_{k(j),n(j)}|}{r_{k(j)}}. \tag{3.9}$$

Suppose that series (1.1) converges at the point z. Then series (1.1) converges at each point of the set $E = \{z\} \cup D$. This is why general term of series (1.1) is bounded on the set E. This gives Condition 1) of Lemma 2.5. By construction,

$$H(\psi, E) = \max\{H(\psi, D), \operatorname{Re}(z e^{-i\psi})\}, \quad \psi \in [0, 2\pi].$$

Therefore, $J(D) = J(E)$. Then by (3.6) we obtain Condition 2) of Lemma 2.5. Condition 3) of Lemma 2.5 holds owing to the same arguing as in Theorem 2.7. Finally, Condition 4) holds trivially since the set E has the unique isolated point $z \neq 0$. Then according to Lemma 2.5, $d \in Q(D_0, \Lambda)$, where $D_0 = E(\Theta(\Lambda))$.

By (3.8) in the domain D_0 there exists a point z_0 such that

$$\operatorname{Re}(z_0 e^{i\varphi}) > h_0(d, -\varphi). \tag{3.10}$$

We choose an index p, for which the compact set $K_{0,p} \in \mathcal{K}(D_0)$ contains z_0. Since $d \in Q(D_0, \Lambda)$, then

$$|d_{k,n}| \leqslant B p^{-n} \exp(-r_k H(-\varphi_k, K_{0,p})) \leqslant B \exp(-r_k H(-\varphi_k, K_{0,p})), \quad k \geqslant 1, \quad n = \overline{0, n_k - 1},$$

where $B > 0$. Since $z_0 \in K_{0,p}$, then

$$\operatorname{Re}(z_0 e^{i\varphi_k}) \leqslant H(-\varphi_k, K_{0,p}), \quad k \geqslant 1.$$

In view of the said above this implies

$$|d_{k,n}| \leqslant B \exp(-r_k \operatorname{Re}(z_0 e^{i\varphi_k})), \quad k \geqslant 1.$$

Then by (3.10) we have:

$$\lim_{j\to\infty} \frac{-\ln |d_{k(j),n(j)}|}{r_{k(j)}} \geqslant \lim_{j\to\infty} \frac{-\ln B + r_{k(j)} \operatorname{Re}(z_0 e^{i\varphi_k})}{r_{k(j)}} = \operatorname{Re}(z_0 e^{i\varphi}) > h_0(d, -\varphi).$$

This contradicts (3.9). The proof is complete. □

Remark 3.5. *In the particular case for series (3.5) we have the formula*

$$h_0(d,0) = \varliminf_{k\to\infty} \frac{-\ln|d_k|}{k} = \varliminf_{k\to\infty} (-\ln \sqrt[k]{|d_k|}).$$

Making the change $w = e^z$ transforming series (3.5) into the power series, we obtain the following formula for the convergence radius of the latter

$$R = \exp h_0(d,0) = \varliminf_{k\to\infty} \frac{1}{\sqrt[k]{|d_k|}}.$$

Thus, we have obtained the Cauchy-Hadamard formula for the power series.

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