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ON C^1-CONVERGENCE OF PIECEWISE POLYNOMIAL SOLUTIONS TO A FOURTH ORDER VARIATIONAL EQUATION

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Abstract. In the present work we consider a boundary value problem in a polygonal domain for a fourth order variational equation. We assume that this domain is partitioned into finitely many triangles forming its triangulation. We introduce a class of piecewise polynomial functions of a given degree and for a considered equation we define the notion of a piecewise polynomial solution on a triangle net. We prove a theorem on existence and uniqueness of such solution. Moreover, we establish that under certain conditions for the triangulation of the domain, the second derivatives of the piecewise polynomial solutions are estimated by a constant independent of the fineness of the partition. This fact allows us to prove C^1-convergence of piecewise polynomial solutions to the equations as the fineness of grid tends to zero.

Keywords: biharmonic functions, triangular grid, piecewise polynomial function, approximation error.

Mathematics Subject Classification: 35A15, 65N12

1. Introduction

We consider a functional of form

$$I(f)=\int\limits_{\Omega} G(x,f,\nabla f,D^2 f)\,dx,\qquad D^2 f=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)_{i,j=1}^{n}, \tag{1.1}$$

which is defined for functions $f\in C^2(\overline{\Omega})$ and the function G reads as

$$G=G(x,u,\xi_1,\ldots,\xi_n,\eta_{11},\eta_{12},\ldots,\eta_{nn}).$$

We assume that the function $G(x,u,\xi,\eta)$ has continuous derivatives up to the third order in all its variables.

For the functional $I(f)$ we can write a corresponding Euler-Lagrange equation of the variational problem

$$\begin{aligned}Q[f]\equiv\sum_{i,j=1}^{n}\Big(G'_{\eta_{ij}}(x,f,\nabla f,D^2 f)\Big)''_{x_ix_j}&\\ -\sum_{i=1}^{n}\big(G'_{\xi_i}(x,f,\nabla f,D^2f)\big)'_{x_i}&+G'_u(x,f,\nabla f,D^2 f)=0.\end{aligned} \tag{1.2}$$

We note that a particular case of equation (1.2) is the biharmonic equation

$$\frac{\partial^4 f}{\partial x_1^4}+2\frac{\partial^4 f}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 f}{\partial x_2^4}=0,$$

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with the corresponding functions $G = \eta_{11}^2 + 2\eta_{12}^2 + \eta_{22}^2$ and $G = (\eta_{11} + \eta_{22})^2$. As one more example we adduce a free energy functional for a deformed plate, see [1, Ch. II], playing an important role in the elasticity theory,

$$\iint\limits_{\Omega} \left\{ \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \right)^2 + 2(1-\sigma) \left(\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} \right) \right\} dxdy.$$

A huge amount of works both in Russian and foreign journals are devoted to a numerical solving of fourth order equations, including a biharmonic one. For instance, in work [2], in solving a boundary value problem for a variational nonlinear inequality with a biharmonic operator, a Newton-Kantorovich method was used, at each step of which a linear problem is solved by a difference method. In [3] a sharp estimate was established for the error in calculating the eigenvalues of a discrete problem for a biharmonic operator in a rectangular domain. The author showed that this error is of second order, moreover, it was possible to calculate its main part as of an infinitesimal quantity. We also note paper [4], where the author proved an estimate for the convergence of approximate solutions in the grid norm W_2^2 in terms of the grid step $h \to 0$. As in the previous article, the results were obtained in a rectangular area. It is interesting to mention the article [5], in which integral representations were derived for exact solutions of the Dirichlet boundary value problem for a certain family of elliptic equations, which also included the biharmonic equation. We also cite paper [6], where the existence and uniqueness of a solution of a Robin-type problem for an inhomogeneous biharmonic equation in the unit ball were shown. For a rectangular area, the author of [7] succeeded to reduce the Neumann problem for a certain class of fourth order linear equations to Fredholm integral equations and to prove their solvability under certain conditions on the coefficients of the equation. In paper [8] the fourth order elliptic equation was considered in a rectangular domain under mixed boundary conditions. Its solution was based on iterative factorization of an operator that is energetically equivalent to the operator of the problem being solved; the original problem was discretized by using the finite element method. The problem of convergence of approximate solutions obtained by the finite element method was quite thoroughly investigated in paper [9]. The author proved the unique solvability of the corresponding variational problems and the convergence of the solutions constructed by the variational method in the space W_p^1. In the next work [10] a new version of the collocation and least squares (CLS) method for the numerical solution of the inhomogeneous biharmonic equation was developed. The idea was based on projecting the original problem into the space of polynomials of the fourth and eighth degrees. Note that the method gives very good results even for a small number of grid points. However, in the case of non-linear equations, it needs to be revised. Work [11] was also devoted to an approximate solving of the biharmonic equation. In this work, the authors presented the results on calculations by using some difference scheme with a fairly good degree of accuracy. Moreover, just as in the previous article, the area in which the solution is sought had a general form. It is also worth noting that in this work there is a comparative analysis with the results of another work [12], noting the advantages and disadvantages of the method. In [13] the deviation of a piecewise-cubic near-solution of a biharmonic equation was defined and a general formula for its calculation was provided. Based on this concept, the authors of this article obtained an approximation of this equation. A number of numerical calculations were carried out in order to confirm experimentally the obtained formula.

In this paper we obtain conditions including those for a triangular computational grid, which would guarantee a uniform boundedness of at least the second derivatives of the approximate solutions in grid cells as the sizes of these cells tend to zero. Using this fact, we succeed to prove the C^1-convergence of piecewise polynomial solutions to the exact solution of the corresponding boundary value problem.

2. Piecewise polynomial solutions

Let a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ be given. We consider an arbitrary partition of this polygon into non-degenerate triangles $T_1, T_2, \ldots, T_N$ and let $M_1, M_2, \ldots, M_p$ be the vertices of these triangles. We shall assume that the set of these triangles forms a triangulation of the domain Ω. By Γ_l we denote the sides of all triangles, $l = 1, 2, \ldots, L$, while the maximal diameter of all triangles is denoted by h, that is, $h = \max\limits_{1\leqslant k\leqslant N} \operatorname{diam} T_k$.

By $\alpha > 0$ we denote the minimum of the angles in all triangles T_k. We formulate one more condition for the triangle grid. We assume that there exists a constant $C_1 > 0$ independent of h such that

$$h \cdot \sum_{\Gamma_l} |\Gamma_l| \leqslant C_1, \tag{2.1}$$

where the sum is taken over all internal sides Γ_l of the triangles in the triangulation. This condition was formulated in work [14] and it gives a needed accuracy in calculating integral $I(f)$ under replacing f by a piecewise polynomial function.

We fix a natural number m. In the domain we consider a function $f(x_1, x_2)$, all derivatives of which of order $4m + 2$ are bounded by some constant M. For each triangle T_k we construct a polynomial of degree $4m + 1$ as follows, see [15]. Let A_1^k, A_2^k, A_3^k be the vertices of this triangle. On each of sides $[A_i^k, A_j^k]$ we select the set of points $\{B_{l,ij}^k\}_{l=1}^r$, $r = 1, \ldots, m$, such that for each fixed r these points partition the side on which they lie into $r + 1$ equal parts. For constructing an interpolation polynomial P_{4m+1} on T_k we define the values of a function and all its derivatives up to the order $2m$ at the vertices of the triangle and r derivatives of rth order $(r = 1, \ldots, m)$ along the normal to each side of the triangle

$$\frac{\partial^r P_{4m+1}(A_i^k)}{\partial x_1^{r-l} \partial x_2^l} = \frac{\partial^r f(A_i^k)}{\partial x_1^{r-l} \partial x_2^l}, \qquad 0 \leqslant r \leqslant 2m, \qquad 0 \leqslant l \leqslant r, \qquad i = 1, 2, 3, \tag{2.2}$$

$$\frac{\partial^r P_{4m+1}(B_{l,ij}^k)}{\partial n_{ij}^r} = \frac{\partial^r f(B_{l,ij}^k)}{\partial n_{ij}^r}, \quad 1 \leqslant r \leqslant m, \qquad 1 \leqslant l \leqslant r, \qquad i, j = 1, 2, 3, \qquad i \neq j. \tag{2.3}$$

In [15] and [16] for conditions (2.2), (2.3) and chosen in some ways of remaining conditions, the estimates were proved:

$$\left| \frac{\partial^s (f(x_1, x_2) - P_{4m+1}(x_1, x_2))}{\partial x_1^l \partial x_2^{s-l}} \right| \leqslant C(m) M h^{4m+2-s} (\sin \alpha)^{-s}. \tag{2.4}$$

The obtained piecewise polynomial function belongs to the class $C^m(\Omega)$, see [15].

In view of the said above, we introduce the following notations. The set of m times continuously differentiable piecewise polynomial functions with zero boundary conditions is denoted by $P_{0,4m+1}^m$, that is, $v \in P_{0,4m+1}^m$, if $v \in C^m(\Omega)$, in each triangle T_k the function v is a polynomial of degree $4m + 1$ and satisfies the conditions

$$v = 0, \qquad \nabla v = 0 \quad \text{on} \quad \partial\Omega.$$

The set of all m times continuously differentiable piecewise polynomial functions of degree $4m + 1$, that is, with no boundary conditions, is denoted by P_{4m+1}^m.

We note that if a function $f \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies equation (1.2), then for each function $\psi \in C_0^1(\Omega)$ such that $\psi \in C^2(T_k)$ in each triangl T_k, the identity

$$\begin{aligned} \sum_{k=1}^{2} \int_{T_k} \Bigg(& \sum_{ij=1}^{2} G'_{\eta_{ij}}(x, f, \nabla f, D^2 f) \psi_{x_i x_j} \\ & + \sum_{i=1}^{2} G'_{\xi_i}(x, f, \nabla f, D^2 f) \psi_{x_i} + G'_u(x, f, \nabla f, D^2 f) \psi \Bigg) \, dx = 0 \end{aligned} \tag{2.5}$$

holds. This identity is obtained by applying the Gauss-Ostrogradsky formula and by taking into consideration that the function ψ is continuously differentiable in the domain Ω. Then the arising integrals over the sides of the boundary ∂T_k mutually cancel out for each pair of neighbouring triangles.

We introduce a quantity

$$\begin{aligned}\delta(u', u'', \xi', \xi'', \eta', \eta'') =& G(x, u'', \xi'', \eta'') - G(x, u', \xi', \eta') - G'_u(x, u', \xi', \eta')(u'' - u') \\ &- \sum_{i=1}^{2} G'_{\xi_i}(x, u', \xi', \eta')(\xi'_i - \xi''_i) - \sum_{i,j=1}^{2} G'_{\eta_{ij}}(x, u', \xi', \eta')(\eta'_{ij} - \eta''_{ij}).\end{aligned}$$

We observe that the condition $\delta(u', u'', \xi', \xi'', \eta', \eta'') \geqslant 0$ for all $u', u'', \xi'\xi'', \eta', \eta''$ is equivalent to the property that the function $G(x, u, \xi, \eta)$ is convex down in the variables u, ξ, η. In what follows we assume that the function $\delta(u', u'', \xi', \xi'', \eta', \eta'')$ obeys one of the following conditions. The first is that there exists a constant $\mu > 0$ such that

$$\delta(u', u'', \xi', \xi'', \eta', \eta'') \geqslant \mu \sum_{i,j=1}^{2} (\eta''_{ij} - \eta'_{ij})^2 \tag{2.6}$$

for all $u', u'', \xi', \xi'', \eta', \eta''$. The second condition is weaker: for some constant $\mu > 0$ the inequality

$$\delta(u', u'', \xi', \xi'', \eta', \eta'') \geqslant \mu \left(\sum_{i=1}^{2} (\eta''_{ii} - \eta'_{ii}) \right)^2 \tag{2.7}$$

holds for all $u', u'', \xi', \xi'', \eta', \eta''$.

Using identity (2.5), it is easy to see that if a function $f \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies equation (1.2), then the identity

$$\begin{aligned}&\sum_{k=1}^{N} \int_{T_k} \delta(f, g, \nabla f, \nabla g, D^2 f, D^2 g)\, dx \\ &\quad = \sum_{k=1}^{N} \int_{T_k} G(x, g, \nabla g, D^2 g)\, dx - \sum_{k=1}^{N} \int_{T_k} G(x, f, \nabla f, D^2 f)\, dx\end{aligned} \tag{2.8}$$

holds true, where a function $g \in C^1(\Omega)$ is such that $g \in C^2(T_k)$ in each triangle T_k and

$$g|_{\partial\Omega} = f|_{\partial\Omega}, \qquad \nabla g|_{\partial\Omega} = \nabla f|_{\partial\Omega}. \tag{2.9}$$

For a shorter writing we shall employ the notation $I(g)$ for the above described functions g meaning the following:

$$I(g) = \sum_{k=1}^{N} \int_{T_k} G(x, g, \nabla g, D^2 g)\, dx.$$

Example 2.1. *Let*

$$G(x, u, \xi_1, \xi_2, \eta_{11}, \eta_{12}, \eta_{21}, \eta_{22}) = \sum_{i,j,k,l=1}^{2} a_{ij}^{kl} \eta_{ij} \eta_{kl},$$

where the set of twice continuous differentiable functions $a_{ij}^{kl} = a_{ij}^{kl}(x)$ is such that

$$a_{ij}^{kl} = a_{kl}^{ij}$$

and for each non-zero matrix η_{ij} *the inequality holds:*

$$\sum_{i,j,k,l=1}^{2} a_{ij}^{kl}\eta_{ij}\eta_{kl} > 0.$$

We note that in this case equation (1.2) reads as

$$\sum_{i,j,k,l=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ij}^{kl}\frac{\partial^2 f}{\partial x_k \partial x_l}\right) = 0.$$

For such function G, identity(2.8) is as follows:

$$\sum_{s=1}^{N}\int\limits_{T_s}\sum_{i,j,k,l=1}^{2} a_{ij}^{kl}(f_{x_i x_j} - g_{x_i x_j})(f_{x_k x_l} - g_{x_k x_l})\,dx = I(g) - I(f).$$

In particular, the identity holds:

$$\sum_{s=1}^{N}\int\limits_{T_s}(\Delta f - \Delta g)^2\,dx = \sum_{s=1}^{N}\int\limits_{T_s}(\Delta g)^2\,dx - \sum_{s=1}^{N}\int\limits_{T_s}(\Delta f)^2\,dx.$$

We note that identity (2.8) gives the following property. If the function f satisfies equation (1.2), then functional (1.1) attains its minimum on f among the functions of form g, $g \in C^1(\Omega)$ such that $g \in C^2(T_k)$ in each T_k and satisfying (2.9).

In view of this we introduce the notion of a piecewise polynomial solution to equation (1.2). Let $f \in C^{4m+2}(\Omega) \cap C^1(\overline{\Omega})$ be a solution to equation (1.2) and let an arbitrary function $\varphi \in C^{4m+2}(\Omega) \cap C^1(\overline{\Omega})$ be given, which satisfies the boundary conditions

$$\varphi|_{\partial\Omega} = f|_{\partial\Omega}, \qquad \nabla\varphi|_{\partial\Omega} = \nabla f|_{\partial\Omega}.$$

In what follows, in order to avoid introducing new constants, we suppose that the second derivatives of the function φ are also bounded by some constant M.

Definition 2.1. *If on the function* v^* *the minimum of the integral*

$$I(\varphi + v) = \sum_{k=1}^{N}\int\limits_{T_k} G(x, \varphi + v, \nabla\varphi + \nabla v, D^2\varphi + D^2 v)\,dx$$

is attained among all $v \in P_{0,4m+1}^{m}$, *then the function* $f^* = \varphi + v^*$ *is called a piecewise polynomial solution of equation (1.2) satisfying the boundary conditions*

$$f^*|_{\partial\Omega} = \varphi|_{\partial\Omega}, \qquad \nabla f^*|_{\partial\Omega} = \nabla\varphi|_{\partial\Omega}. \tag{2.10}$$

We shall also need the quantity

$$\lambda_q = \inf_{P}\frac{\int\limits_{T_k}|P(x)|\,dx}{|T_k|\max\limits_{T_k}|P(x)|},$$

where the infimum is taken over all polynomials P in T_k of degree q and $|T_k|$ is the area of the triangle T_k. We note that $\lambda_q > 0$ and by means of the linear change of the variables one can show that this quantity depends only on the degree of the polynomials and is independent of the triangle T_k.

Theorem 2.1. *If there exist constants* ν_1, $\nu_2 > 0$ *such that*

$$G(x, u, \xi, \eta) \geqslant \nu_1|\eta|^2 - \nu_2, \qquad |\eta|^2 = \eta_{11}^2 + \eta_{12}^2 + \eta_{21}^2 + \eta_{22}^2, \tag{2.11}$$

for all x, u, ξ, η, *the the piecewise polynomial exists and is unique.*

Proof. Let a sequence $v_r \in P^m_{0,4m+1}$, $r = 1,2,3,\ldots$, be such that $I(\varphi + v_r) \to I_0$, where $I_0 = \inf\limits_{v\in P^m_{0,4m+1}} I(\varphi + v)$. It is clear that for sufficiently large r the inequality $I(\varphi + v_r) \leqslant 2I_0$ holds. Taking into consideration that the second derivatives of the function φ are bounded by a constant M, by condition (2.11) we see easily that

$$\left(\sum_{k=1}^{N}\int\limits_{T_k}|D^2v_r|^2\,dx\right)^{1/2} \leqslant \sqrt{\frac{2I_0+\nu_2}{\nu_1}} + 2M\sqrt{|\Omega|}.$$

We note that in each triangle T_k the function $|D^2v_r|^2$ is a polynomial of degree $8m-2$. This is why

$$|T_k|\max_{T_k}|D^2v_r|^2 \leqslant \frac{1}{\lambda_{8m-2}}\int\limits_{T_k}|D^2v_r|^2\,dx.$$

Then we arrive at the estimate

$$\left(|T_k|\max_{T_k}|D^2v_r|^2\right)^{1/2} \leqslant \frac{1}{\sqrt{\lambda_{8m-2}}}\left(\sqrt{\frac{2I_0+\nu_2}{\nu_1}} + 2M\sqrt{|\Omega|}\right).$$

Since the triangulation is fixed, it follows from the obtained inequality that the sequence $\max\limits_{1\leqslant k\leqslant N}\max\limits_{T_k}|D^2v_r(x)|$ is bounded. Since v_r has zero boundary values, the sequences

$$\max_{\Omega}|\nabla v_r(x)|,\quad \max_{\Omega}|v_r(x)|$$

are also bounded. Then taking into consideration that in each triangle T_k the function v_r is polynomial of a fixed degree and passing to a subsequence if this is needed, we conclude that there exists $v^* \in P^m_{0,4m+1}$ such that

$$\max_{1\leqslant k\leqslant N}(\max_{T_k}|v_r - v^*| + \max_{T_k}|\nabla v_r - \nabla v^*| + \max_{T_k}|D^2v_r - D^2v^*|) \to 0.$$

Then we obviously have $I(\varphi+v^*) = I_0$. Let us show the uniqueness of the piecewise polynomial solution. We note that for a piecewise polynomial solution we can show that the identity

$$\begin{aligned}\sum_{k=1}^{2}\int\limits_{T_k}\Bigg(&\sum_{ij=1}^{2}G'_{\eta_{ij}}(x,f^*,\nabla f^*,D^2f^*)v_{x_ix_j}\\ &+\sum_{i=1}^{2}G'_{\xi_i}(x,f^*,\nabla f^*,D^2f^*)v_{x_i} + G'_u(x,f^*,\nabla f^*,D^2f^*)v\Bigg)\,dx = 0\end{aligned} \tag{2.12}$$

holds for each $v \in P^m_{0,4m+1}$. This is why if f_1^* and f_2^* are two piecewise polynomial solutions obeying boundary condition (2.10), then

$$\sum_{k=1}^{N}\int\limits_{T_k}\delta(f_1^*,f_2^*,\nabla f_1^*,\nabla f_2^*,D^2f_1^*,D^2f_2^*)\,dx = I(f_2^*) - I(f_1^*),$$

$$\sum_{k=1}^{N}\int\limits_{T_k}\delta(f_2^*,f_1^*,\nabla f_2^*,\nabla f_1^*,D^2f_2^*,D^2f_1^*)\,dx = I(f_1^*) - I(f_2^*).$$

Summing up these identity, we obtain:

$$\sum_{k=1}^{N}\int\limits_{T_k}\big(\delta(f_1^*,f_2^*,\nabla f_1^*,\nabla f_2^*,D^2f_1^*,D^2f_2^*) + \delta(f_2^*,f_1^*,\nabla f_2^*,\nabla f_1^*,D^2f_2^*,D^2f_1^*)\big)\,dx = 0.$$

Then it follows from conditions (2.6) and (2.7) that $f_1^* = f_2^*$. □

3. Convergence of piecewise polynomial solutions

We are going to study the behavior of such approximate solutions f^* as the grid fineness h tends to zero. In view of this we consider an additional condition on the triangulation. Namely, we assume the minimal angle α is separated from zero by a constant C_2 independent of h,

$$\alpha \geqslant C_2 > 0. \tag{3.1}$$

First we are going to show that the second derivatives of these solutions are bounded by some constant independent of h under the formulated condition on the triangle grid.

Theorem 3.1. *If condition (2.1) holds as well as one of inequalities (2.6) or (2.7), then in each triangle T_k the piecewise polynomial solution f^* satisfies the estimate*

$$\sqrt{\max_{T_k} \sum_{i,j=1}^{2} (f^*_{x_i x_j})^2} \leqslant 2M + \frac{1}{\lambda_{8m-2}} \left(4M + \frac{1}{\sin^2 \alpha} \sqrt{\frac{2C}{\mu}} (\operatorname{diam} \Omega)^{2m-1/2} \right), \tag{3.2}$$

where C is some constant independent of h.

Proof. We first assume that condition (2.6) is satisfied. Then for functions $g \in C^1(\Omega)$ such that $g \in C^2(T_k)$ in each triangle T_k and satisfying condition (2.9), by (2.8) we obtain

$$\sum_{k=1}^{N} \int_{T_k} \sum_{i,j=1}^{2} (f_{x_i x_j} - g_{x_i x_j})^2 \leqslant \frac{1}{\mu} \left(I(g) - I(f) \right). \tag{3.3}$$

This is why by inequality (3.3) for each triangle, by letting $g = f^*$ we have:

$$\sqrt{\int_{T_k} \sum_{i,j=1}^{2} (f^*_{x_i x_j})^2 \, dx} \leqslant \sqrt{\int_{T_k} \sum_{i,j=1}^{2} (f_{x_i x_j})^2 \, dx} + \sqrt{\frac{1}{\mu} (I(f^*) - I(f))}.$$

We divide both sides of the inequality by the square root of the area of the triangle $|T_k|$ and using that $|f_{x_i x_j}| \leqslant M$, we obtain:

$$\sqrt{\frac{1}{|T_k|} \int_{T_k} \sum_{i,j=1}^{2} (f^*_{x_i x_j})^2 \, dx} \leqslant 2M + \sqrt{\frac{I(f^*) - I(f)}{|T_k| \mu}}.$$

Using the boundedness of the second derivatives of the function φ by the constant M and applying the inequality

$$\sqrt{\frac{1}{|T_k|} \int_{T_k} \sum_{i,j=1}^{2} (f^*_{x_i x_j})^2 \, dx} \geqslant \sqrt{\frac{1}{|T_k|} \int_{T_k} \sum_{i,j=1}^{2} (v^*_{x_i x_j})^2 \, dx} - \sqrt{\frac{1}{|T_k|} \int_{T_k} \sum_{i,j=1}^{2} (\varphi_{x_i x_j})^2 \, dx},$$

we have:

$$\sqrt{\frac{1}{|T_k|} \int_{T_k} \sum_{i,j=1}^{2} (v^*_{x_i x_j})^2 \, dx} \leqslant 4M + \sqrt{\frac{I(f^*) - I(f)}{|T_k| \mu}}.$$

Since the expression $\sum\limits_{i,j=1}^{2} (v^*_{x_ix_j})^2$ is a polynomial of degree $8m-2$, we have:

$$\max_{T_k} \sum_{i,j=1}^{2} (v^*_{x_ix_j})^2 \leqslant \frac{1}{\lambda_{8m-2}} \frac{1}{|T_k|} \int\limits_{T_k} \sum_{i,j=1}^{2} (v^*_{x_ix_j})^2 \, dx.$$

Thus, we arrive at the inequality

$$\sqrt{\max_{T_k} \sum_{i,j=1}^{2} (v^*_{x_ix_j})^2} \leqslant \frac{1}{\sqrt{\lambda_{8m-2}}} \left(4M + \sqrt{\frac{I(f^*) - I(f)}{|T_k|\mu}} \right).$$

By the obtained inequality we see that the issue on the boundedness of the second derivatives of the piecewise polynomial solution is reduced to studying the behavior of the quantity

$$\frac{I(f^*) - I(f)}{|T_k|}$$

as $h \to 0$. We denote by v_f the function from $P^m_{0,4m+1}$ constructed by the values and derivatives of the function $f - \varphi$. In this case, since $I(\varphi + v^*) \leqslant I(\varphi + v_f)$, the inequality holds:

$$\frac{I(f^*) - I(f)}{|T_k|} \leqslant \frac{I(\varphi + v_f) - I(f)}{|T_k|}.$$

We consider a functional

$$\tilde{I}(g) = I(\varphi + g).$$

We showed in work [14] that the value of the functional on the function $f \in C^{4m+2}(\Omega)$ is approximated by piecewise polynomial functions of degree $4m+1$ up to $O(h^{4m+1})$ for a triangle grid satisfying condition (2.1). Therefore, since we assume that $(f - \varphi) \in C^{4m+2}(\Omega)$, there exists a constant $C > 0$ independent of h and such that

$$|\tilde{I}(v_f) - \tilde{I}(f - \varphi)| \leqslant Ch^{4m+1}.$$

It is easy to see that $|T_k| \geqslant 0.5h^2 \sin^2 \alpha$. Then we arrive at the inequality

$$\sqrt{\max_{T_k} \sum_{i,j=1}^{2} (f^*_{x_ix_j})^2} \leqslant 2M + \frac{1}{\lambda_{8m-2}} \left(4M + \frac{1}{\sin^2 \alpha} \sqrt{\frac{2C}{\mu}} (\operatorname{diam} \Omega)^{2m-1/2} \right).$$

Now we suppose that instead of inequality (2.6), a weaker inequality (2.7) is satisfied. Then by (2.8) we obtain:

$$\sum_{k=1}^{N} \int\limits_{T_k} (\Delta(f - g))^2 \, dx \leqslant \frac{I(g) - I(f)}{\mu}.$$

Then we employ Calderon-Zygmund inequality [17, Cor. 9.10] for a function $f - g$. Namely, the identity

$$||D^2 u||_2 = ||\Delta u||_2$$

holds, where $u \in W^2_0(\Omega)$. Let us clarify the possibility of using this identity. If $m > 1$, then the assumptions of Corollary 9.10 in [17] are satisfied. As $m = 1$, we have $g \in C^1(\Omega)$ and $g \in C^2(T_k)$ for each triangle T_k. If in each triangle the second derivatives of the function g

are bounded, then taking into consideration that $f = g$ on $\partial\Omega$, we get $f - g \in W_0^2(\Omega)$. Thus, letting $g = f^*$, we arrive at the inequality

$$\sum_{k=1}^{N} \int\limits_{T_k} \sum_{i,j=1}^{2} (f_{x_i x_j} - f^*_{x_i x_j})^2\, dx \leqslant \frac{I(f^*) - I(f)}{\mu}.$$

Further arguing to similar to ones made in the case of condition (2.6) and they also lead to estimate (3.2). □

Theorem 3.2. *Let $f \in C^{4m+2}(\Omega) \cap C^1(\overline{\Omega})$ be a solution to equation (1.2) satisfying the boundary conditions*

$$f|_{\partial\Omega} = \varphi|_{\partial\Omega}, \qquad \nabla f|_{\partial\Omega} = \nabla\varphi|_{\partial\Omega}$$

and f^* be a piecewise polynomial solution with the same boundary conditions. If conditions (2.1), (3.1) are satisfied as well as one of inequalities (2.6) or (2.7), then

$$\lim_{h\to 0} \max_{\Omega} |f^*(x) - f(x)| = 0, \qquad \lim_{h\to 0} \sum_{i=1}^{2} \max_{\Omega} |f^*_{x_i} - f_{x_i}| = 0.$$

Proof. Under the formulated conditions on the triangle grid we have showed that as it becomes finer, in each triangle T_k the second derivatives of approximate solutions remain bounded by a constant independent of h. Let us show that a corollary of such behavior of the solutions is their uniform convergence to the exact solution in the space C^1. Let $|f^*_{x_i x_j} - f_{x_i x_j}| \leqslant K$, where K is independent of h. We then employ Sobolev inequality [17, Thm. 7.10] for $p > 2$ and prove inequality (3.3):

$$\begin{aligned}\max_{\Omega} |f^*_{x_i} - f_{x_i}| &\leqslant C_0 |\Omega|^{1/2-1/p} ||D^2(f^* - f)||_p \\ &\leqslant C_0 |\Omega|^{1/2-1/p} K^{1-2/p} \left(\int\limits_{\Omega} \sum_{i,j=1}^{2} (f^*_{x_i x_j} - f_{x_i x_j})^2\, dx \right)^{1/p} \\ &\leqslant C_0 |\Omega|^{1/2-1/p} K^{1-2/p} \left(\frac{I(f^*) - I(f)}{\mu} \right)^{1/p} \leqslant C_0 |\Omega|^{1/2-1/p} K^{1-2/p} \left(\frac{C}{\mu} \right)^{1/p} h^{\frac{4m+1}{p}}.\end{aligned}$$

Thus, the uniform convergence of the first derivatives holds as $h \to 0$. Since $f^* = f$ on $\partial\Omega$, we obtain

$$\lim_{h\to 0} \max_{\Omega} |f^*(x) - f(x)| = 0.$$

□

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