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ONE-DIMENSIONAL L_p-HARDY-TYPE INEQUALITIES FOR SPECIAL WEIGHT FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We establish one-dimensional L_p-Hardy inequalities with additional terms and use them for justifying their multidimensional analogues in convex domains with finite volumes. We obtain variational inequalities with power-law weights being generalizations of the corresponding inequalities presented earlier in papers by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev and J. Tidblom. We formulate and prove inequalities valid for arbitrary domains, and then we simplify them substantially for the class of convex domains. The constants in the additional terms in these spatial inequalities depend on the volume or on the diameter of the domain. As a corollary of the obtained results we get estimates for the first eigenvalue of the p-Laplacian subject to the Dirichlet boundary conditions.

Keywords: Hardy inequality, additional term, one-dimensional inequality, distance function, volume of a domain, diameter of a domain, first eigenvalue of the Dirichlet problem.

Mathematics Subject Classification: 26D15, 46E35

1. Introduction

The present paper is devoted to the generalizations of a Hardy type inequality proved by V.I. Levin in paper [1]. Namely, this is the following sharp inequality

$$\int_0^1 \frac{y^2(t)}{t^2(2-t)^2}dt < \int_0^1 y'^2(t)\,dt, \tag{1.1}$$

valid for each not identically zero absolutely continuous function y such that $y(0) = 0$ and $y' \in L^2[0,1]$. We are going to establish L_p-analogues of (1.1).

An interest to inequality (1.1) is due to the fact that it is a strengthening of the classical Hardy inequality

$$\int_0^1 \frac{y^2(t)}{t^2}dt < 4\int_0^1 y'^2(t)dt \tag{1.2}$$

on the unit segment for the same class of functions. The constant 1 in inequality (1.1) and constant 4 in (1.2) are sharp but there exists no extremal function, on which these inequalities become identities.

It should be noted that close to (1.1) inequalities are also employed to establish sufficient conditions for the univalence of meromorphic in a circle functions in terms of an estimate for the absolute value of the Schwartz derivative. At a first glance, it is rather complicated

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to understand how the Hardy inequalities for absolutely continuous functions are applied in justifying the sufficient conditions for the univalence of analytic functions. It turns out that there exists a relation, see [2]–[5], of the univalence of the function with a non-oscillating of a solution to special differential equations, which is close to Hardy type inequalities.

We can easily transform the weight function $t^{-2}(2-t)^{-2}$ and rewrite (1.1) as

$$\int_0^1 \frac{y^2(t)}{t^2}dt + 2\int_0^1 \frac{y^2(t)}{t(2-t)}dt + \int_0^1 \frac{y^2(t)}{(2-t)^2} < 4\int_0^1 y'^2(t)dt, \tag{1.1'}$$

which clearly shows how inequality (1.2) is strengthened: by means of additional terms. In recent decades, plenty of works were published devoted to inequalities with additional terms, see, for instance [6]–[24], but in the literature, and even in paper [1], this results by V.I. Levin is almost not mentioned as a strengthening of the Hardy inequality by means of additional terms.

An analogue of inequality (1.1′) on the segment $[0,2b]$ for absolutely continuous functions such that $y(0)=y(2b)=0$ is rather interesting by its form and meaning. Namely, this is the inequality

$$\int_0^{2b} \frac{y^2(t)}{\rho^2(t)}dt + 2\int_0^{2b} \frac{y^2(t)}{\rho(t)\mu(t)}dt + \int_0^{2b} \frac{y^2(t)}{\mu^2(t)} < 4\int_0^{2b} y'^2(t)dt, \tag{1.1''}$$

where $\rho(t)=\min\{t,2b-t\}$ and $\mu(t)=2b-\rho(t)$. A weaker version of (1.1″)

$$\int_0^{2b} \frac{y^2(t)}{\rho^2(t)}dt + \int_0^{2b} \frac{y^2(t)}{\rho(t)\mu(t)}dt \leqslant 4\int_0^{2b} y'^2(t)dt \tag{1.3}$$

was implicitly employed by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev in paper [13] for proving the following multidimensional inequality

$$\frac{1}{4}\int_\Omega \frac{|g(x)|^2}{\delta(x)^2}dx + \frac{1}{4}\frac{K(n)}{|\Omega|^{2/n}}\int_\Omega |g(x)|^2dx \leqslant \int_\Omega |\nabla g(x)|^2dx, \tag{1.4}$$

valid for all g from a known family of continuously differentiable functions $C_0^1(\Omega)$ with compact supports in an open convex domain $\Omega\subset\mathbb{R}^n$ with a finite volume $|\Omega|$, where $|\mathbb{S}^{n-1}|$ is the area $n-1$-dimensional unit sphere and

$$K(n) = n\left[\frac{|\mathbb{S}^{n-1}|}{n}\right]^{2/n}.$$

Multidimensional inequality (1.4) has a series of differences from the one-dimensional case: the integration is made over an n-dimensional domain Ω of the Euclidean space $\mathbb{R}^n$, the powers t are replaced by the powers of the function $\delta(x)$, which the distance from the point $x\in\Omega$ to the boundary $\partial\Omega$ of the domain Ω, that is,

$$\delta(x)=\operatorname{dist}(x,\partial\Omega),$$

while the derivative of the function is replaced by its gradient

$$\nabla g(x) = \left(\frac{\partial g(x)}{\partial x_1},\dots,\frac{\partial g(x)}{\partial x_n}\right).$$

It is clear that if we succeed to strengthen inequality (1.1″), then employing an approach from paper [13] (see also [23], [24]), we can obtain inequalities of type (1.4) with sharper constants.

In paper [11] H. Brezis and M. Marcus showed that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with a finite diameter $D(\Omega)$, then for each function $g \in C_0^1(\Omega)$ the inequality holds:

$$\frac{1}{4}\int\limits_{\Omega} \frac{|g(x)|^2}{\delta(x)^2}dx + \frac{1}{4D^2(\Omega)}\int\limits_{\Omega} |g(x)|^2 dx \leqslant \int\limits_{\Omega} |\nabla g(x)|^2 dx \tag{1.5}$$

We also mention a result by F.G. Avkhadiev and K.-J. Wirths from paper [6]. They proved that for all continuously differentiable functions g with compact supports in a convex domain Ω with a finite inner radius $\delta_0(\Omega)$ a sharp inequality

$$\frac{s^2-\nu^2q^2}{4}\int\limits_{\Omega} \frac{|g(x)|^2}{\delta(x)^{s+1}}dx + \frac{q^2\lambda^2}{4\delta_0^q(\Omega)}\int\limits_{\Omega} \frac{g^2(x)}{\delta(x)^{s+1-q}}dx \leqslant \int\limits_{\Omega} \frac{|\nabla g(x)|^2}{\delta(x)^{s-1}}dx \tag{1.6}$$

holds, where $s > 0$, $q > 0$, $\nu \in \left[0, \frac{s}{q}\right]$ and a constant λ is a solution to the following Lamb type equation for the Bessel function J_ν of order ν:

$$sJ_\nu(\lambda) + q\lambda J_\nu'(\lambda) = 0.$$

The constants $(s^2-\nu^2q^2)/4$ and $q^2\lambda^2/4$ in this inequality are sharp. We just mention that as $\nu > 0$ there exists an extremal function, on which the identity is attained, while for $\nu = 0$ F.G. Avkhadiev and K.-J. Wirts constructed a minimizing sequence, by which they showed the sharpness and unattainability of the constant.

Following papers [6]–[8], we call the quantity λ a Lamb constant, see also [16]–[18]. Let us clarify the name "Lamb constant" and "Lamb equation". The matter is that a particular case of this equation was first considered by H. Lamb in paper [26]. Later it was developed and called in this way by F.G. Avkhadiev and K.-J. Wirts in paper [6]. This is the reason why we call general equations of such form parametric Lamb equations, while its roots are called Lamb constants.

A problem on adding an additional term in the Hardy inequality is related with classical estimates for the first eigenvalue $\lambda_1(\Omega)$ for the Dirichlet Laplacian and the following Poincaré inequality:

$$\lambda_1(\Omega)\int\limits_{\Omega} |g(x)|^2 dx \leqslant \int\limits_{\Omega} |\nabla g(x)|^2 dx \quad \forall g \in C_0^1(\Omega).$$

Poincaré estimate $\lambda_1(\Omega) > \pi^2/D^2(\Omega)$ and a famous isoperimetric Rayleigh-Faber-Crane inequality are well known:

$$\lambda_1(\Omega) > \frac{\omega^{2/n}}{|\Omega|^{2/n}} j_{n/2-1}^2,$$

where j_ν is the first zero of the Bessel function J_ν of order ν, see [28].

As a corollary of this result by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian in the case of convex domains with a fixed volume can be estimated as follows:

$$\lambda_1(\Omega) \geqslant \frac{1}{4}\frac{K(n)}{|\Omega|^{2/n}}.$$

In the present paper we improve the constant in the previous estimate more than in three times. Namely, we obtain that

$$\lambda_1(\Omega) \geqslant \frac{5\lambda_1^2 K(n)}{8|\Omega|^{2/n}},$$

where $\lambda_1 \approx 1.25578$.

Avkhadiev-Wirths inequality (1.6) in the case of n-dimensional convex domain is a second way of proving well-known estimates for the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian, see [27]:

$$\lambda_1(\Omega) \geqslant \frac{\pi^2}{4\delta_0^2(\Omega)} \geqslant \frac{\pi^2}{D^2(\Omega)}.$$

In paper [22] J. Tidblom established L_p-analoigues of the results by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev. For each function g in an appropriate Sobolev space as $p > 1$ the following inequality was proved in a convex domain Ω

$$\left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx + \frac{a(p,n)}{|\Omega|^{p/n}} \int\limits_\Omega |g(x)|^p dx \leqslant \int\limits_\Omega |\nabla g(x)|^p dx, \tag{1.7}$$

where

$$a(p,n) = \frac{(p-1)^{p+1}}{p^p} \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{p/n} \frac{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

The symbol Γ denotes the Euler Gamma function. In particular, as $p = 2$, J. Tidblom has the constant from inequality (1.4):

$$a(2,n) = \frac{1}{4}\frac{K(n)}{|\Omega|^{2/n}}.$$

It was shown in work [12] by S. Filippas, V.G. Maz'ya, A. Tertikas that in convex domain $\Omega \subset \mathbb{R}^n$ as $1 < p < n$ and $p \leqslant q < \frac{np}{n-p}$, a sharp constant $C(\Omega)$ in L_p-inequality

$$\left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx + C(\Omega)\left(\int\limits_\Omega |g(x)|^q dx\right)^{p/q} \leqslant \int\limits_\Omega |\nabla g(x)|^p dx$$

can be estimated from two sides by the inner radius as follows:

$$c_1(p,q,n)(\delta_0(\Omega))^{n-p-\frac{np}{q}} \geqslant C(\Omega) \geqslant c_2(p,q,n)(\delta_0(\Omega))^{n-p-\frac{np}{q}},$$

where $c_1(p,q,n)$ and $c_2(p,q,n)$ are some constants, the existence of which was justified.

Similar to the L_2-case, the problem on adding an additional term in L_p-inequality is related with estimates for the first eigenvalue $\lambda_p(\Omega)$ for the Dirichlet p-Laplacian and with the following Poincaré inequality:

$$\lambda_p(\Omega) \int\limits_\Omega |g(x)|^p dx \leqslant \int\limits_\Omega |\nabla g(x)|^p dx \quad \forall g \in C_0^1(\Omega).$$

As a corollary of the result by J. Tidblom we obtain

$$\lambda_p(\Omega) \geqslant \frac{(p-1)^{p+1}}{p^p} \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{p/n} \frac{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{1}{|\Omega|^{p/n}}.$$

It is interesting to the compare the results of this paper with inequalities from [20], [23] and [24]. For instance, in paper [24] there were obtained a generalization and strengthening of inequality (1.7), but as a corollary the authors obtain the same constant $a(p,n)$ in the additional term. In paper [20] the constant $a(p,n)$ is strengthened as $p \geqslant 2$. As a corollary of our main results in the case $p \in [2, p_0]$ with $p_0 \approx 2.314$, we obtain sharper estimates for $\lambda_p(\Omega)$. Namely, we show that

$$\lambda_p(\Omega) \geqslant \frac{7p\lambda_1^2}{8(p-1)^2} \frac{(p-1)^p}{p^p} \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{p/n} \frac{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{1}{|\Omega|^{p/n}},$$

where λ_1 is a first positive root of Lamb type equation

$$(p-1)J_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Thus, in the present paper we establish L_p-versions of (1.1) and apply them for justifying spatial inequalities of form (1.4), (1.5) and (1.7) with better constants in the additional terms. As a corollary of the multidimensional inequalities we obtain estimates for the first eigenvalue $\lambda_p(\Omega)$ of the Dirichlet p-Laplacian.

2. Lamb equation and Lamb constant

In this section we provide needed preliminary facts. They mostly concern the properties of two special functions.

2.1. First function. Suppose that $q \in (0,\infty)$, $s \in (0,\infty)$ and $\nu \geqslant 0$. We consider a function $F_{\nu,s,q}$ introduced as follows:

$$F_{\nu,s,q}(t) = t^{\frac{s}{2}} \sqrt{(2-t)} J_\nu \left(\lambda \left(\frac{t}{2-t} \right)^{\frac{q}{2}} \right), \qquad t \in [0,1],$$

where the constant λ is a first positive solution of Lamb type equation

$$(s-1) J_\nu(\lambda) + 2q\lambda J'_\nu(\lambda) = 0, \quad \lambda \in (0, j_\nu), \tag{2.1}$$

for the Bessel function J_ν of order ν. We recall that the Bessel function can be defined by a converging series

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k+1+\nu)}.$$

Hereinafter byh j_ν we denote the first positive root of the Bessel function J_ν. A detailed information on the properties of the Bessel functions and its zeroes can be found in the monograph by G.N. Watson [25].

We mention just some properties of this function, which we shall use in what follows. For instance, it is known that

a) $u(t) = J_\nu(t)$ is a canonical solution of the Bessel differential equation:

$$t^2 u''(t) + t u'(t) + \left(t^2 - \nu^2\right) u(t) = 0;$$

b) for sufficiently small t, the following asymptotic formula holds:

$$J_\nu(t) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{t}{2} \right)^\nu.$$

We proceed to some properties of the function $F_{\nu,s,q}$. In view of the definition of the Bessel function, we have $F_{\nu,s,q}(t) > 0$ for sufficiently small t and

$$\lambda \left(\frac{t}{2-t} \right)^{\frac{q}{2}} \in (0, j_\nu)$$

for $t \in (0,1]$. This is why $F_{\nu,s,q}(t)$ is strictly positive also for $t \in (0,1]$.

Straightforward calculations give the following expression for the derivative of this function:

$$t^{1-\frac{s}{2}} F'_{\nu,s,q}(t) = \frac{s(2-t)-t}{2\sqrt{2-t}} J_\nu \left(\lambda \left(\frac{t}{2-t} \right)^{\frac{q}{2}} \right) + q\lambda \frac{t^{\frac{q}{2}}}{(2-t)^{\frac{1}{2}+\frac{q}{2}}} J'_\nu \left(\lambda \left(\frac{t}{2-t} \right)^{\frac{q}{2}} \right).$$

For the convenience we rewrite this identity in a more compact form:

$$v(t) F'_{\nu,s,q}(t) = w(t) J_\nu(z(t)) + 2z(t) J'_\nu(z(t)),$$

where

$$v(t) = \frac{2t^{1-\frac{s}{2}}\sqrt{2-t}}{q}, \quad w(t) = -\frac{s+1}{q} t + \frac{2s}{q} \quad \text{and} \quad z(t) = \lambda \left(\frac{t}{2-t} \right)^{\frac{q}{2}}.$$

For each fixed $t \in (0,1]$ we consider a Lamb type equation

$$w(t)J_\nu(z(t)) + 2z(t)J'_\nu(z(t)) = 0.$$

In paper [6] F.G. Avkhadiev and K.-J. Wirths showed that the solution $z(t)$ of this equation increases monotonically as $w(t)$ increases and $z(t) < j_\nu$. Since the function $w(t)$ is decreasing, we obtain that the solution $z(1) = \lambda$ of the equation

$$w(1)J_\nu(z(1)) + 2z(1)J'_\nu(z(1)) = 0$$

is minimal and lies in the interval $(0, j_\nu)$. This fact will be employed essentially in what follows. For instance, this implies that $F'_{\nu,s,q}(t) > 0$ as $t \in (0,1)$ and $F'_{\nu,s,q}(1) = 0$.

Indeed, we have $F'_{\nu,s,q}(t) > 0$ for sufficiently small t. If we suppose the opposite, that $F'_{\nu,s,q}(t) \leqslant 0$ for some t, then there exists a point $t_0 \in (0,1)$, for which $F'_{\nu,s,q}(t_0) = 0$, i.e., there exists a solution $z(t_0) < z(1) = \lambda$ of the equation

$$w(t_0)J_\nu(z(t_0)) + 2z(t_0)J'_\nu(z(t_0)) = 0.$$

This contradicts the minimality of λ.

Employing a formula relating the Bessel function with its derivative, see, for instance, [25],

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z),$$

we obtain:

$$\begin{aligned}\frac{F'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} &= \frac{s(2-t)-t}{2t(2-t)} + q\lambda\frac{t^{\frac{q}{2}-1}}{(2-t)^{1+\frac{q}{2}}}\frac{J'_\nu\left(\lambda\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)}{J_\nu\left(\lambda\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)} \\ &= \frac{s(2-t)-t}{2t(2-t)} - q\nu\frac{2-t}{t} + q\lambda\frac{t^{\frac{q}{2}-1}}{(2-t)^{1+\frac{q}{2}}}\frac{J_{\nu-1}\left(\lambda\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)}{J_\nu\left(\lambda\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)}.\end{aligned}$$

Therefore, Lamb equation (2.1) can be rewritten as follows:

$$(s - 2\nu q - 1)J_\nu(\lambda) + 2q\lambda J_{\nu-1}(\lambda) = 0, \quad \lambda \in (0, j_\nu).$$

Applying property a) of the Bessel function, we also have an identity for the second derivative of the function $F_{\nu,s,q}$:

$$\frac{F''_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} + (1-s)\frac{F'_{\nu,s,q}(t)}{tF_{\nu,s,q}(t)} = -\frac{s^2 - \nu^2 q^2}{t^2} - (1-\nu^2q^2)\frac{4-t}{t(2-t)^2} - \frac{4\lambda^2 q^2}{(2-t)^{2+q}}. \tag{2.2}$$

Finally, employing the expansion of the Bessel into a series, we obtain:

$$\lim_{t\to 0}\frac{tF'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} = \frac{s+\nu q}{2}, \tag{2.3}$$

see also [6]–[8] for more details.

2.2. Second function. Now assume that $q \in (0,\infty)$ and $s \in (0,\infty)$. We shall also need a function $\Phi_{s,q}$ defined as follows:

$$\Phi_{s,q}(t) = t^{\frac{s}{2}} J_0\left(\lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right), \quad t \in [0,1],$$

where a constant λ_1 is a first positive solution of the equation

$$sJ_0(\lambda_1) - 2q\lambda J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Straightforward calculation give:

$$2(2-t)t^{1-\frac{s}{2}}\Phi'_{s,q}(t) = s(2-t)J_0\left(\lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right) - 2q\lambda_1\frac{t^{\frac{q}{2}}}{(2-t)^{\frac{q}{2}}}J_1\left(\lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right).$$

Here we have employed that $J_0'(z) = -J_1(z)$. We rewrite the latter identity as follows:

$$v(t)\Phi'_{s,q}(t) = w(t)J_0\left(z(t)\right) - 2z(t)J_1\left(z(t)\right),$$

where

$$v(t) = \frac{2}{q}(2-t)t^{1-\frac{s}{2}}, \quad w(t) = -\frac{s}{q}t + \frac{2s}{q}, \quad z(t) = \lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}.$$

For each fixed $t \in (0,1]$ we consider the equation

$$w(t)J_0\left(z(t)\right) - 2z(t)J_1\left(z(t)\right) = 0.$$

By analogy with the function $F_{\nu,s,q}$, since the function $w(t)$ is decreasing, we obtain that the solution $\lambda_1 = z(1)$ of the equation

$$w(1)J_0\left(z(1)\right) - 2z(1)J_1\left(z(1)\right) = 0$$

is minimal.

It is clear that $\Phi_{s,q}(0) = 0$, $\Phi_{s,q}(t) > 0$ and $\Phi'_{s,q}(t) > 0$ for sufficiently small t. Employing the definition of the Lamb constant λ_1, we have

$$\Phi'_{s,q}(1) = 0, \quad \Phi_{s,q}(t) > 0 \quad \text{as} \quad t \in (0,1], \quad \Phi'_{s,q}(t) > 0 \quad \text{as} \quad t \in (0,1).$$

It should be also noted that applying property a) of the Bessel function, we can obtain the following identity

$$\frac{\Phi''_{s,q}(t)}{\Phi_{s,q}(t)} + (1-s)\frac{\Phi'_{s,q}(t)}{t\Phi_{s,q}(t)} = -\frac{s^2}{4t^2} - \frac{\lambda_1^2 q^2}{t^{2-q}(2-t)^{2+q}} - \lambda_1 q\frac{t^{-1+q/2}}{(2-t)^{2+q/2}}\frac{J_1\left(\lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)}{J_0\left(\lambda_1\left(\frac{t}{2-t}\right)^{\frac{q}{2}}\right)},$$

in which, as we shall show, an increasing as $z \in [0,2]$ function $J_1(z)/(zJ_0(z))$ is involved.

The following statement holds true.

Lemma 2.1. *A continuous function $h(t) = \frac{J_1(t)}{tJ_0(t)}$ is increasing as $t \in [0,2]$ and*

$$\inf_{t\in[0,2]} h(t) = \frac{1}{2}.$$

Proof. Let us show that $h'(t) \geqslant 0$ as $t \in [0,2]$. Employing the following known identities for the Bessel function and its derivative, see, for instance, [25],

$$J_0'(t) = -J_1(t), \qquad tJ_1'(t) - J_1(t) = -tJ_2(t),$$

$$J_1^2(t) - J_0(t)J_2(t) = \frac{4}{t^2}\sum_{j=0}^{\infty}(2+2j)J_{2+2j}^2(t),$$

we get

$$h'(t) = \frac{tJ_1'(t)J_0(t) - J_1(t)J_0(t) - tJ_0'(t)J_1(t)}{t^2J_0^2(t)} = \frac{J_0(t)\left(tJ_1'(t) - J_1(t)\right) + tJ_1^2(t)}{t^2J_0^2(t)}$$

$$= \frac{J_1^2(t) - J_0(t)J_2(t)}{tJ_0^2(t)} = \frac{4}{t^3}\sum_{j=0}^{\infty}(2+2j)J_{2+2j}^2(t) \geqslant 0.$$

Therefore, the function $h(t)$ is increasing and taking into consideration property b), we find:

$$\inf_{t\in[0,2]} h(t) = \lim_{t\to 0} h(t) = \frac{1}{2}.$$

□

In view of this lemma we obtain:

$$\frac{\Phi''_{s,q}(t)}{\Phi_{s,q}(t)}+(1-s)\frac{\Phi'_{s,q}(t)}{t\Phi_{s,q}(t)}\leqslant-\frac{s^2}{4t^2}-\frac{\lambda_1^2q^2}{t^{2-q}(2-t)^{2+q}}-\frac{\lambda_1^2q}{2}\frac{t^{q-1}}{(2-t)^{2+q}}.$$

Employing also the expansion of the Bessel function into a series, we have

$$\lim_{t\to 0}\frac{t\Phi'_{s,q}(t)}{\Phi_{s,q}(t)}=\frac{s}{2}.$$

3. One-dimensional inequalities

In this section we obtain one-dimensional inequalities on the unit segment $[0,1]$ and on a segment of form $[0,2b]$. We shall essentially employ the properties of the functions defined in the previous section.

The following statement holds.

Lemma 3.1. *Let $p\geqslant 2$, $s>0$, $q\in(0,+\infty)$, $\nu\in[0,s/q]$ and an absolutely continuous on the segment $[0,1]$ function y be such that $y(0)=0$,*

$$|y'(t)|t^{(p+1-s)/p}\in L^p[0,1].$$

Then the inequality holds:

$$\frac{p^p}{(s^2-\nu^2q^2)^{p/2}}\int_0^1\frac{|y'(t)|^p}{t^{s-p+1}}dt\geqslant\int_0^1\frac{|y(t)|^p}{t^{s+1}}\left(1+\frac{1-\nu^2q^2}{s^2-\nu^2q^2}\frac{p}{2}\frac{(4-t)t}{(2-t)^2}+\frac{2p\lambda^2q^2}{s^2-\nu^2q^2}\frac{t^q}{(2-t)^{2+q}}\right)dt,$$

where λ is the first positive root of the equation

$$-1-2\nu q+s+2qz\frac{J_{\nu-1}(z)}{J_\nu(z)}=0.$$

Proof. Without loss of generality we can assume that y is a positive non-decreasing function. Indeed, if g is an arbitrary absolutely continuous function such that $g(0)=0$ and

$$y(t)=\int_0^t|g'(\tau)|d\tau$$

and the identity holds:

$$\int_a^b y^p(t)w(t)dt\leqslant C_1\int_a^b y'^p(t)v(t)dt$$

with some constant C_1 and weight functions w and v, then due to the identity

$$|g(t)|\leqslant\int_0^t|g'(\tau)|dt=y(t),\quad y'(t)=|g'(t)|,$$

we have an inequality for an arbitrary case:

$$\int_a^b|g(t)|^pw(t)dt\leqslant\int_a^b y^p(t)w(t)dt\leqslant C_1\int_a^b y'^p(t)v(t)dt=C_1\int_a^b|g'(t)|^pv(t)dt.$$

It is easy to show that

$$0 \leqslant P := \int\limits_0^1 \frac{y^{p-2}(t)}{t^{s-1}} \left(y'(x) - \frac{2}{p} \frac{F'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} y(t) \right)^2 dt$$

$$= \int\limits_0^1 \frac{y^{p-2}(t) y'^2(y)}{y^{s-1}} dt - \frac{4}{p^2} \int\limits_0^1 \frac{F'_{\nu,s,q}(t)}{F_{\nu,s,q}(t) t^{s-1}} dy^p(t) + \frac{4}{p^2} \int\limits_0^1 \frac{y^p(t)}{t^{s-1}} \frac{F'^2_{\nu,s,q}(t)}{F^2_{\nu,s,q}(t)} dt.$$

Integrating by parts, we obtain:

$$P = \int\limits_0^1 \frac{y^{p-2}(t) y'^2(t)}{t^{s-1}} dt - y^p(1) \frac{F'_{\nu,s,q}(1)}{F_{\nu,s,q}(1)} + \lim_{t \to 0} \frac{y^p(t)}{t^{s-1}} \frac{F'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)}$$

$$+ \frac{4}{p^2} \int\limits_0^1 \frac{y^p(t)}{t^{s-1}} \left(\frac{F''_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} + (1-s) \frac{F'_{\nu,s,q}(t)}{t F_{\nu,s,q}(t)} \right) dt.$$

Employing the definition of the constant λ, we find:

$$y^p(1) \frac{F'_{\nu,s,q}(1)}{F_{\nu,s,q}(1)} = \frac{y^p(1)}{2} \left(-1 - 2\nu q + s + 2q\lambda \frac{J_{n-1}(\lambda)}{J_n(\lambda)} \right) = 0.$$

For each absolutely continuous function $y : [0,1] \to \mathbb{R}$ such that $y(0) = 0$ and

$$|y'(t)| t^{(p-s-1)/p} \in L^p[0,1]$$

by applying the Hölder inequality we obtain:

$$|y(t)|^p \leqslant \left(\int\limits_0^t |y'(\tau)| d\tau \right)^p \leqslant \left(\int\limits_0^t \tau^{\frac{s-p+1}{p-1}} d\tau \right)^{p-1} \int\limits_0^t \frac{|y'(\tau)|^p}{\tau^{s-p+1}} d\tau = \left(\frac{p-1}{s} \right)^{p-1} t^s \int\limits_0^t \frac{|y'(\tau)|^p}{\tau^{s-p+1}} d\tau.$$

Therefore, taking into consideration (2.3), we get:

$$\lim_{t \to 0} \frac{y^p(t)}{t^{s-1}} \frac{F'_{\nu,s,q}(t)}{F_{\nu,s,q}(t)} = 0.$$

Employing identity (2.2), we obtain:

$$p^2 \int\limits_0^1 \frac{y^{p-2}(t) y'^2(t)}{t^{s-1}} dt \geqslant \int\limits_0^1 \frac{y^p(t)}{t^{s-1}} \left(\frac{s^2 - \nu^2 q^2}{t^2} + (1 - \nu^2 q^2) \frac{4-t}{t(2-t)^2} + \frac{4\lambda^2 q^2}{t^{2-q}(2-t)^{2+q}} \right) dt.$$

Thus,

$$\frac{p^2}{s^2 - \nu^2 q^2} \int\limits_0^1 \frac{y^{p-2}(t) y'^2(t)}{t^{s-2}} dt \geqslant \int\limits_0^1 \frac{y^p(t)}{t^{s-1}} \left(\frac{1}{t^2} + \frac{1 - \nu^2 q^2}{s^2 - \nu^2 q^2} \frac{4-t}{t(2-t)^2} + \frac{4\lambda^2 q^2 t^{q-2}}{(s^2 - \nu^2 q^2)(2-t)^{2+q}} \right) dt.$$

Applying a theorem on arithmetic mean written in the following form [29]

$$a^{p_1} b^{p_2} \leqslant \left(\frac{p_1 a + p_2 b}{p_1 + p_2} \right)^{p_1 + p_2},$$

for the quantity

$$a = \frac{y^p(t)}{t^s}, \quad b = \frac{p^p}{(s^2 - \nu^2 q^2)^{p/2}} \frac{y'^p(t)}{t^{s+1-p}}, \quad p_1 = 1 - \frac{2}{p} \quad \text{and} \quad p_2 = \frac{2}{p},$$

we have

$$\frac{p^p}{(s^2-\nu^2q^2)^{p/2}}\int\limits_0^1 \frac{y'^p(t)}{t^{s-p+1}}dt \geqslant \int\limits_0^1 \frac{y^p(t)}{t^{s-1}}\left(\frac{1}{t^2}+\frac{1-\nu^2q^2}{s^2-\nu^2q^2}\frac{p}{2}\frac{4-t}{t(2-t)^2}+\frac{2p\lambda^2q^2t^{q-2}}{(s^2-\nu^2q^2)(2-t)^{2+q}}\right)dt.$$

This completes the proof. □

As $s=p-1$ and $q=1$, by Lemma 3.1 we obtain the following statement.

Corollary 3.1. *Let $p\geqslant 2$, $\nu\in[0,p-1]$ and an absolutely continuous on the segment $[0,1]$ function y be such that $y(0)=0$, $|y'(t)|\in L^p[0,1]$. Then the inequality holds:*

$$c_p\int\limits_0^1 |y'(t)|^pdt \geqslant \int\limits_0^1 \frac{|y(t)|^p}{t^{p-2}}\left(\frac{1}{t^2}+\frac{p(1-\nu^2)}{2((p-1)^2-\nu^2)}\frac{4-t}{t(2-t)^2}+\frac{2p\lambda^2}{((p-1)^2-\nu^2q^2)t(2-t)^3}\right)dt,$$

where $c_p=p^p((p-1)^2-\nu^2)^{-\frac{p}{2}}$ and λ is a first positive root of the equation

$$-1-2\nu+s+2z\frac{J_{\nu-1}(z)}{J_\nu(z)}=0.$$

We proceed to inequalities on a segment $[0,2b]$ in terms of the functions

$$\rho(t)=\min\{t,2b-t\} \quad\text{and}\quad \mu(t)=2b-\rho(t).$$

The following theorem holds.

Theorem 3.1. *Assume that $0<b<\infty$, $p\in[2,\infty)$ and $\nu\in[0,p-1]$. If $y:[0,2b]\to\mathbb{R}$ is an absolutely continuous function such that $y(0)=y(2b)=0$ and $|y'(t)|\in L^p[0,2b]$, then the following inequality holds:*

$$c_p\int\limits_0^{2b} |y'(t)|^pdt \geqslant \int\limits_0^{2b} \frac{|y(t)|^p}{\rho^{p-2}(t)}\left(\frac{1}{\rho^2(t)}+\frac{c_1}{\rho(t)\mu(t)}+\frac{c_2}{\mu^2(t)}+\frac{p\lambda^2}{2((p-1)^2-\nu^2)}\frac{\rho(t)}{\mu^3(t)}\right)dt,$$

where

$$c_p=\frac{p^p}{((p-1)^2-\nu^2)^{\frac{p}{2}}},\, c_1=\frac{p(2+\lambda^2)-2\nu^2}{2(p-1)^2-\nu^2}, \qquad c_2=\frac{p(1+2\lambda^2)-2\nu^2}{2((p-1)^2-\nu^2)}$$

and λ is the first positive root of the equation

$$-1-2\nu+s+2z\frac{J_{\nu-1}(z)}{J_\nu(z)}=0.$$

Proof. The inequality in Corollary 3.1 can be transformed as follows:

$$c_p\int\limits_0^1 |y'(t)|^pdt \geqslant \int\limits_0^1 \frac{|y(t)|^p}{t^{p-2}}\left(\frac{1}{t^2}+\frac{c_1}{t(2-t)}+\frac{c_2}{(2-t)^2}+\frac{p\lambda^2}{2((p-1)^2-\nu^2)}\frac{t}{(2-t)^3}\right)dt.$$

By means of the change of variable $t=\tau/b$ in the latter inequality we obtain

$$c_p\int\limits_0^b |y'(\tau)|^pd\tau \geqslant \int\limits_0^b \frac{|y(\tau)|^p}{\tau^{p-2}}\left(\frac{1}{\tau^2}+\frac{c_1}{\tau(2b-\tau)}+\frac{c_2}{(2b-t)^2}+\frac{p\lambda^2}{2((p-1)^2-\nu^2)}\frac{t}{(2b-\tau)^3}\right)d\tau.$$

Combining the latter inequality with the following corresponding inequality on the interval $[b,2b]$

$$c_p\int\limits_b^{2b} |y'(\tau)|^pd\tau \geqslant \int\limits_b^{2b} \frac{|y(\tau)|^p}{(2b-\tau)^{p-2}}\left(\frac{1}{(2b-\tau)^2}+\frac{c_1}{\tau(2b-\tau)}+\frac{c_2}{\tau^2}+\frac{p\lambda^2}{2((p-1)^2-\nu^2q^2)}\frac{2b-\tau}{\tau^3}\right)d\tau,$$

for a function $y \in C^1(b, 2b)$ such that $y(2b) = 0$, we arrive at the statement of the theorem. □

In what follows we shall apply the properties of the above introduced second function for justifying the following auxiliary statement.

Lemma 3.2. *Let $p \geqslant 2$, $s > 0$, $q \in (0, +\infty)$ and an absolutely continuous on the segment $[0,1]$ function y be such that $y(0) = 0$,*

$$|y'(t)|t^{(p+1-s)/p} \in L^p[0,1].$$

Then the inequality

$$\int\limits_0^1 \frac{|y'(t)|^p}{t^{s-p+1}}dt \geqslant \frac{s^p}{p^p}\int\limits_0^1 \frac{|y(t)|^p}{t^{s+1}}\left(1 + \frac{2\lambda_1^2 q^2 p}{s^2}\frac{t^q}{(2-t)^{2+q}} + \frac{p\lambda_1^2 q}{s^2}\frac{t^{q+1}}{(2-t)^{2+q}}\right)dt$$

holds, where λ_1 is the first positive root of the equation

$$sJ_0(\lambda_1) - 2q\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

The proof is similar to that of Lemma 3.1.

For $q = 1$ we have:

Corollary 3.2. *Let $s > 0$, $p \geqslant 2$ and an absolutely continuous on the segment $[0,1]$ function y be such that $y(0) = 0$,*

$$|y'(t)| \in L^p[0,1].$$

Then the inequality

$$\int\limits_0^1 \frac{|y'(t)|^p}{t^{s+1-p}}dt \geqslant \frac{s^p}{p^p}\int\limits_0^1 \frac{|y(t)|^p}{t^{s-1}}\left(\frac{1}{t^2} + \frac{2\lambda_1^2 p}{s^2}\frac{1}{t(2-t)^3} + \frac{p\lambda_1^2}{s^2}\frac{1}{(2-t)^3}\right)dt$$

holds, where λ is the first positive root of the equation

$$sJ_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Theorem 3.2. *Assume that $0 < b < \infty$, $p \in [2, \infty)$ and $\nu \in [0, p-1]$. If $y : [0, 2b] \to \mathbb{R}$ is an absolutely continuous function such that $y(0) = y(2b) = 0$ and $|y'(t)| \in L^p[0, 2b]$, then the following inequality holds:*

$$\left(\frac{p}{s}\right)^p \int\limits_0^{2b} \frac{|y'(t)|^p}{\rho^{s+1-p}(t)}dt \geqslant \int\limits_0^{2b} \frac{|y(t)|^p}{\rho^{s-1}(t)}\left(\frac{1}{\rho^2(t)} + \frac{p\lambda_1^2}{2s^2}\left[\frac{1}{\rho(t)\mu(t)} + \frac{3}{\mu^2(t)} + \frac{2\rho(t)}{\mu^3(t)}\right]\right)dt,$$

where λ_1 is a first positive root of the equation

$$sJ_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \qquad \lambda_1 \in (0, j_0).$$

Proof. The inequality in Corollary 3.2 can be transformed as follows:

$$\left(\frac{p}{s}\right)^p \int\limits_0^1 |y'(t)|^p dt \geqslant \int\limits_0^1 \frac{|y(t)|^p}{t^{s-1}}\left(\frac{1}{t^2} + \frac{p\lambda_1^2}{2s^2}\left[\frac{1}{t(2-t)} + \frac{3}{(2-t)^2} + \frac{2t}{(2-t)^3}\right]\right)dt.$$

In the latter inequality we make the change of the variable $t = \tau/b$ and we get:

$$\left(\frac{p}{s}\right)^p \int\limits_0^b \frac{|y'(\tau)|^p}{t^{s+1-p}}d\tau \geqslant \int\limits_0^b \frac{|y(\tau)|^p}{\tau^{s-1}}\left(\frac{1}{\tau^2} + \frac{p\lambda_1^2}{2s^2}\left[\frac{1}{\tau(2b-\tau)} + \frac{3}{(2b-\tau)^2} + \frac{2\tau}{(2b-\tau)^3}\right]\right)d\tau.$$

Combining this inequality with the following corresponding inequality on the interval $[b, 2b]$

$$\frac{p^p}{s^p}\int\limits_b^{2b}\frac{|y'(\tau)|^p}{(2b-\tau)^{s+1-p}}d\tau \geqslant \int\limits_b^{2b}\frac{|y(\tau)|^p}{(2b-\tau)^{s-1}}\left(\frac{1}{(2b-\tau)^2}+\frac{p\lambda_1^2}{2s^2}\left[\frac{1}{\tau(2b-\tau)}+\frac{3}{\tau^2}+\frac{2(2b-\tau)}{\tau^3}\right]\right)d\tau,$$

for a function $y \in C^1(b, 2b)$ such that $y(2b) = 0$, we arrive at the statement of the theorem. □

4. Multidimensional inequalities

In this section we obtain multidimensional analogues of Hardy type inequalities in arbitrary domains in terms of the mean distance. The mean distance are sometimes called Davies distance. The obtained inequalities become simpler in convex domains.

We first introduce main notations used in the present section. Let Ω be an open connected proper subset in the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, $d\mathbb{S}^{n-1}(\nu)$ be an area differential of the unit sphere and $d\omega(\nu) = \frac{d\mathbb{S}^{n-1}(\nu)}{|\mathbb{S}^{n-1}|}$ be a normed measure on the unit sphere. For each point $x \in \Omega$, $\nu \in \mathbb{S}^{n-1}$ by

$$\tau_\nu(x) := \min\{s > 0 : x + s\nu \notin \Omega\}$$

we denote the distance from a point x to the boundary of the domain Ω along the vector ν,

$$\delta(x) = \inf_{\nu\in\mathbb{S}^{n-1}} \tau_\nu(x)$$

is the distance from a point x to the boundary of the domain Ω,

$$\rho_\nu(x) := \min\{\tau_\nu(x), \tau_{-\nu}(x)\}, \qquad \mu_\nu(x) := \max\{\tau_\nu(x), \tau_{-\nu}(x)\},$$

$$D_\nu(x) := \tau_\nu(x) + \tau_{-\nu}(x), \qquad D(\Omega) = \sup_{x\in\Omega,\nu\in\mathbb{S}^{n-1}} D_\nu(x),$$

and the mean distance is the quantity, see, for instance, [23]

$$\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\int\limits_{\mathbb{S}^{n-1}}\rho_\nu^{-p}(x)d\omega(\nu).$$

By $|\Omega|$ we denote the volume of the domain Ω and Ω_x stands for the elements in the set Ω, which are visible from the point x.

In what follows, for proving the theorems, we apply an approach from paper [13], see also [22]–[24].

Assume that $g \in C_0^1(\Omega)$ is a real-valued function. By ∂_ν we denote a partial derivative along the direction ν. The arguing by E.B. Davies, see [30], together with the one-dimensional inequality in Theorem 3.2 give

$$\left(\frac{p}{s}\right)^p\int\limits_\Omega\frac{|\partial_\nu g(x)|^p}{\rho_\nu^{s+1-p}(x)}dx - \int\limits_\Omega\frac{|g(x)|^p}{\rho_\nu^{s+1}(x)}dx \geqslant \frac{p\lambda_1^2}{2s^2}\int\limits_\Omega\frac{|g(x)|^p}{\rho_\nu^{s-1}(x)}\left(\frac{1}{\rho_\nu(x)\mu_\nu(x)}+\frac{3}{\mu_\nu^2(x)}+\frac{2\rho_\nu(x)}{\mu_\nu^3(x)}\right)dx.$$

We integrate this inequality with respect to the normed measure $d\omega(\nu)$ and use the definition of the derivative along the direction

$$|\partial_\nu g| = |\nu\cdot\nabla g| = |\nabla g||\cos(\nu,\nabla g)|$$

and we obtain

$$\begin{aligned}&\left(\frac{p}{s}\right)^p\int\limits_\Omega|\nabla g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{|\cos(\nu,\nabla g)|^p|}{\rho_\nu^{s+1-p}(x)}d\omega(\nu)dx - \int\limits_\Omega|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx\\ &\geqslant \frac{p\lambda_1^2}{2s^2}\int\limits_\Omega|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{1}{\rho_\nu^{s-1}(x)}\left(\frac{1}{\rho_\nu(x)\mu_\nu(x)}+\frac{3}{\mu_\nu^2(x)}+\frac{2\rho_\nu(x)}{\mu_\nu^3(x)}\right)d\omega(\nu)dx.\end{aligned}\tag{4.1}$$

In [22] J. Tidblom showed that as $p>1$, the relations hold:

$$B(n,p) := \int\limits_{\mathbb{S}^{n-1}} |\cos(\nu,\nabla g)|^p d\omega(\nu) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+p}{2}\right)},$$

$$\int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(x)}\right)^p d\omega(\nu) \geqslant \left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-p/n},$$

and A.A. Balinsky, W.D. Evans, R.T. Lewis established in [23] the following estimates

$$\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)\mu_\nu(x)} d\omega(\nu) \geqslant \left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-2/n}, \qquad \int\limits_{\mathbb{S}^{n-1}} \frac{1}{\mu_\nu(x)^2} d\omega(\nu) \geqslant \frac{1}{2}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-2/n}.$$

Below for $p\geqslant s+1$ we consider four cases.

Case 1: $s\in(0,1]$. Employing the definitions of the functions ρ_ν, μ_ν and applying previous four formulae, we obtain

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^s(x)\mu_\nu(x)} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{\rho_\nu^{1-s}(x)}{\rho_\nu(x)\mu_\nu(x)} d\omega(\nu) \geqslant \delta^{1-s}(x)\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-2/n},$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-1}(x)\mu_\nu(x)^2} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{\rho_\nu^{1-s}(x)}{\mu_\nu^2(x)} d\omega(\nu) \geqslant \frac{\delta^{1-s}(x)}{2}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-2/n},$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{\rho_\nu^{2-s}(x)}{\mu_\nu(x)^3} d\omega(\nu) \geqslant \frac{\delta^{2-s}(x)}{8}\int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^3 d\omega(\nu) \geqslant \frac{\delta^{2-s}(x)}{8}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-\frac{3}{n}}.$$

Therefore,

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_\Omega |\nabla g(x)|^p dx - \int\limits_\Omega |g(x)|^p \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} dx$$
$$\geqslant \frac{5p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{2}{n}} \int\limits_\Omega \frac{|g(x)|^p\delta^{1-s}(x)}{|\Omega_x|^{2/n}} dx + \frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{3}{n}} \int\limits_\Omega |g(x)|^p \frac{\delta^{2-s}(x)}{|\Omega_x|^{3/n}} dx.$$

Case 2: $s\in(1,2]$. Similar to Case 1 we have:

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^s(x)\mu_\nu(x)} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{4\rho_\nu^{1-s}(x)}{(\rho_\nu(x)+\mu_\nu(x))^2} d\omega(\nu) \geqslant \int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu),$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-1}(x)\mu_\nu(x)^2} \geqslant \frac{1}{4}\int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu), \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-2}(x)\mu_\nu(x)^3} \geqslant \frac{\delta^{2-s}(x)}{8}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-\frac{3}{n}}.$$

We hence obtain

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_\Omega |\nabla g(x)|^p dx - \int\limits_\Omega |g(x)|^p \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} dx$$
$$\geqslant \frac{7p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}} \int\limits_\Omega \frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}} dx + \frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{3}{n}} \int\limits_\Omega |g(x)|^p \frac{\delta^{2-s}(x)}{|\Omega_x|^{3/n}} dx.$$

Case 3: $s \in (2,3)$. Since the estimates hold

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^s(x)\mu_\nu(x)} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{4\rho_\nu^{1-s}(x)}{(\rho_\nu(x)+\mu_\nu(x))^2} d\omega(\nu) \geqslant \int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu),$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-1}(x)\mu_\nu(x)^2} \geqslant \frac{1}{4}\int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu), \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-2}(x)\mu_\nu(x)^3} \geqslant \frac{1}{8}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-\frac{s+1}{n}},$$

in this case we have the inequality

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_\Omega |\nabla g(x)|^p dx - \int\limits_\Omega |g(x)|^p \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} dx$$
$$\geqslant \frac{p\lambda_1^2}{s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}} \int\limits_\Omega \frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}} dx.$$

Case 4: $s \in [3,+\infty)$. Similar to the previous cases we obtain

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^s(x)\mu_\nu(x)} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{4\rho_\nu^{1-s}(x)}{(\rho_\nu(x)+\mu_\nu(x))^2} d\omega(\nu) \geqslant \int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu),$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu}{\rho_\nu^{s-1}(x)\mu_\nu(x)^2}) \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{16\rho_\nu^{3-s}(x)}{(\rho_\nu(x)+\mu_\nu(x))^4} d\omega(\nu) \geqslant \int\limits_{\mathbb{S}^{n-1}} \left(\frac{2}{D_\nu(\Omega)}\right)^{s+1} d\omega(\nu),$$

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s-2}(x)\mu_\nu(x)^3} \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{4\rho_\nu^{3-s}(x)}{(\rho_\nu(x)+\mu_\nu(x))^2\mu_\nu^2(x} d\omega(\nu) \geqslant \frac{1}{4}\left[\frac{n}{|\mathbb{S}^{n-1}|}|\Omega_x|\right]^{-\frac{s+1}{n}}.$$

Therefore,

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_\Omega |\nabla g(x)|^p dx - \int\limits_\Omega |g(x)|^p \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} dx$$
$$\geqslant \frac{9p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}} \int\limits_\Omega \frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}} dx.$$

Thus, we have established a theorem.

Theorem 4.1. *Let Ω be an arbitrary domain in the Euclidean space $\mathbb{R}^n$, $g \in C_0^1(\Omega)$, $p \geqslant 2$ and $p \geqslant s+1$. If $s \in (0,1]$, then the inequality*

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_\Omega |\nabla g(x)|^p dx - \int\limits_\Omega |g(x)|^p \int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} dx$$
$$\geqslant \frac{5p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{2}{n}} \int\limits_\Omega \frac{|g(x)|^p\delta^{1-s}(x)}{|\Omega_x|^{2/n}} dx + \frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{3}{n}} \int\limits_\Omega |g(x)|^p \frac{\delta^{2-s}(x)}{|\Omega_x|^{3/n}} dx$$

holds. If $s \in (1,2]$, *then the inequality*

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_{\Omega}|\nabla g(x)|^p dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx+\frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{3}{n}}\int\limits_{\Omega}|g(x)|^p\frac{\delta^{2-s}(x)}{|\Omega_x|^{3/n}}dx$$

holds. If $s \in (2,3)$, *then the inequality*

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_{\Omega}|\nabla g(x)|^p dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx$$
$$\geqslant\frac{p\lambda_1^2}{s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx$$

holds. If $s \in [3,+\infty)$, *then the inequality*

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega)B(n,p)\int\limits_{\Omega}|\nabla g(x)|^p dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx$$
$$\geqslant\frac{9p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx$$

holds. Here λ_1 *is the first positive root of a Lamb type equation*

$$sJ_0(\lambda_1)-2q\lambda_1 J_1(\lambda_1)=0,\quad \lambda_1\in(0,j_0).$$

If we let $p \leqslant s+1$, similar to the proof of Theorem 4.1 we get the following statement.

Theorem 4.2. *Let* Ω *be an arbitrary domain in the Euclidean space* $\mathbb{R}^n$, $g \in C_0^1(\Omega)$, $p \geqslant 2$ *and* $p \leqslant s+1$. *If* $s \in (1,2]$, *then the inequality*

$$\left(\frac{p}{s}\right)^p B(n,p)\int\limits_{\Omega}\frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx+\frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{3}{n}}\int\limits_{\Omega}|g(x)|^p\frac{\delta^{2-s}(x)}{|\Omega_x|^{3/n}}dx$$

holds. If $s \in (2,3)$, *then the inequality*

$$\left(\frac{p}{s}\right)^p B(n,p)\int\limits_{\Omega}\frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx\frac{p\lambda_1^2}{s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx$$

holds. If $s \in [3,+\infty)$, *then the inequality*

$$\left(\frac{p}{s}\right)^p B(n,p)\int\limits_{\Omega}\frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx-\int\limits_{\Omega}|g(x)|^p\int\limits_{\mathbb{S}^{n-1}}\frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)}dx\geqslant\frac{9p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{s+1}{n}}\int\limits_{\Omega}\frac{|g(x)|^p}{|\Omega_x|^{\frac{s+1}{n}}}dx$$

holds. Here λ_1 *is the first positive root of a Lamb type equation*

$$sJ_0(\lambda_1)-2q\lambda_1 J_1(\lambda_1)=0,\quad \lambda_1\in(0,j_0).$$

Previous theorems can be considered in narrower classes of domains. We can suppose that the domain Ω is regular in the Davies sense or to impose the external cone condition or the convexity condition. In these cases the formulae simplify essentially, see for more details [20], [23]. For instance, if Ω is a convex domain, then $|\Omega_x| = |\Omega|$ and, as it was shown in [22], the inequality holds:

$$\int\limits_{\mathbb{S}^{n-1}} \frac{d\omega(\nu)}{\rho_\nu^{s+1}(x)} \geqslant \frac{B(n,s+1)}{\delta^{s+1}(x)}.$$

As corollaries of Theorems 4.1 and 4.2 we respectively obtain the following statements.

Theorem 4.3. *Let Ω be a bounded convex domain in the Euclidean space $\mathbb{R}^n$, $g \in C_0^1(\Omega)$, $p \geqslant 2$ and $p \geqslant s+1$. If $s \in (0,1]$, then the inequality*

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega) B(n,p) \int\limits_\Omega |\nabla g(x)|^p dx - B(n,s+1) \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)} dx$$
$$\geqslant \frac{5p\lambda_1^2}{4s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{2}{n}} \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s-1}(x)} dx + \frac{p\lambda_1^2}{8s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{3}{n}} \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s-2}(x)} dx$$

holds. If $s \in (1,2]$, then the inequality

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega) B(n,p) \int\limits_\Omega |\nabla g(x)|^p dx - B(n,s+1) \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)} dx$$
$$\geqslant \frac{7p\lambda_1^2}{8s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}} \int\limits_\Omega |g(x)|^p dx + \frac{p\lambda_1^2}{8s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{3}{n}} \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s-2}(x)} dx$$

holds. If $s \in (2,3)$, then the inequality

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega) B(n,p) \int\limits_\Omega |\nabla g(x)|^p dx - B(n,s+1) \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)} dx$$
$$\geqslant \frac{p\lambda_1^2}{s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}} \int\limits_\Omega |g(x)|^p dx$$

holds. If $s \in [3,+\infty)$, then the inequality

$$\left(\frac{p}{s}\right)^p D^{s+1-p}(\Omega) B(n,p) \int\limits_\Omega |\nabla g(x)|^p dx - B(n,s+1) \int\limits_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)} dx$$
$$\geqslant \frac{9p\lambda_1^2}{4s^2} \left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}} \int\limits_\Omega |g(x)|^p dx$$

holds. Here λ_1 is the first positive root of a Lamb type equation

$$sJ_0(\lambda_1) - 2q\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Theorem 4.4. *Let Ω be a bounded convex domain in the Euclidean space $\mathbb{R}^n$, $g \in C_0^1(\Omega)$, $p \geqslant 2$ and $p \leqslant s+1$. If $s \in (1,2]$, then the inequality*

$$\left(\frac{p}{s}\right)^p B(n,p)\int_\Omega \frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx - B(n,s+1)\int_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}}\int_\Omega |g(x)|^p dx + \frac{p\lambda_1^2}{8s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{3}{n}}\int_\Omega \frac{|g(x)|^p}{\delta^{s-2}(x)}dx$$

holds. If $s \in (2,3)$, then the inequality

$$\left(\frac{p}{s}\right)^p B(n,p)\int_\Omega \frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx - B(n,s+1)\int_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)}dx \geqslant \frac{p\lambda_1^2}{s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}}\int_\Omega |g(x)|^p dx$$

holds. If $s \in [3,+\infty)$, then the inequality

$$\left(\frac{p}{s}\right)^p B(n,p)\int_\Omega \frac{|\nabla g(x)|^p}{\delta^{s+1-p}(x)}dx - B(n,s+1)\int_\Omega \frac{|g(x)|^p}{\delta^{s+1}(x)}dx \geqslant \frac{9p\lambda_1^2}{4s^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{s+1}{n}}\int_\Omega |g(x)|^p dx$$

holds. Here λ_1 is the first positive root of a Lamb type equation

$$sJ_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

To compare with known results, we consider the case $s = p-1$. The following statement holds.

Corollary 4.1. *Let Ω be a bounded convex domain in the Euclidean space $\mathbb{R}^n$, $g \in C_0^1(\Omega)$, $p \geqslant 2$. If $p \in (2,3]$, then the inequality*

$$\left(\frac{p}{p-1}\right)^p B(n,p)\int_\Omega |\nabla g(x)|^p dx - B(n,p)\int_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{8(p-1)^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\int_\Omega |g(x)|^p dx + \frac{p\lambda_1^2}{8(p-1)^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{3}{n}}\int_\Omega \frac{|g(x)|^p}{\delta^{p-3}(x)}dx$$

holds. If $p \in (3,4)$, then the inequality

$$\left(\frac{p}{p-1}\right)^p B(n,p)\int_\Omega |\nabla g(x)|^p dx - B(n,p)\int_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx \geqslant \frac{p\lambda_1^2}{(p-1)^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\int_\Omega |g(x)|^p dx$$

holds. If $p \in [4,+\infty)$, then the inequality

$$\left(\frac{p}{p-1}\right)^p B(n,p)\int_\Omega |\nabla g(x)|^p dx - B(n,p)\int_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx \geqslant \frac{9p\lambda_1^2}{4(p-1)^2}\left(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\right)^{\frac{p}{n}}\int_\Omega |g(x)|^p dx$$

holds.

Using the definiftion of $\lambda_p(\Omega)$ and Corollary 4.1, we obtain one more corollary.

Corollary 4.2. *In convex domains Ω with a fixed volume as $p \in (2,3]$, the estimate holds:*

$$\lambda_p(\Omega) \geqslant \frac{7p\lambda_1^2}{8(p-1)^2}\frac{(p-1)^p}{p^p}\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{p/n}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}\frac{1}{|\Omega|^{p/n}},$$

where λ_1 is the first positive root of a Lamb type equation

$$(p-1)J_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Remark 4.1. *Numerical calculations show that*

$$\frac{7p\lambda_1^2}{8(p-1)^2} \geqslant p-1$$

as $p \in [2, p_0]$, *where* $p_0 \approx 2.314$.

In in Theorem 4.3 we let $s = 1$ and $p = 2$, we get a result strengthening inequality (1.4) by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev. The following statement holds true.

Corollary 4.3. *Let* Ω *be a bounded convex domain in the Euclidean space* $\mathbb{R}^n$, $g \in C_0^1(\Omega)$. *Then the inequality holds:*

$$\int\limits_\Omega |\nabla g(x)|^2 dx \geqslant \frac{1}{4}\int\limits_\Omega \frac{g^2(x)}{\delta^2(x)}dx + \frac{5\lambda_1^2 K(n)}{8|\Omega|^{2/n}}\int\limits_\Omega g^2(x)dx + \frac{\lambda_1^2(1)K(n)n}{16|\mathbb{S}^{n-1}||\Omega|^{3/n}}\int\limits_\Omega g^2(x)\delta(x)dx,$$

where $\lambda_1 \approx 1.25578$.

Corollary 4.4. *In convex domains* Ω *with a fixed volume*

$$\lambda_1(\Omega) \geqslant \frac{5\lambda_1^2 K(n)}{8|\Omega|^{2/n}},$$

where $\lambda_1 \approx 1.25578$.

There is also an inequality with additional terms depending only on the diameter of the domain. The following theorem is true.

Theorem 4.5. *Let* Ω *be a bounded convex domain in the Euclidean space* $\mathbb{R}^n$ *and* $p \geqslant 2$. *Then for each function* $g \in C_0^1(\Omega)$ *the inequality holds:*

$$B(n,p)\left(\frac{p}{s}\right)^p \int\limits_\Omega |\nabla g(x)|^p dx - B(n,p)\int\limits_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{2(p-1)^2 D^p(\Omega)}\int\limits_\Omega |g(x)|^p dx + \frac{p\lambda_1^2}{(p-1)^2 D^p(\Omega)}\int\limits_\Omega |g(x)|^p \delta(x)dx.$$

Here λ_1 *is the first positive root of a Lamb type equation*

$$(p-1)J_0(\lambda_1) - 2\lambda_1 J_1(\lambda_1) = 0, \quad \lambda_1 \in (0, j_0).$$

Proof. Employing inequality (4.1) and the definition of the diameter of the domain $D(\Omega)$, as $s = p-1$ by obvious inequalities

$$\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu(x)\mu_\nu(x)}d\omega(\nu) \geqslant \int\limits_{\mathbb{S}^{n-1}} \frac{4}{(\rho_\nu(x)+\mu_\nu(x))^2}d\omega(\nu) \geqslant \frac{4}{D^2(\Omega)},$$
$$\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\mu_\nu(x)^2}d\omega(\nu) \geqslant \frac{1}{D^2(\Omega)},$$

we obtain

$$B(n,p)\left(\frac{p}{s}\right)^p \int\limits_\Omega |\nabla g(x)|^p dx - B(n,p)\int\limits_\Omega \frac{|g(x)|^p}{\delta^p(x)}dx$$
$$\geqslant \frac{7p\lambda_1^2}{2(p-1)^2 D^p(\Omega)}\int\limits_\Omega |g(x)|^p dx + \frac{p\lambda_1^2}{(p-1)^2 D^p(\Omega)}\int\limits_\Omega |g(x)|^p \delta(x)dx.$$

This implies the statement of the theorem. □

Corollary 4.5. *In convex domains* Ω *with a fixed diameter*

$$\lambda_p(\Omega) \geqslant \frac{7p\lambda_1^2}{2(p-1)^2 D^p(\Omega) B(n,p)},$$

where $\lambda_1 \approx 1.25578$.

Inequalities similar to the results of this section can be also obtained by using Theorem 3.1. The arguing and jusfitication are almost the same but the constants in inequalities will be different. These results sometimes have certain advantages. For instance, in convex domains Ω with a fixed volume one can obtain the estimate

$$\lambda_1(\Omega) \geqslant \frac{j_1'^2}{2} \frac{K(n)}{|\Omega|^{2/n}},$$

where ${j'}_1 \approx 1.84118$ is the first positive root of the derivative ${J_1}'$ of the Bessel function J_1.

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