ISSN 2304-0122 Ufa Mathematical Journal. Vol. 14. No 1 (2022). P. 87-94.

doi:10.13108/2022-14-1-87

NON SELF-ADJOINT WELL-DEFINED RESTRICTIONS AND EXTENSIONS WITH REAL SPECTRUM

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Abstract. In this paper we study the spectral properties of relatively bounded well-defined perturbations of the well-defined restrictions and extensions. The work is devoted to the study of the similarity of a well-defined restriction to some self-adjoint operator in the case when the minimal operator is symmetric. We show that the system of eigenvectors forms a Riesz basis in the case of discrete spectrum. The resulting theorem is applied to the Sturm-Liouville operator and the Laplace operator.

Singular perturbations for differential operators have been studied by many authors for the mathematical substantiation of solvable models of quantum mechanics, atomic physics, and solid state physics. For the Sturm-Liouville operator with a potential from the Sobolev space $W_2^\alpha[0,1]$ with $-1 \leqslant \alpha \leqslant 0$, the Riesz basis property of the system of eigenvectors in the Hilbert space $L_2(0,1)$ was proved. In all those cases, the problems were self-adjoint. In this paper, we consider non-self-adjoint singular perturbation problems for the Sturm-Liouville operator with a potential from the Sobolev space $W_2^\alpha[0,1]$ with $-2 \leqslant \alpha \leqslant 0$. We also obtained a similar result for the Laplace operator. A new method has been developed that allows investigating the considered problems. It is shown that the spectrum of a non-self-adjoint singularly perturbed operator is real and the corresponding system of eigenvectors forms a Riesz basis in the considered Hilbert space.

Keywords: maximal (minimal) operator, correct restriction, correct extension, real spectrum, non self-adjoint operator, perturbation.

Mathematics Subject Classification: 47A55, 35P05, 34L05

1. Introduction

In a Hilbert space H, we consider a linear operator L with a domain $D(L)$ and a range $R(L)$. By the *kernel* of the operator L we mean the set

$$\operatorname{Ker} L = \big\{f \in D(L) : \ Lf = 0\big\}.$$

The linear equation

$$Lu = f \tag{1.1}$$

is said to be *well-solvable* on $R(L)$ if $\|u\| \leqslant C\|Lu\|$ for all $u \in D(L)$ (where $C > 0$ is independent on u) and *everywhere solvable* if $R(L) = H$. If (1.1) is simultaneously well and everywhere solvable, then we say that L is a *properly defined operator*. A well-solvable operator L_0 is said to be *minimal* if $\overline{R(L_0)} \neq H$. A closed operator $\widehat{L}$ is called a *maximal operator* if $R(\widehat{L}) = H$ and $\operatorname{Ker}\widehat{L} \neq \{0\}$. An operator A is called a *restriction* of an operator B and B is said to be an *extension* of A if $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$.

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Submitted February 9, 2021.

Note that if a well-defined restriction L of a maximal operator $\widehat{L}$ is known, then the inverses of all well-defined restrictions of $\widehat{L}$ are of the form [1]

$$L_K^{-1}f = L^{-1}f + Kf, \tag{1.2}$$

where K is an arbitrary bounded linear operator from H into $\operatorname{Ker}\widehat{L}$.

Let L_0 be some minimal operator, and let M_0 be another minimal operator related to L_0 by the equation $(L_0u, v) = (u, M_0v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\widehat{L} = M_0^*$ and $\widehat{M} = L_0^*$ are maximal operators such that $L_0 \subset \widehat{L}$ and $M_0 \subset \widehat{M}$. A well-defined restriction L of a maximal operator $\widehat{L}$ such that L is simultaneously a well-defined extension of the minimal operator L_0 is called a *boundary well-defined extension*. The existence of at least one boundary well-defined extension L was proved by Vishik in [2], that is, $L_0 \subset L \subset \widehat{L}$.

The inverse operators to all possible well-defined restrictions L_K of the maximal operator $\widehat{L}$ have the form (1.2). Hence, $D(L_K)$ is dense in H if and only if $\operatorname{Ker}(I + K^*L^*) = \{0\}$. All possible well-defined extensions M_K of M_0 have inverses of the form

$$M_K^{-1}f = (L_K^*)^{-1}f = (L^*)^{-1}f + K^*f,$$

where K is an arbitrary bounded linear operator in H with $R(K) \subset \operatorname{Ker}\widehat{L}$ such that

$$\operatorname{Ker}(I + K^*L^*) = \{0\}.$$

Lemma 1.1 (Hamburger [3]). *Let A be a bounded linear transformation in H and N a linear manifold. If we write $A(N) = M$ then*

$$A^*(M^\perp) = N^\perp \cap R(A^*).$$

Proposition 1.1 ([4]). *Well-defined restrictions L_K of the maximal operator $\widehat{L}$ are well-defined extensions of the minimal operator L_0 if and only if*

$$R(K) \subset \operatorname{Ker}\widehat{L} \quad \textit{and} \quad R(M_0) \subset \operatorname{Ker}K^*.$$

The main result of this work is as follows.

Theorem 1.1. *Let L_0 be symmetric minimal operator in a Hilbert space H, L be a self-adjoint well-defined extension of L_0, and L_K be a well-defined restriction of the maximal operator $\widehat{L}(\widehat{L} = L_0^*)$. If*

$$R(K^*) \subset D(L), \quad I + KL \geqslant 0,$$

and $I + KL$ is invertible, where L and K are the operators in representation (1.2), *then L_K is similar to a self-adjoint operator.*

Corollary 1.1. *If K satisfies the assumptions of Theorem 1.1, then the spectrum of L_K is real, that is, $\sigma(L_K) \subset \mathbb{R}$.*

Corollary 1.2. *If K satisfies the assumptions of Theorem 1.1 and L^{-1} is the compact operator, then the system of the eigenvectors of L_K forms a Riesz basis in H.*

Corollary 1.3. *The results of Theorem 1.1 are also valid if conditions "$I + KL \geqslant 0$ and $I + KL$ is invertible" are replaced with the condition "$KL \geqslant 0$".*

Corollary 1.4. *The results of Theorem 1.1, Corollary 1.1-1.3 are also valid for the L_K^*.*

2. Preliminaries

In this section, we present some results on well-defined restrictions and extensions which are used in Section 3.

If A is a bounded linear transformation from a complex Hilbert space H into itself, then the numerical range of A is by definition the set

$$W(A) = \{(Ax, x) : x \in H, \ \|x\| = 1\}.$$

It is well known and easy to prove that if $\sigma(A)$ denotes the spectrum of A, then

$$\sigma_p(A) \subset W(A), \quad \sigma(A) \subset \overline{W(A)},$$

for the point spectrum $\sigma_p(A)$ and the spectrum $\sigma(A)$ of A, where the bar indicates the closure. The numerical range of an unbounded operator A in a Hilbert space H is defined as

$$W(A) = \{(Ax, x) : x \in D(A), \ \|x\| = 1\},$$

and similarly to the bounded case, $W(A)$ is convex and satisfies $\sigma_p(A) \subset W(A)$. In general, the conclusion $\sigma(A) \subset \overline{W(A)}$ does not surely hold for unbounded operators A (see [5]).

Theorem 2.1 (Theorem 2 in [6]). *The following are equivalent conditions on an operator T:*
(1) *T is similar to a self-adjoint operator.*
(2) *$T = PA$, where P is positive and invertible and A is self-adjoint.*
(3) *$S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$.*

Theorem 2.2 (Theorem 1 in [7]). *Let A and B be operators on the complex Hilbert space H. If $0 \notin \overline{W(A)}$, then*

$$\sigma(A^{-1}B) \subset \overline{W(B)}/\overline{W(A)}.$$

Corollary 2.1 (Corollary in [7]). *If $A > 0$, $B \geqslant 0$ and $C = C^*$, then $\sigma(AB)$ is positive and $\sigma(AC)$ is real.*

Theorem 2.3 (Theorem A in [8]). *The numerical range $W(T)$ of T is convex and*

$$W(aT + b) = aW(T) + b$$

for all complex numbers a and b.

3. Proof of Theorem 1.1

We transform (1.2) to the form

$$L_K^{-1} = L^{-1} + K = (I + KL)L^{-1}. \tag{3.1}$$

Then L_K is defined as the restriction of the maximal operator $\widehat{L}$ on the domain

$$D(L_K) = \{u \in D(\widehat{L}) : (I - K\widehat{L})u \in D(L)\}.$$

Now let us prove Theorem 1.1. It was proved in [9] that KL is bounded on $D(L)$ (that is, $\overline{KL} \in B(H)$) if and only if

$$R(K^*) \subset D(L^*).$$

It follows from $\overline{D(L)} = H$ that $\overline{KL}$ is bounded on H. In what follows, instead of $\overline{KL}$, we shall write KL. Then, by virtue of Theorem 2.1, and taking into account the conditions of Theorem 1.1 that $I + KL \geqslant 0$ and $I + KL$ is invertible, we arrive at the statement of Theorem 1.1.

The proof of Corollary 1.1 follows from Theorem 1.1 or Corollary 2.1. Corollary 1.2 can be easily obtained from the fact that the operator

$$C = (I + KL)^{\frac{1}{2}} L^{-1} (I + KL)^{\frac{1}{2}}$$

is self-adjoint and

$$L_K^{-1} = (I + KL)^{\frac{1}{2}} C (I + KL)^{-\frac{1}{2}} = (I + KL) L^{-1}. \tag{3.2}$$

Let us proof Corollary 1.3. By Theorem 2.3, we get that $0 \notin W(I + KL)$. Then $I + KL \geqslant 0$ and $I + KL$ is invertible.

The statement of Corollary 1.4 follows from (3.2), since C is a self-adjoint operator and in the case Corollary 1.2 the self-adjoint operator C is compact.

4. Non self-adjoint perturbations for some differential operators

Example 1. We consider the Sturm-Liouville equation

$$\widehat{L} y = -y'' + q(x) y = f \tag{4.1}$$

on the interval $(0, 1)$, where $q(x)$ is the real-valued function of $L^2(0, 1)$. We denote by L_0 the minimal operator and by $\widehat{L}$ the maximal operator generated by the differential equation (4.1) in the space $L_2(0, 1)$. It is clear that

$$D(L_0) = \mathring{W}_2^2(0, 1)$$

and

$$D(\widehat{L}) = \left\{ y \in L^2(0, 1) : \ y, \ y' \in AC[0, 1], \ y'' - q(x) y \in L^2(0, 1) \right\}.$$

Then $\operatorname{Ker} \widehat{L} = \{a_{11} c(x) + a_{12} s(x)\}$, where a_{11}, a_{12} are arbitrary constants, and the functions $c(x)$ and $s(x)$ are defined as follows

$$c(x) = 1 + \int_0^x \mathscr{K}(x, t; 0)\, dt, \quad s(x) = x + \int_0^x \mathscr{K}(x, t; \infty) t\, dt,$$

where

$$\mathscr{K}(x, t; 0) = \mathscr{K}(x, t) + \mathscr{K}(x, -t), \quad \mathscr{K}(x, t; \infty) = \mathscr{K}(x, t) - \mathscr{K}(x, -t),$$

and $\mathscr{K}(x, t)$ is the solution of the following Goursat problem

$$\begin{cases} \dfrac{\partial^2 \mathscr{K}(x, t)}{\partial x^2} - \dfrac{\partial^2 \mathscr{K}(x, t)}{\partial t^2} = q(x) \mathscr{K}(x, t), \\ \mathscr{K}(x, -x) = 0, \quad \mathscr{K}(x, x) = \dfrac{1}{2} \displaystyle\int_0^x q(t) dt, \end{cases}$$

in the domain

$$\Omega = \left\{ (x, t) : \ 0 < x < 1, \ -x < t < x \right\}.$$

Note that $c(0) = s'(0) = 1, \ c'(0) = s(0) = 0$ and Wronskian satisfies the identity

$$W(c, s) \equiv c(x) s'(x) - c'(x) s(x) = 1.$$

As a fixed boundary well-defined extension L, we take the operator corresponding to the Dirichlet problem for equation (4.1) on $(0, 1)$. Then

$$D(L) = \left\{ y \in W_2^2(0, 1) : \ y(0) = 0, \ y(1) = 0 \right\}.$$

Hence, the inverse of all correct restrictions L_K of the maximal operator $\widehat{L}$ is of the form

$$\begin{aligned} y \equiv L_K^{-1} f = &\int_0^x \big[c(x)s(t) - s(x)c(t)\big] f(t)\,dt \\ &- \frac{s(x)}{s(1)} \int_0^1 \big[c(1)s(t) - s(1)c(t)\big] f(t)\,dt \\ &+ c(x)\int_0^1 f(t)\overline{\sigma_1(t)}dt + s(x)\int_0^1 f(t)\overline{\sigma_2(t)}\,dt, \end{aligned}$$

where $\sigma_1(x)$, $\sigma_2(x) \in L_2(0,1)$ and this determines uniquely the operator K in (1.2) as follows

$$Kf = c(x)\int_0^1 f(t)\overline{\sigma_1(t)}dt + s(x)\int_0^1 f(t)\overline{\sigma_2(t)}dt, \quad \text{for all} \quad f \in L_2(0,1).$$

K is a bounded operator in $L_2(0,1)$ acting from $L_2(0,1)$ into $\operatorname{Ker}\widehat{L}$. The operator$L_K$ is the restriction of $\widehat{L}$ on the domain

$$\begin{aligned} D(L_K) = \Big\{ y \in W_2^2(0,1) : y(0) = &\int_0^1 \big(-y''(t) + q(t)y(t)\big)\overline{\sigma_1(t)}dt; \\ y(1) = c(1)y(0) + s(1) &\int_0^1 \big(-y''(t) + q(t)y(t)\big)\overline{\sigma_2(t)}dt \Big\}. \end{aligned}$$

By the condition

$$R(K^*) \subset D(L^*) = D(L)$$

we have

$$KLy = c(x)\int_0^1 y(t)[-\overline{\sigma}_1''(t) + q(t)\overline{\sigma}_1(t)]dt + s(x)\int_0^1 y(t)[-\overline{\sigma}_2''(t) + q(t)\overline{\sigma}_2(t)]dt,$$

where

$$y \in D(L), \ \ \sigma_1, \sigma_2 \in W_2^2(0,1), \ \ \sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0.$$

If $I + KL \geqslant 0$ and $I + KL$ is invertible, then the spectrum of the operator L_K consists only of real eigenvalues $\{\lambda_k\}_{k=1}^\infty$ and the corresponding eigenfunctions $\{\varphi_k\}_{k=1}^\infty$ forms a Riesz basis in $L^2(0,1)$, since L^{-1} is a compact self-adjoint positive operator. In particular, if

$$\sigma_1(x) = \alpha(L^{-1}c)(x), \ \ \sigma_2(x) = \beta(L^{-1}s)(x), \ \ \alpha,\ \beta \geqslant 0,$$

then $KL \geqslant 0$. Therefore, by Corollary 1.3, the results of Theorem 1.1 are valid for L_K. In this case, L_K^{-1} has the form

$$y = L_K^{-1}f = L^{-1}f + c(x)\int_0^1 f(t)(L^{-1}c)(t)dt + s(x)\int_0^1 f(t)(L^{-1}s)(t)dt.$$

Then $(L_K^{-1})^* = (L_K^*)^{-1}$ has form

$$v(x) = (L^{-1}f)(x) + \alpha(L^{-1}c)(x)\int_0^1 f(t)c(t)dt + \beta(L^{-1}s)(x)\int_0^1 f(t)s(t)dt.$$

Thus, we have

$$(L_K^* v)(x) = -v''(x) + q(x)v(x) + a(x)v'(0) + b(x)v'(1) = f(x),$$

$$D(L_K^*) = \{v \in W_2^2(0,1) :\ v(0) = v(1) = 0\},$$

where
$$a(x) = \frac{\alpha\beta(c,s)s(x) - \alpha(1+\beta\|s\|^2)c(x)}{(1+\alpha\|c\|^2)(1+\beta\|s\|^2) - \alpha\beta|(c,s)|^2},$$
$$b(x) = \frac{\alpha[c(1)(1+\beta\|s\|^2) - \beta s(1)(s,c)]c(x) - \beta[\alpha c(1)(c,s) - s(1)(1+\alpha\|c\|^2)]s(x)}{(1+\alpha\|c\|^2)(1+\beta\|s\|^2) - \alpha\beta|(c,s)|^2},$$

$a(x)$, $b(x) \in \operatorname{Ker} \widehat{L}$ and $(\cdot,\cdot)$ is scalar product in $L^2(0,1)$. The operator L_K^* acts as
$$L_K^* = L^* + Q,$$
where
$$L^* = -\frac{d^2}{dx^2} + q(x),$$
$$(Qv)(x) = a(x) < \delta'(x), v(x) > +b(x) < \delta'(x-1), v(x) >= a(x)v'(0) + b(x)v'(1),$$

that is, the function $Q \in W_2^{-2}(0,1)$. Thus, we have constructed an example of a non self-adjoint singularly perturbed Sturm-Liouville operator with a real spectrum and the system of eigenvectors that forms a Riesz basis in $L^2(0,1)$.

Example 2. In the Hilbert space $L^2(\Omega)$, where is a bounded domain in $\mathbb{R}^m$ with an infinitely smooth boundary $\partial\Omega$, let us consider the minimal L_0 and maximal $\widehat{L}$ operators generated by the Laplace operator
$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}\right). \tag{4.2}$$
The closure L_0, in the space $L^2(\Omega)$ of Laplace operator (4.2) with the domain $C_0^\infty(\Omega)$, is *the minimal operator corresponding to the Laplace operator*. The operator $\widehat{L}$, adjoint to the minimal operator L_0 corresponding to Laplace operator, is *the maximal operator corresponding to the Laplace operator*. Then
$$D(\widehat{L}) = \{u \in L^2(\Omega) : \ \widehat{L}u = -\Delta u \in L^2(\Omega)\}.$$
Denote by L the operator, corresponding to the Dirichlet problem with the domain
$$D(L) = \{u \in W_2^2(\Omega) : \ u|_{\partial\Omega} = 0\}.$$
We have (1.2), where K is an arbitrary linear operator bounded in $L^2(\Omega)$ with
$$R(K) \subset \operatorname{Ker} \widehat{L} = \{u \in L^2(\Omega) : \ -\Delta u = 0\}.$$
Then the operator L_K is defined by
$$\widehat{L}u = -\Delta u,$$
on
$$D(L_K) = \{u \in D(\widehat{L}) : \ [(I - K\widehat{L})u]|_{\partial\Omega} = 0\},$$
where I is the identity operator in $L^2(\Omega)$. Note that L^{-1} is a self-adjoint compact operator. If K satisfies the conditions of Theorem 1.1, then L_K is non self-adjoint operator with a real positive spectrum (i.e., $\sigma(L_K) \subset \mathbb{R}_+$), and the system of eigenvectors L_K forms a Riesz basis in $L^2(\Omega)$. In particular, if
$$Kf = \varphi(x) \int_\Omega f(t)\psi(t)dt,$$
where $\varphi \in W_{2,loc}^2(\Omega) \cap L^2(\Omega)$ is a harmonic function and $\psi \in L^2(\Omega)$, then $K \in B(L^2(\Omega))$ and $R(K) \subset \operatorname{Ker} \widehat{L}$. From $R(K^*) \subset D(L)$ it follows that $\psi \in W_2^2(\Omega)$ and $\psi|_{\partial\Omega} = 0$. From the

condition $KL \geqslant 0$ we have that $\psi(x) = \alpha(L^{-1}\varphi)(x), \quad \alpha \in \mathbb{R}_+$. Hence the operator L_K is the restriction of $\widehat{L}$ to the domain

$$D(L_K) = \Big\{u \in D(\widehat{L}) : \Big(u - \frac{\varphi}{1+\|\varphi\|^2}\int\limits_\Omega u(y)\varphi(y)dy\Big)\Big|_{\partial\Omega} = 0\Big\}.$$

The inverse of L_K^{-1} has the form

$$u = L_K^{-1}f = L^{-1}f + \varphi\int\limits_\Omega f(y)(L^{-1}\varphi)(y)dy. \tag{4.3}$$

We find the adjoint operator L_K^*. From (4.3) we have

$$v = (L_K^{-1})^*g = L^{-1}g + L^{-1}\varphi\int\limits_\Omega g(y)\varphi(y)dy, \quad \text{for all } g \in L^2(\Omega).$$

Then

$$L_K^*v = -\Delta v + \frac{\varphi}{1+\|\varphi\|^2}\int\limits_\Omega (\Delta v)(y)\varphi(y)dy = g,$$

$$D(L_K^*) = D(L) = \big\{v \in W_2^2(\Omega) : v\big|_{\partial\Omega} = 0\big\}.$$

By virtue of Corollary 1.4, the spectrum of the operator L_K^* consists only of real positive eigenvalues and the corresponding eigenfunctions forms a Riesz basis in $L^2(\Omega)$. Note that

$$(L_K^*v)(x) = -(\Delta v)(x) + \frac{\varphi(x)}{1+\|\varphi\|^2}F(u) = g(x),$$

where $F \in W_2^{-2}(\Omega)$, since

$$F(u) = \int\limits_\Omega (\Delta v)(y)\varphi(y)dy.$$

This is understood in the sense of the definition of the space $H^{-s}(\Omega)$, $s > 0$ as in Theorem 12.1 (see [10]).

Thus, we have provided the examples of a non self-adjoint singularly perturbed operator with a real spectrum. Moreover, the corresponding eigenvectors forms a Riesz basis in $L^2(\Omega)$.

BIBLIOGRAPHY

1. B.K. Kokebaev, M. Otelbaev, A.N. Shynibekov. *About expansions and restrictions of operators in Banach space* // Uspekhi Matem. Nauk. **37**:4, 116–123 (1982) (in Russian).
2. M.I. Vishik. *On general boundary problems for elliptic differential equations* // Tr. Mosk. Matem. Obsch. 1 (1952), 187–246. [Am. Math. Soc., Transl. II. 24, 107–172 (1963).]
3. H.L. Hamburger, *Five notes on a generalization of quasi-nilpotent transformations in Hilbert space* // Proc. London Math. Soc. **3**:1, 494–512 (1951).
4. B.N. Biyarov. *Spectral properties of correct restrictions and extensions of the Sturm-Liouville operator* // Diff. Uravn. **30**:12, 2027–2032 (1994). [Diff. Equat. **30**:12, 1863–1868 (1994).]
5. D.Y. Wu and A. Chen. *Spectral inclusion properties of the numerical range in a space with an indefinite metric* // Linear Alg. Appl. **435**:5, 1131–1136 (2011).
6. H. Radjavi and J.P. Williams. *Products of self-adjoint operators* // Michigan Math. J. **16**:2, 177–185 (1969).
7. J.P. Williams. *Spectra of products and numerical ranges* // J. Math. Anal. Appl. **17**:2, 214–220 (1967).

8. M. Radjabalipour and H. Radjavi. *On the geometry of numerical ranges* // Pacific J. Math. **61**:2, 507–511 (1975).
9. B.N. Biyarov, D.A. Svistunov, G.K. Abdrasheva. *Correct singular perturbations of the Laplace operator* // Eurasian Math. J. **11**:4, 25–34 (2020).
10. J.L. Lions, E. Magenes. *Non-Homogeneous boundary value problems and applications I.* Springer-Verlag, Berlin (1972).

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