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ON DISCRETE SPECTRUM OF ONE TWO-PARTICLE LATTICE HAMILTONIAN

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Abstract. Linear self-adjoint operators in the Friedrichs models arise in various fields, for instance, in the perturbation theory of spectra of self-adjoint operators, in the quantum field theory, in theory of two- and three-particle discrete Schrödinger operators, in hydrodynamics, etc. An operator H in the Friedrichs model is a sum of two operators in the Hilbert space $L_2(\Omega)$, that is, $H = H_0 + \varepsilon K$, $\varepsilon > 0$, where H_0 is the operator by multiplication by a function and K is a compact integral operator. For the operators in the Friedrichs models we need to solve the following problems:

1) Under which conditions the discrete spectrum is an empty set?
2) Under which conditions the discrete spectrum is a non-empty set?
3) Find conditions ensuring that an operator in the Friedrichs model is a finite set;
4) Find sufficient conditions guaranteeing that an operator in the Friedrichs model is an infinite set.

It is known that if a kernel of an integral operator in the model is degenerate, then the discrete spectum of the correponding operator in the Friedrichs model is a finite set. Therefore, a necessary condition for the operator in the Friedrichs model to possess an infinite discrete spectrum is the non-degeneracy of the integral operator in the model. In the paper we consider linear bounded self-adjoint operator in the Friedrichs model, for which the integral operator has a non-degenerate kernel. In this work, we study the first and the fourth questions. We obtain one sufficient condition guaranteeing that the operators in Friedrichs model possesses an infinite discrete spectrum. We study the spectrum of one two-particle discrete Schrödinger operator $Q(\varepsilon)$ on the lattice $\mathbb{Z}^\nu \times \mathbb{Z}^\nu$, in which the Fourier transform of the operator $Q(\varepsilon)$ is represented as $H = H_0 + \varepsilon K$, $\varepsilon > 0$. It is shown that the structure of the Schrödinger operator $Q(\varepsilon)$ highly depends on the dimension ν of the lattice. It is proven that in the case $\nu = 1, 2$, for all $\varepsilon > 0$ the discrete spectrum of the Schrödinger operator $Q(\varepsilon)$ is infinite, while in the case $\nu \geqslant 3$, for sufficiently small $\varepsilon > 0$, the discrete spectrum of the Schrödinger operator $Q(\varepsilon)$ is an empty set.

Keywords: Friedrichs model, two-partical Hamiltonian, self-adjoint operator, spectrum, essential spectrum, discrete spectrum, non-degenerate kernel.

Mathematics Subject Classification: 47A10, 47A11, 47A13, 47A25, 47B38

1. Introduction

The studying of spectra is the main problem in the theory of Schrödinger operators. Let $u(x)$ be a real-valued continuous function on $\Omega_\nu = [0,1]^\nu$, $\nu \in \mathbb{N}$, K be an integral operator in the Hilbert space $L_2(\Omega_\nu)$ with the kernel $k(x,s) \in L_2(\Omega_\nu^2)$, where $k(x,s) = \overline{k(s,x)}$. A series of questions in quantum mechanics and statistical physics (see [1]–[5]) lead to studying a discrete spectrum of an operator H in a Hilbert space $L_2(\Omega_\nu)$ acting by the rule:

$$H = H_0 - K, \tag{1.1}$$

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where

$$(H_0 f)(x) = u(x)f(x), \qquad (Kf)(x) = \int\limits_{\Omega_\nu} k(x,s)f(s)d\mu(s).$$

Here the integral is treated in the Lebesgue sense, $\mu(\,\cdot\,)$ is the Lebesgue measure on $\mathbb{R}^\nu$. The classical Weyl theorem on a compact perturbation [6] yields that the essential spectrum $\sigma_{ess}(H)$ of the operator H coincides with the set of the values of the function $u(x)$, that is, $\sigma_{ess}(H) = [u_{min}, u_{max}]$, where $u_{min} = \min\limits_{x\in\Omega_\nu} u(x)$, $u_{max} = \max\limits_{x\in\Omega_\nu} u(x)$.

An operator of form (1.1) is called an operator in the Friedrichs model. It should be noted that the Friedrichs model is applied in various fields of science. In 1937, in work [7], K. Freidrichs proposed to consider this model in the perturbation theory of essential spectra of self-adjoint operators. Then K. Friedrichs showed [8] that the studying of a one-particle Schrödinger operator is reduced to studying the operator in the Friedrichs model. In paper [2], while studying the spectra of stochastic operators arising the lattice gas models, properties of the operators in the Friedrichs model were used. Moreover, solving problems related with waves propagation and tsunami problem, is reduced to studying the spectra of the operators in the Friedrichs model [9]. The spectral properties of self-adjoint operators in the Friedrichs models are widely used in studying spectra of two-particle and three-particle discrete Schrödinger operators [5], [10]–[12] and so forth.

A series of publications is devoted to studying the spectra of the operators in the Friedrichs model, see, for instance, [13]–[20] and others. By the minimax and maximin principle, it was proved in [14] that if the kernel of the integral operator K degenerates, then the discrete spectrum in Friedrichs model (1.1) is finite. This implies that in order to the operator in model (1.1) to have an infinite discrete spectrum, the kernel of the integral operator K should be non-degenerate.

In the present paper we consider a two-particle Hamiltonian $Q(\varepsilon)$, $\varepsilon > 0$ on the lattice $\mathbb{Z}^\nu \times \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ is an ν-dimensional integer lattice. The Fourier transform of the Hamiltonian $Q(\varepsilon)$ is a self-adjoint operator in the Friedrichs model with a non-degenerate kernel. In the case $\nu = 1, 2$, for all $\varepsilon > 0$ we prove the existence of infinitely many negative eigenvalues of the Hamiltonian $Q(\varepsilon)$. In the case $\nu \geqslant 3$ we prove that for sufficiently small ε, the Hamiltonian $Q(\varepsilon)$ possesses no negative eigenvalues.

2. Formulation of problem and auxiliary statements

Let $\mathcal{H}$ be a separable Hilbert space, $A : \mathcal{H} \to \mathcal{H}$ be a linear bounded self-adjoint operator. By $\sigma(A)$, $\sigma_{ess}(A)$ and $\sigma_{disc}(A)$ we respectively denote the spectrum, essential spectrum and discrete spectrum of an operator A, see [21]. We also introduce the following notations:

$$E_{min}(A) = \inf\{\lambda : \lambda \in \sigma_{ess}(A)\}, \quad E_{max}(A) = \sup\{\lambda : \lambda \in \sigma_{ess}(A)\}.$$

A number $E_{min}(A)$ (a number $E_{max}(A)$) is called the bottom (top) of the essential spectrum of an operator A.

By $\{\mu_n(A)\}_{n\in\mathbb{N}}$ we denote a bounded increasing sequence of real numbers constructed by the minimax principle for a given self-adjoint operator A, see [14]. Then each number $\mu_n(A)$, $n \in \mathbb{N}$, is an eigenvalue of the operator A and $\lim\limits_{n\to\infty} \mu_n(A) = E_{min}(A)$, that is,

$$\{\mu_n(A)\}_{n\in\mathbb{N}} = \sigma_{disc}(A) \cap (-\infty, E_{min}(A)),$$

or there exists $n_0 \in \mathbb{N}$ such that each number $\mu_k(A)$, $k \in \{1, 2, \ldots, n_0\}$ is an eigenvalue of the operator A and $\mu_k(A) = E_{min}(A)$ for all $k > n_0$, that is,

$$\{\mu_1(A), \mu_2(A), \ldots, \mu_{n_0}(A)\} = \sigma_{disc}(A) \cap (-\infty, E_{min}(A)).$$

A linear bounded self-adjoint operator A is called positive if $(Ax,x) \geqslant 0$ for all $x \in \mathcal{H}$; we write this as $A \geqslant 0$ or $0 \leqslant A$.

Lemma 2.1 ([6], [14])**.** *Let $A, B : \mathcal{H} \to \mathcal{H}$ be linear bounded self-adjoint operators, $E_{min}(A) = E_{min}(B)$ and $A \leqslant B$. Then $\mu_n(A) \leqslant \mu_n(B)$, $n \in \mathbb{N}$, where*

$$\mu_k(A) = \sup_{L\subset\mathcal{H},\ \dim L = k-1} \ \inf_{\|x\|=1,\ x\perp L} (Ax,x), \quad k \in \mathbb{N}.$$

In a Hilbert space $L_2(\Omega_\nu \times \Omega_\nu)$ we consider an operator H_1 in the Friedrichs model

$$H_1 = H_0 - K, \tag{2.1}$$

where

$$(H_0 f)(x,y) = u(x,y)f(x,y), \quad (Kf)(x,y) = \int\limits_{\Omega_\nu}\int\limits_{\Omega_\nu} k(x,y;s,t)f(s,t)d\mu(s)d\mu(t).$$

Here $u(x,y) \in C(\Omega_\nu \times \Omega_\nu)$ is non-negative and $0 \in \mathrm{Ran}(u)$, $k(x,y;s,t) \in L_2(\Omega_\nu^2 \times \Omega_\nu^2)$ and $k(x,y;s,t) = \overline{k(s,t;x,y)}$.

Let the operator K possess infinitely many positive eigenvalues $\eta_1 > \eta_2 > \ldots > \eta_n > \ldots$, $\eta_n \to 0$, $n \to \infty$ and $\{g_n(x,y)\}_{n\in\mathbb{N}}$ be an associated sequence of orthonormalized eigenfunctions. The self-adjointness of the operator H_1 (2.1) implies that $\sigma(H_1) \subset \mathbb{R}$, while the positivity of the operator K yields that $\sigma(H_1) \cap (u_{\max}, \infty) = \emptyset$. This is why the discrete spectrum of the operator H_1 can be located only in the half-line $(-\infty, 0)$.

For each $\xi < 0$ we define integral operators

$$P(\xi) = K^{\frac{1}{2}} r_0(\xi) K^{\frac{1}{2}}, \qquad R(\xi) = K^{\frac{1}{2}} r_0^{\frac{1}{2}}(\xi),$$

where $r_0(\xi)$ is the resolvent of the multiplier H_0. The representation $P(\xi) = R(\xi)(R(\xi))^*$ yields the positivity of the operator $P(\xi)$. The solution f_0 of the equation $H_1 f = \xi f$ and fixed points φ of the operator $P(\xi)$ are related by the identities

$$f_0 = r_0(\xi)K^{\frac{1}{2}}\varphi, \qquad \varphi = K^{\frac{1}{2}} f_0. \tag{2.2}$$

Lemma 2.2 ([20])**.** *A number $\xi < 0$ is an eigenvalue of the operator H_1 if and only if the number $\lambda = 1$ is an eigenvalue of the operator $P(\xi)$.*

It follows from Lemma 2.2 that $\dim \mathrm{Ker}(H_1 - \xi I) = \dim \mathrm{Ker}(P(\xi) - I)$, $\xi < 0$. We let

$$\Phi(\xi) = \int\limits_{\Omega_\nu}\int\limits_{\Omega_\nu} \frac{dxdy}{u(x,y) - \xi}, \quad \xi < 0.$$

Theorem 2.1. *Let $u(x,y) = u_0(y)$ and $k(x,y;s,t) = k_0(x,s)$ in the model H_1 (2.1).*

If $\lim\limits_{\xi\to 0-0} \Phi(\xi) = M < +\infty$ and for some index $n_0 \in \mathbb{N}$ the condition $M\eta_{n_0} > 1$ holds, then the operator H_1 (2.1) possesses n_0 negative eigenvalues $\xi_1 < \xi_2 < \ldots < \xi_{n_0}$, $n_0 \in \mathbb{N}$, and the associated eigenfunctions are of the form:

$$f_k(x,y) = \frac{g_k^0(x)}{u_0(y) - \xi_k}, \quad k \in \{1, 2, \ldots, n_0\}. \tag{2.3}$$

Proof. In the case $\xi < 0$ for the kernel $p(\xi, x; s)$ of the integral operator $P(\xi)$ the identity $p(\xi; x, s) = \Phi(\xi)k_0(x,s)$ holds. Therefore,

$$P(\xi) = \Phi(\xi)K. \tag{2.4}$$

This means that the eigenfunctions of the operator K are also the eigenfunctions of the operator $P(\xi)$. Under the assumptions of Theorem 2.1, non-zero eigenvalues of the operator K are the numbers η_n, $n \in \mathbb{N}$, and the associated eigenfunctions as $g_n(x,y) = g_n^0(x) \in L_2(\Omega_\nu)$, $n \in \mathbb{N}$.

Then it follows from (2.4) that the numbers

$$\lambda_n(\xi) = \eta_n \Phi(\xi), \quad n \in \mathbb{N}, \tag{2.5}$$

are the eigenvalues of the operator $P(\xi)$.

By Lemma 2.2 we obtain:

$$\Phi(\xi) = \frac{1}{\eta_n}, \quad n \in \mathbb{N}. \tag{2.6}$$

It is clear that the function $\Phi(\xi)$ is positive and increases on $(-\infty, 0)$ since

$$\Phi'(\xi) = \int\limits_{\Omega_\nu} \frac{dy}{(u_0(y) - \xi)^2} \geqslant 0.$$

Moreover,

$$\lim_{\xi \to 0-0} \Phi(\xi) = M, \qquad \lim_{\xi \to -\infty} \Phi(\xi) = 0.$$

If $M\eta_{n_0} > 1$, then equation (2.6) has n_0 negative solutions $\xi_1 < \xi_2 < \ldots < \xi_{n_0} < 0$. Lemma 2.2 implies that each of the numbers ξ_k, $k \in \{1, 2, \ldots, n_0\}$ is an eigenvalue of the operator H_1. Since the operator $P(\xi)$ has the eigenfunctions $g_k(x, y) = g_k^0(x)$, $k \in \{1, 2, \ldots, n_0\}$, by relation (2.2) we conclude that with an eigenvalue ξ_k, $k \in \{1, 2, \ldots, n_0\}$ of the operator H_1, the eigenfunctions $f_k(x, y)$ of form (2.3) are associated. □

Theorem 2.1 implies the following proposition.

Proposition 2.1. *Let $u(x, y) = u_0(y)$ and $k(x, y; s, t) = k_0(x, s)$ in model (2.1).*

a) If $\lim\limits_{\xi \to 0-0} \Phi(\xi) = +\infty$, then the operator H_1 possesses infinitely many negative eigenvalues ξ_n, $n \in \mathbb{N}$.

b) If $\lim\limits_{\xi \to 0-0} \Phi(\xi) = M < \infty$ and $M\eta_{n_1} < 1$, then the operator H_1 possesses no negative eigenvalues.

On $(-\infty, 0)$ we define the following functions

$$\Phi_1(\nu; \xi) = \int\limits_{\mathbb{T}^\nu} \frac{dy_1 dy_2 \ldots dy_\nu}{\sum\limits_{k=1}^{\nu} (1 - \cos y_k) - \xi}, \quad \text{where} \quad \mathbb{T} = [-\pi, \pi].$$

Employing the properties of the trigonometrci function $\cos y$, we prove the following:

Lemma 2.3. *a) If $\nu = 1, 2$, then $\lim\limits_{\xi \to 0-0} \Phi_1(\nu; \xi) = +\infty$.*

b) If $\nu \geq 3$, then $\lim\limits_{\xi \to 0-0} \Phi_1(\nu; \xi) < \infty$.

3. Description of two-particle lattice Hamiltonian $Q(\varepsilon)$ on lattice $\mathbb{Z}^\nu \times \mathbb{Z}^\nu$

We consider a two-particle lattice Hamiltonian [22]

$$Q(\varepsilon) = Q_0 - \varepsilon Q_1, \quad \varepsilon > 0, \tag{3.1}$$

acting in the space $l_2(\mathbb{Z}^\nu \times \mathbb{Z}^\nu)$ $(\nu \in \mathbb{N})$, where the kinetic energy Q_0 is defined by a convolution with a function of general form:

$$(Q_0 \phi)(m, n) = \sum_{k, l \in \mathbb{Z}^\nu} v_0(m - k, n - l) \phi(k, l),$$

while the potential energy Q_1 equals

$$(Q_1 \phi)(m, n) = v_1(m, n) \phi(m, n).$$

Let the kinetic energy reads as $v_0(m,n)=u_1(m)u_2(n)$, where

$$u_1(m)=\begin{cases}-2a_1\nu & \text{as} \quad m=0,\\ \ a_1 & \text{as} \quad |m|=1,\\ \ 0 & \text{for other values} \quad m\in\mathbb{Z}^\nu,\end{cases}$$

$$u_2(n)=\begin{cases}-2a_2\nu & \text{as} \quad n=0,\\ \ a_2 & \text{as} \quad |n|=1,\\ \ 0 & \text{for other values} \quad n\in\mathbb{Z}^\nu,\end{cases}$$

where $a_1,\, a_2>0$ and $|m|=|m_1|+|m_2|+\ldots+|m_\nu|$, $m\in\mathbb{Z}^\nu$.

We define a potential function

$$v_1(m,n)=\begin{cases}\alpha_0 & \text{as} \quad m=n=0,\\ \beta_q & \text{as} \quad m=0,\ n\in\{\pm qe_j\},\ q\in\mathbb{N},\\ \alpha_p & \text{as} \quad m\in\{\pm pe_j\},\ n=0,\ p\in\mathbb{N},\\ 0 & \text{for other values} \quad m,n\in\mathbb{Z}^\nu,\end{cases}$$

where $\alpha_0,\alpha_p,\beta_q>0$, $p,q\in\mathbb{N}$, $\sum\limits_{p\in\mathbb{N}}\alpha_p<\infty$, $\sum\limits_{q\in\mathbb{N}}\beta_q<\infty$, $e_j=(\underbrace{0,0,\ldots,1}_{j},0,0,\ldots,0)\in\mathbb{Z}^\nu$.

Let $\mathbb{T}=(-\pi,\pi]$ and $\mathcal{F}:l_2(\mathbb{Z}^\nu\times\mathbb{Z}^\nu)\to L_2(\mathbb{T}^\nu\times\mathbb{T}^\nu)$ be a Fourier transform, under which the functions $\phi(m,n)$ on lattice $\mathbb{Z}^\nu\times\mathbb{Z}^\nu$ are transformed into functions $f(x,y)$ on $\mathbb{T}^\nu\times\mathbb{T}^\nu$ by the rule

$$f(x,y)=\frac{1}{(2\pi)^\nu}\sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu}\phi(p^{(1)},q^{(1)})exp(i[(p^{(1)},x)+(q^{(1)},y)]).$$

Lemma 3.1. *The Fourier transform $\widetilde{H}_2(\varepsilon)$ of the operator $Q(\varepsilon)$ (3.1) acts in $L_2(\mathbb{T}^\nu\times\mathbb{T}^\nu)$ by formula*

$$\widetilde{H}_2(\varepsilon)f(x,y)=H_0^{(2)}f(x,y)-\varepsilon K_2f(x,y), \tag{3.2}$$

here

$$H_0^{(2)}f(x,y)=u_0^{(2)}(x,y)f(x,y), \qquad K_2f(x,y)=\int\limits_{\mathbb{T}^\nu}\int\limits_{\mathbb{T}^\nu}k_2(x,y;s,t)f(s,t)dsdt$$

and

$$u_0^{(2)}(x,y)=4a_1a_2\sum_{k=1}^{\nu}(1-\cos x_k)\sum_{k=1}^{\nu}(1-\cos y_k),$$

the kernel $k_2(x,y;s,t)$ is non-degenerate:

$$k_2(x,y;s,t)=\lambda_0\varphi_0(x,y)+\sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(1)}(x_i)\varphi_p^{(1)}(s_i)+\sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(2)}(x_i)\varphi_p^{(2)}(s_i)$$
$$+\sum_{q=1}^{\infty}\lambda_q^{(2)}\sum_{i=1}^{\nu}\varphi_q^{(1)}(y_i)\varphi_q^{(1)}(t_i)+\sum_{q=1}^{\infty}\lambda_q^{(2)}\sum_{i=1}^{\nu}\varphi_q^{(2)}(y_i)\varphi_q^{(2)}(t_i),$$

where

$$\lambda_0=(2\pi)^\nu\alpha_0, \qquad \lambda_p^{(1)}=(2\pi)^{2\nu}\alpha_p, \qquad \lambda_q^{(2)}=(2\pi)^{2\nu}\beta_q;$$

$$\varphi_0(x,y)=\frac{1}{(2\pi)^\nu}, \qquad \varphi_p^{(1)}(x_i)=\frac{\cos px_i}{\sqrt{2^{2\nu-1}\pi^\nu}}, \qquad \varphi_p^{(2)}(x_i)=\frac{\sin px_i}{\sqrt{2^{2\nu-1}\pi^\nu}}, \qquad i=1,2,\ldots,\nu.$$

Proof. I. We first consider the Fourier transform of the operator Q_0:

$$\mathcal{F} : Q_0 \to H_0^{(2)}.$$

We let

$$Q_0\phi(m,n) = \psi_1(m,n), \qquad \psi_1(m,n) \in l_2(\mathbb{Z}^\nu \times \mathbb{Z}^\nu), \qquad \zeta_{p^{(1)},q^{(1)}}(x,y) = e^{i[(p^{(1)},x)+(q^{(1)},y)]}.$$

Then we have:

$$\begin{aligned}
\mathcal{F} : \psi_1(m,n) \to & g_1(x,y) = \frac{1}{(2\pi)^\nu} \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \psi(p^{(1)},q^{(1)}) e^{i[(p^{(1)},x)+(q^{(1)},y)]} \\
= & \frac{1}{(2\pi)^\nu} \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \left[\sum_{k,l\in\mathbb{Z}^\nu} v_0(p^{(1)}-k, q^{(1)}-l)\phi(k,l) \right] \zeta_{p^{(1)},q^{(1)}}(x,y) \\
= & \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{k,l\in\mathbb{Z}^\nu} v_0(p^{(1)}-k, q^{(1)}-l) \left(\int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t)\overline{\zeta_{k,l}(s,t)} ds dt \right) \zeta_{p^{(1)},q^{(1)}}(x,y) \\
= & \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t) \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{k,l\in\mathbb{Z}^\nu} v_0(p^{(1)}-k, q^{(1)}-l)\zeta_{p^{(1)},q^{(1)}}(x,y)\overline{\zeta_{k,l}(s,t)} ds dt \\
= & \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t) \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{\substack{p^{(1)}-k=0\\ q^{(1)}-l=0\\ k,l\in\mathbb{Z}^\nu}} v_0(p^{(1)}-k, q^{(1)}-l)\zeta_{p^{(1)},q^{(1)}}(x,y)\overline{\zeta_{k,l}(s,t)} ds dt \\
& + \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t) \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{\substack{|p^{(1)}-k|=1\\ q^{(1)}-l=0\\ k,l\in\mathbb{Z}^\nu}} v_0(p^{(1)}-k, q^{(1)}-l)\zeta_{p^{(1)},q^{(1)}}(x,y)\overline{\zeta_{k,l}(s,t)} ds dt \\
& + \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t) \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{\substack{p^{(1)}-k=0\\ |q^{(1)}-l|=1\\ k,l\in\mathbb{Z}^\nu}} v_0(p^{(1)}-k, q^{(1)}-l)\zeta_{p^{(1)},q^{(1)}}(x,y)\overline{\zeta_{k,l}(s,t)} ds dt \\
& + \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t) \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} \sum_{\substack{|p^{(1)}-k|=1\\ |q^{(1)}-l|=1\\ k,l\in\mathbb{Z}^\nu}} v_0(p^{(1)}-k, q^{(1)}-l)\zeta_{p^{(1)},q^{(1)}}(x,y)\overline{\zeta_{k,l}(s,t)} ds dt \\
= & T_1 f(x,y) + T_2 f(x,y) + T_3 f(x,y) + T_4 f(x,y), \quad f \in L_2(\mathbb{T}^\nu \times \mathbb{T}^\nu).
\end{aligned}$$

Here by T_1, T_2, T_3, T_4 we denote the operators involved in the latter expression according the orders in the terms. For each operator T_k, $k = 1,2,3,4$, the following identity holds:

$$T_1 f(x,y) = 4a_1a_2\nu^2 f(x,y), \quad T_2 f(x,y) = -4a_1a_2\nu \sum_{k=1}^{\nu} \cos x_k f(x,y),$$

$$T_3 f(x,y) = -4a_1a_2\nu \sum_{k=1}^{\nu} \cos y_k f(x,y), \quad T_4 f(x,y) = 4a_1a_2 \left(\sum_{k=1}^{\nu} \cos x_k \right) \left(\sum_{k=1}^{\nu} \cos y_k \right) f(x,y).$$

Therefore,

$$\begin{aligned} g_1(x,y) =& \Big(4a_1a_2\nu^2 - 4a_1a_2\nu\sum_{k=1}^{\nu}\cos x_k - 4a_1a_2\nu\sum_{k=1}^{\nu}\cos y_k \\ &+ 4a_1a_2\sum_{k=1}^{\nu}\cos x_k\sum_{k=1}^{\nu}\cos y_k\Big) f(x,y) \\ =& 4a_1a_2\sum_{k=1}^{\nu}(1-\cos x_k)\sum_{k=1}^{\nu}(1-\cos y_k) f(x,y). \end{aligned}$$

II. We consider the Fourier transform of the operator Q_1:

$$\mathcal{F}: Q_1 \to K_2.$$

We let $Q_1\phi(m,n) = \psi_2(m,n)$, $\psi_2(m,n) \in l_2(\mathbb{Z}^\nu \times \mathbb{Z}^\nu)$. Then we have:

$$\begin{aligned} \mathcal{F}: \psi_2(m,n) \to g_2(x,y) &= \frac{1}{(2\pi)^\nu}\sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu}\psi(p^{(1)},q^{(1)})e^{i[(p^{(1)},x)+(q^{(1)},y)]} \\ &= \frac{1}{(2\pi)^\nu}\sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} v_1(p^{(1)},q^{(1)})\phi(p^{(1)},q^{(1)})e^{i[(p^{(1)},x)+(q^{(1)},y)]} \\ &= \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} v_1(p^{(1)},q^{(1)})\left(\int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} f(s,t)e^{-i[(p^{(1)},s)+(q^{(1)},t)]}dsdt\right)e^{i[(p^{(1)},x)+(q^{(1)},y)]} \\ &= \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu}\sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} v_1(p^{(1)},q^{(1)})e^{-i(p^{(1)},s-x)-i(q^{(1)},t-y)}f(s,t)dsdt \\ &= \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} k_2(x,y;s,t)f(s,t)dsdt. \end{aligned}$$

For the kernel $k_2(x,y;s,t)$ of the integral operator K_2 we obtain:

$$\begin{aligned} k_2(x,y;s,t) &= \sum_{p^{(1)},q^{(1)}\in\mathbb{Z}^\nu} v_1(p^{(1)},q^{(1)})e^{-i(p^{(1)},s-x)-i(q^{(1)},t-y)} \\ &= \alpha_0 + \sum_{q=1}^{\infty}\beta_q e^{-i(\pm qe_j,t-y)} + \sum_{p=1}^{\infty}\alpha_p e^{-i(\pm pe_j,s-x)} \\ &= \alpha_0 + I_1(y,t) + I_2(x,s), \end{aligned}$$

where

$$\begin{aligned} I_1(y,t) &= \sum_{q=1}^{\infty}\beta_q e^{-i(\pm qe_j,t-y)} = 2\sum_{q=1}^{\infty}\beta_q\sum_{i=1}^{\nu}\cos qy_i\cos qt_i + 2\sum_{q=1}^{\infty}\beta_q\sum_{i=1}^{\nu}\sin qy_i\sin qt_i, \\ I_2(x,s) &= \sum_{p=1}^{\infty}\alpha_p e^{-i(\pm pe_j,s-x)} = 2\sum_{p=1}^{\infty}\alpha_p\sum_{i=1}^{\nu}\cos px_i\cos ps_i + 2\sum_{p=1}^{\infty}\alpha_p\sum_{i=1}^{\nu}\sin px_i\sin ps_i. \end{aligned}$$

We therefore have:

$$k_2(x,y;s,t) = \lambda_0\varphi_0(x,y) + \sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(1)}(x_i)\varphi_p^{(1)}(s_i) + \sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(2)}(x_i)\varphi_p^{(2)}(s_i)$$
$$+ \sum_{q=1}^{\infty}\lambda_q^{(2)}\sum_{i=1}^{\nu}\varphi_q^{(1)}(y_i)\varphi_q^{(1)}(t_i) + \sum_{q=1}^{\infty}\lambda_q^{(2)}\sum_{i=1}^{\nu}\varphi_q^{(2)}(y_i)\varphi_q^{(2)}(t_i).$$

Thus, the Fourier transform $\widetilde{H}_2(\varepsilon)$ of the operator $Q(\varepsilon) : l_2(\mathbb{Z}^\nu \times \mathbb{Z}^\nu) \to l_2(\mathbb{Z}^\nu \times \mathbb{Z}^\nu)$ acts in $L_2(\mathbb{T}^\nu \times \mathbb{T}^\nu)$ by formula (3.2). The proof is complete. □

4. Discrete spectrum of Hamiltonian $\widetilde{H}_2(\varepsilon)$

According Lemma 3.1, the discrete Schrödinger operator $\widetilde{H}_2(\varepsilon)$ (3.2) is an operator in the Friedrichs model with a non-degenerate kernel. We have $\sigma_{ess}(\widetilde{H}_2(\varepsilon)) = [0, 16a_1a_2\nu^2]$.

Theorem 4.1. *Let $\nu = 1, 2$. For all $\varepsilon > 0$ the two-particle Hamiltonian $\widetilde{H}_2(\varepsilon)$ (3.2) possesses infinitely many eigenvalues.*

Proof. Let β be an arbitrary positive number, for which $\beta \geqslant 8\nu a_1 a_2$. In the space $L_2(\mathbb{T}^\nu \times \mathbb{T}^\nu)$ we define the operator $\widetilde{H}_1(\varepsilon)$ in the Friedrichs model as follows:

$$\widetilde{H}_1(\varepsilon) = H_0^{(1)} - \varepsilon K_1. \tag{4.1}$$

Here

$$H_0^{(1)}f(x,y) = \beta\sum_{k=1}^{\nu}(1-\cos y_k)f(x,y), \quad K_1 f(x,y) = \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} k_1(x;s)f(s,t)dsdt,$$

where

$$k_1(x;s) = \lambda_0\varphi_0(x) + \sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(1)}(x_i)\varphi_p^{(1)}(s_i) + \sum_{p=1}^{\infty}\lambda_p^{(1)}\sum_{i=1}^{\nu}\varphi_p^{(2)}(x_i)\varphi_p^{(2)}(s_i).$$

Let $\nu = 1, 2$. It is obvious that $E_{min}(\widetilde{H}_1(\varepsilon)) = E_{min}(\widetilde{H}_2(\varepsilon)) = 0$. According to Lemma 2.3 and by Proposition 2.1, the operator $\widetilde{H}_1(\varepsilon)$ (4.1) has infinitely many negative eigenvalues since

$$\lim_{\xi\to 0-0}\int_{\mathbb{T}^\nu}\frac{dy}{\beta\sum\limits_{k=1}^{\nu}(1-\cos y_k)-\xi} = +\infty$$

and $\dim \operatorname{Ran} K_1 = \infty$. On the other hand, $\widetilde{H}_2(\varepsilon) \leqslant \widetilde{H}_1(\varepsilon)$. By Lemma 2.1 this implies the statement of the theorem. □

For each $\xi < 0$ in the Hilbert space $L_2(\mathbb{T}^\nu \times \mathbb{T}^\nu)$ we define an integral operator $W(\xi)$:

$$W(\xi)f(x,y) = \int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} k_\xi(x,y;s,t)f(s,t)dsdt, \quad \text{where} \quad k_\xi(x,y;s,t) = \frac{k_2(x,y;s,t)}{u_0^{(2)}(s,t)-\xi}.$$

We consider the equation for the eigenvalues $\xi < 0$:

$$u_0^{(2)}(x,y)f(x,y) - \varepsilon\int_{\mathbb{T}^\nu}\int_{\mathbb{T}^\nu} k_2(x,y;s,t)f(s,t)dsdt = \xi f(s,t), \quad f(s,t) \neq 0.$$

We define a function $g(x,y) = \frac{f(x,y)}{u_0^{(2)}(x,y)-\xi} \in L_2(\Omega_\nu^2)$. We obtain

$$\varepsilon W(\xi)g(x,y) = g(x,y),$$

that is, the number $\lambda = 1$ is an eigenvalue of the operator $\varepsilon W(\xi)$.

We define a sequence of continuous functions on $(\mathbb{T}^\nu)^{4n}$, $n \in \mathbb{N}$

$$F_n\left(\xi \left| \begin{matrix} x_1, x_2 \dots, x_n, \; y_1, y_2 \dots, y_n \\ s_1, s_2 \dots, s_n, \; t_1, t_2 \dots, t_n \end{matrix} \right.\right) = \begin{vmatrix} k_\xi(x_1, y_1; s_1, t_1) & \dots & k_\xi(x_1, y_1; s_n, t_n) \\ k_\xi(x_2, y_2; s_1, t_1) & \dots & k_\xi(x_2, y_2; s_n, t_n) \\ \dots & \dots & \dots \\ k_\xi(x_n, y_n; s_1, t_1) & \dots & k_\xi(x_n, y_n; s_n, t_n) \end{vmatrix} \tag{4.2}$$

and we let

$$d_n(\xi) = \int\limits_{(\mathbb{T}^\nu)^n} \int\limits_{(\mathbb{T}^\nu)^n} F_n\left(\xi \left| \begin{matrix} s_1, s_2 \dots, s_n, \; t_1, t_2 \dots, t_n \\ s_1, s_2 \dots, s_n, \; t_1, t_2 \dots, t_n \end{matrix} \right.\right) ds_1 ds_2 \dots ds_n dt_1 dt_2 \dots dt_n.$$

The following expression

$$\Delta(\varepsilon; \xi) = 1 + \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} d_n(\xi), \quad \xi \in \mathbb{C} \backslash \sigma_{ess}(\widetilde{H}_2(\varepsilon)) \tag{4.3}$$

is the Fredholm determinant for the operator $I - \varepsilon W(\xi)$, where I is the identity mapping.

Lemma 4.1 ([23])**.** *The number $\xi \in \mathbb{C} \backslash \sigma_{ess}(\widetilde{H}_2(\varepsilon))$ is an eigenvalues of the operator $\widetilde{H}_2(\varepsilon)$ if and only if $\Delta(\varepsilon; \xi) = 0$.*

For $\xi \in (-\infty, 0)$ we define the following functions:

$$\Phi_2(\nu; \xi) = \int\limits_{\mathbb{T}^\nu} \int\limits_{\mathbb{T}^\nu} \frac{dx dy}{4a_1 a_2 \sum\limits_{k=1}^{\nu} (1 - \cos x_k) \sum\limits_{k=1}^{\nu} (1 - \cos y_k) - \xi}, \quad \text{where} \quad \mathbb{T} = [-\pi, \pi].$$

Let $\nu \geqslant 3$. Then by employing Lemma 2.3 one can prove that $\lim\limits_{\xi \to 0-0} \Phi_2(\nu; \xi) < \infty$.

We let:

$$M_\nu = \frac{2\nu}{2^{2\nu-1} \pi^{2\nu}} \left(\sum_{p=1}^{\infty} \lambda_p^{(1)} + \sum_{q=1}^{\infty} \lambda_q^{(2)} \right), \quad A_\nu = \lim_{\xi \to 0-0} \Phi_2(\nu; \xi).$$

Then we have $\Phi_2(\nu; \xi) \leqslant A_\nu$ for all $\xi < 0$.

Theorem 4.2. *Let $\nu \geqslant 3$. If $\varepsilon < \frac{1}{2A_\nu M_\nu}$, then the operator $\widetilde{H}_2(\varepsilon)$ (3.2) has no discrete spectrum.*

Proof. Let $\xi < 0$. We have

$$\Delta(\varepsilon; \xi) = 1 + \widetilde{\Delta}(\varepsilon; \xi), \quad \text{where} \quad \widetilde{\Delta}(\varepsilon; \xi) = \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} d_n(\xi).$$

For the kernel $k_2(x, y; s, t)$ of the operator K_2 the inequality

$$k_2(x, y; s, t) \leqslant M_\nu, \quad \forall x, y, s, t \in \mathbb{T}^\nu \tag{4.4}$$

holds. Employing Hadamard inequality and (4.4), we obtain

$$\left| F_n\left(\xi \left| \begin{matrix} x_1, x_2 \dots, x_n, \; y_1, y_2 \dots, y_n \\ s_1, s_2 \dots, s_n, \; t_1, t_2 \dots, t_n \end{matrix} \right.\right) \right| \leqslant \frac{\sigma_1 \sigma_2 \dots \sigma_n}{\prod\limits_{k=1}^{n} (u_0^{(2)}(s_k, t_k) - \xi)} \leqslant \frac{(M_\nu \sqrt{n})^n}{\prod\limits_{k=1}^{n} (u_0^{(2)}(s_k, t_k) - \xi)},$$

where

$$\sigma_i = \sqrt{k_2^2(x_i, y_i; s_1, t_1) + k_2^2(x_i, y_i; s_2, t_2) + \dots + k_2^2(x_i, y_i; s_n, t_n)}, \qquad i = 1, 2, \dots, n.$$

Therefore,

$$|d_n(\xi)| \leqslant \int\limits_{(\mathbb{T}^\nu)^n} \int\limits_{(\mathbb{T}^\nu)^n} \frac{(M_\nu \sqrt{n})^n}{\prod\limits_{k=1}^{n} (u_0^{(2)}(s_k,t_k) - \xi)} ds_1 ds_2 \ldots ds_n dt_1 dt_2 \ldots dt_n$$
$$= (M_\nu\sqrt{n})^n \int\limits_{\mathbb{T}^\nu}\int\limits_{\mathbb{T}^\nu} \frac{ds_1 dt_1}{u_0^{(2)}(s_1,t_1)-\xi} \int\limits_{\mathbb{T}^\nu}\int\limits_{\mathbb{T}^\nu} \frac{ds_2 dt_2}{u_0^{(2)}(s_2,t_2)-\xi} \cdots \int\limits_{\mathbb{T}^\nu}\int\limits_{\mathbb{T}^\nu} \frac{ds_n dt_n}{u_0^{(2)}(s_n,t_n)-\xi}$$
$$\leqslant (M_\nu \sqrt{n} A_\nu)^n.$$

For a scalar sequence $c_n = \frac{(\sqrt{n})^n}{n!}$ the inequality $1 \geqslant c_n \geqslant c_{n+1}$, $n \in \mathbb{N}$, holds. Employing this inequality, we find:

$$\sum_{n=1}^{\infty} \frac{(\varepsilon A_\nu M_\nu \sqrt{n})^n}{n!} \leqslant \sum_{n=1}^{\infty} (\varepsilon A_\nu M_\nu)^n.$$

Let $\varepsilon < \frac{1}{2A_\nu M_\nu}$, then the series $\sum\limits_{n=1}^{\infty} (\varepsilon A_\nu M_\nu)^n$ is the sum of a decaying geometric progression. Therefore, we obtain

$$\sum_{n=1}^{\infty} (\varepsilon A_\nu M_\nu)^n = \frac{\varepsilon A_\nu M_\nu}{1-\varepsilon A_\nu M_\nu} < 1.$$

Thus,

$$\left|\widetilde{\Delta}(\varepsilon;\xi)\right| < 1, \qquad \forall \xi \in (-\infty, 0).$$

This implies that if $\varepsilon < \frac{1}{2A_\nu M_\nu}$, then $\Delta(\varepsilon;\xi) \neq 0$ for all $\xi < 0$. Then according Lemma 4.1, the operator $\widetilde{H}_2(\varepsilon)$ has no negative eigenvalues. The proof is complete. □

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