ISSN 2304-0122 Ufa Mathematical Journal. Vol. 14. No 1 (2022). P. 95-103.

doi:10.13108/2022-14-1-95

COMMUTATIVITY CONDITIONS IN PSEUDO-MICHAEL ALGEBRAS

A. NAZIRI-KORDKANDI

Abstract. We consider the commutativity conditions in unital pseudo-Michael algebras. These kinds of algebras have interesting properties regarding the commutativity criteria. We prove several results, which generalize known results in the case of unital Arens-Michael algebras to the pseudo-convex cases. In this paper, we first derive some specific results for the differentiable and entire functions in pseudo-Michael algebras. Then we show how such results can be applied to obtain commutativity conditions for these algebras. In Section 3, we give simple conditions implying commutativity in the unital pseudo-Michael algebras. These conditions are equivalent to similar cases in unital locally m-convex algebras, in particular, in Banach algebras. The most outstanding results in this direction are due to Toma, who generalized the commutativity criteria of Banach algebras to locally m-convex algebras. Our conditions ensure the commutativity of pseudo-Michael algebras. In the proofs of some theorems, we apply the exponential functions and Liouville theorem for bounded holomorphic functions. The use of them allows us to give a very striking short proof. Finally as a consequence, we show that some commutativity results hold for k-Banach algebras.

Keywords: Pseudo-Michael algebra, k-seminorm, k-Banach algebra, Commutative.

Mathematics Subject Classification: 46H05, 46H20

1. Introduction

Non-normed topological algebras were initially introduced around the year 1950 for studying certain classes of these algebras that appeared naturally in mathematics and physics. Some results concerning such topological algebras had been obtained before 1950. In 1952, Arens and Michael [3, 5] independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras. Here we mention the predictions made by the famous Soviet mathematician M.A. Naimark, an expert in the area of Banach algebras, in 1950 regarding the importance of non-normed algebras and the development of their related theory. During his study concerning cosmology, G. Lassner [3] realized that the theory of normed topological algebras was insufficient for his study purposes.

An important class of topological algebras namely pseudo-Michael algebras (complete Hausdorff locally m-pseudo-convex algebras) have interesting properties regarding the commutativity criterions. Hence, we can extent and prove some commutativity conditions of Banach and Locally m-convex algebras to pseudo-Michael algebras (see [6, 8]). In this paper, we first derive some specific results regarding the differentiable and entire functions in pseudo-Michael algebras. Then the commutativitiy conditions are also investigated in these algebras.

Throughout this paper, all algebras are assumed to be unital and the units are denoted by e.

A. Naziri-Kordkandi, Commutativity conditions in pseudo-michael algebras.

Submitted January 28, 2021.

2. Definitions and known results

In this section, we present a collection of definitions and known results, which are included in the list of our references.

Definition 2.1. [1] *Let A be a Hausdorff topological linear space. Let $G \subseteq \mathbb{C}$ be an open set. A function $f : G \to A$ is called differentiable if for each $\lambda \in G$,*

$$\lim_{h\to 0} \frac{f(\lambda + h) - f(\lambda)}{h}$$

exists in the topology of A. We denote this limit by $f'(A)$ and call f' the derivative of f.

Definition 2.2. *Let A be a Hausdorff topological linear space. A function $f : G \to A$ is called analytic on G if around each point $\lambda_0 \in G \subseteq \mathbb{C}$, there is a neighborhood $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r_0\}$ in which f has a power series representation*

$$f(\lambda) = \sum_{n=0}^{\infty} x_n(\lambda - \lambda_0)^n,$$

where $x_n \in A$ and the series converges in the topology of A; for more information see [1].

Definition 2.3. *Let A be an algebra. The set of all invertible elements of A is denoted by* $\mathrm{Inv}(A)$.

Definition 2.4. *For an algebra A, the spectrum $\mathrm{sp}_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible in A. The spectral radius $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(x)\}$.*

Lemma 2.1. [2, Sect. 1.5.32] *If A is an algebra, then*

$$\mathrm{Rad}(A) = \{x \in A : r_A(xy) = 0 \ \textit{for any}\ y \in A\},$$

where $\mathrm{Rad}(A)$ is the Jacobson radical of A.

Definition 2.5. *By a topological algebra we mean an algebra over $\mathbb{C}$ endowed with a topology that makes the multiplication separately continuous.*

Definition 2.6. [1] *A topological linear space A is said to be ample if A^* (topological dual of A) separates points of A. A topological algebra is called ample if it is ample as a topological linear space.*

Definition 2.7. *Let A be a Hausdorff topological algebra. The exponential function in A is defined by*

$$\exp(a) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad (x^0 = e, \qquad 0! = 1)$$

whenever the series on the right converges.

Definition 2.8. *A topological algebra A is said to be a Q-algebra if and only if $\mathrm{Inv}(A)$ is open.*

Corollary 2.1. [4, Sect. 4.2] *If A is a Q-algebra, then $\mathrm{sp}_A(x)$ is compact for each $x \in A$.*

Definition 2.9. [1] *A k-seminorm on A, with $k \in (0, 1]$, is a function*

$$p : A \to \mathbb{R}^+ \cup \{0\}$$

such that, for each $x, y \in A$,

$$p(x + y) \leqslant p(x) + p(y), \qquad \textit{for each} \quad \lambda \in \mathbb{C}, \quad p(\lambda x) \leqslant |\lambda|^k p(x)$$

If, in addition, the function satisfies

$$p(xy) \leqslant p(x)p(y),$$

then the k-seminorm is called submultiplicative.

A k-seminorm p is also called a pseudo-seminorm and k is called the homogenity index of p. Occasionally, we shall employ the symbol k_p, to indicate the index corresponding to p. A pseudo-seminorm p is a pseudo-norm if $p(x) = 0$ implies $x = 0$.

If p is a k-seminorm (k-norm) on a linear space A, then the resulting topological linear space $A = (A, p)$ is called a k-seminormed (k-normed) linear space. A topological algebra whose topology is induced by a k-seminorm (k-norm) p is called a k-seminormed (k-normed) algebra.

A complete k-normed algebra is called a k-Banach algebra. A pseudo-Banach algebra is just a k-Banach algebra for some k, $0 < k \leqslant 1$.

Definition 2.10. [1] *A locally pseudo-convex space A is a topological linear space equipped with a family $\mathcal{P} = (p_\alpha)_{\alpha \in I}$ of pseudo-seminorms on A which define its topology. If each $p_\alpha \in \mathcal{P}$ is a k-seminorm, then A is called a locally k-convex space.*

A locally pseudo-convex algebra A is a topological algebra such that its underlying topological linear space is locally pseudo-convex. If its underlying topological linear space is locally k-convex, then A is called a locally k-convex algebra. A is called a locally m-pseudo-convex algebra (or locally m-(k-convex) algebra) if p_α is submultiplicative for each $\alpha \in I$.

If $p_1, p_2, \ldots, p_n$ are k_{p_j}-seminorms on A and $k = \min\limits_{1 \leqslant j \leqslant n} \{k_{p_j}\}$, then the function $q = p_1 \vee p_2 \vee \ldots \vee p_k$ defined as

$$q(x) = \max_{1 \leqslant j \leqslant n} \left\{ p_1^{\frac{k}{k_{p_1}}}(x), p_2^{\frac{k}{k_{p_2}}}(x), \ldots, p_n^{\frac{k}{k_{p_n}}}(x) \right\}$$

is also a pseudo-seminorm on A (with homogenity index k). We say that the family of pseudo-seminorm $\mathcal{P} = (p_\alpha)_{\alpha \in I}$ is saturated if $p_{\alpha_1} \vee p_{\alpha_2} \vee \cdots \vee p_{\alpha_n} \in \mathcal{P}$ for each finite family $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\} \subset \mathcal{P}$.

Proposition 2.1. [1, Sect. 4.3.11] *Let $(A, \mathcal{P})$ and $(B, \mathcal{Q})$ be two locally m-pseudo-convex spaces where $\mathcal{P} = (p_\alpha)_{\alpha \in I}$ is a saturated family of k_α-seminorms on A defining its topology, and $\mathcal{Q} = (q_\beta)_{\beta \in \Gamma}$ is a family of k_β-seminorms on B defining its topology. Then a linear transformation $T : A \to B$ is continuous if and only if for each $q_\beta \in \mathcal{Q}$, there exists a $p_\alpha \in \mathcal{P}$ and a constant $C = C_{\alpha,\beta} > 0$ such that*

$$q_\beta(T(x)) \leqslant C \cdot p_\alpha(x)^{\frac{k_\beta}{k_\alpha}} \text{ for each } x \in A. \tag{2.1}$$

Here, k_β and k_α are the homogenity indexes of q_β and p_α, respectively.

In the case where $A = (A, p)$ is a k_p-seminormed space and $B = (B, q)$ is a k_q-seminormed space, relation (2.1) takes the form (see [3])

$$q(T(x)) \leqslant Cp(x)^{\frac{k_q}{k_p}} \quad \text{for each} \quad x \in A.$$

Definition 2.11. [1] *We call a complete Hausdorff locally m-pseudo-convex algebra A as a pseudo-Michael algebra.*

Corollary 2.2. [1, 6.4.9] *Every pseudo-Michael algebra A is spectral, i.e. $\mathrm{sp}_A(x) \neq \emptyset$, for each $x \in A$.*

Definition 2.12. *Let A be a pseudo-Michael algebra. A is said to be commutative if and only if $xy = yx$ for all $x, y \in A$, and A is said to be semicommutative if and only if $xyz = zxy$ for any $x, y, z \in A$; for more information see* [8].

Proposition 2.2. [1, Sect. 5.2.2] *If A is a pseudo-Michael algebra, in particular, a k-Banach algebra, then $D_{\exp} = A$, where $D_{\exp}$ is the domain of* exp. *Moreover, the series in Definiton 2.7, converges absolutely for all $x \in A$.*

3. Main results

In this section, we first present some results regarding the differentiable and entire functions in pseudo-Michael algebras. Then we show how they can be applied for obtaining commutativity conditions for these algebras.

Proposition 3.1. *Let A be an ample Hausdorff topological linear space and $f' \equiv 0$ on G, where $G \subseteq \mathbb{C}$ is an open set. Then f is constant.*

Proof. Define $g : G \to \mathbb{C}$ by

$$g(\lambda) = \varphi \circ f(\lambda), \quad \text{where } \varphi \in A^*.$$

Then we have

$$g'(\lambda) = \varphi(f'(\lambda)) = 0.$$

Since A is ample, we infer that f is constant. □

Lemma 3.1. *Let A be a pseudo-Michael algebra. If $f : \mathbb{C} \to A$ is defined by*

$$f(\lambda) = \exp(\lambda a), \qquad a \in A,$$

then $f'(\lambda) = af(\lambda)$.

Proof. By [1, Sect. 5.1.8], f is differentiable on $\mathbb{C}$. Then the result follows as in the classical case of mathematical analysis (for instance, see [7, Sect. 8.6]). □

Lemma 3.2. *Let A be pseudo-Michael algebra. Then the function $f : \mathbb{C} \to A$ defined by*

$$f : \lambda \longmapsto f(\lambda) = \exp(\lambda x) y \exp(-\lambda x)$$

is entire for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

Proof. We have

$$f(\lambda) = \left(e + \lambda x + \frac{(\lambda x)^2}{2!} + \cdots \right) y \left(e - \lambda x + \frac{(\lambda x)^2}{2!} + \cdots \right).$$

By continuity of multiplication in A. We get

$$f(\lambda) = \left(e + \lambda x + \frac{(\lambda x)^2}{2!} + \cdots \right) \left(y - \lambda yx + y\frac{(\lambda x)^2}{2!} + \cdots \right).$$

By [1, Sect. 5.2.2], the first series is absolutely convergent. The second series is also absolutely convergent by the ratio test. Now, we multiply the two series and get:

$$f(\lambda) = y + \lambda(xy - yx) + \lambda^2 \left(\frac{x^2 y}{2} - xyx + \frac{yx^2}{2} \right) + \cdots .$$

Since this series converges absolutely for all $\lambda \in \mathbb{C}$, the function $f(\lambda)$ is entire. □

Theorem 3.1. *Let A be a pseudo-Michael Q-algebra. Assume that $a, b, c \in A$ satisfy the following conditions:*

$$ab - ba = c, \quad ac = ca \quad \text{and} \quad bc = cb.$$

Then $r_A(c) = 0$.

Proof. Let the function f be as defined in Lemma 3.2. Then we have

$$\begin{aligned} f'(\lambda) &= a\exp(\lambda a)b\exp(-\lambda a) - \exp(\lambda a)ba\exp(-\lambda a) \\ &= \exp(\lambda a)(ab-ba)\exp(-\lambda a) \\ &= \exp(\lambda a)c\exp(-\lambda a) = c \end{aligned}$$

for all $\lambda \in \mathbb{C}$. Also, $f(0) = b$. This implies that

$$f(\lambda) = \exp(\lambda a)b\exp(-\lambda a) = b + \lambda c \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Since $r_A(xy) = r_A(yx)$ by [1, Sect. 1.8.12], we have

$$r_A(b) = r_A(b+\lambda c), \quad \text{for all } \lambda \in \mathbb{C}.$$

As b and c commute, by [1, Sect. 7.2.23], we obtain

$$\begin{aligned} |\lambda| r_A(c) = r_A(\lambda c) &= r_A(\lambda c + b - b) \\ &\leqslant r_A(\lambda c + b) + r_A(b) \\ &= r_A(b) + r_A(b) = 2r_A(b). \end{aligned}$$

Since A is a Q-algebra, $\mathrm{sp}_A(b)$ is comact [4, 4.2]. By Corollary 2.2, it is non-empty and hence $0 < r_A(b) < \infty$. Thus $r_A(c) = 0$, as $\lambda \to \infty$. □

Remark 3.1. *Let A be a pseudo-Michael algebra. Since every complete m-convex algebra is a pseudo-Michael algebra* [1], *the example discussed in* [8] *shows that the semicommutativity of A doesn't imply the commutativity of A in general.*

Theorem 3.2. [1, Sect. 7.4.8] *Let $(A, (p_\alpha)_{\alpha\in I})$ be a pseudo-Michael algebra with projective limit decomposition $A = \varprojlim \hat{A}_\alpha$, where each $\hat{A}_\alpha$ is a pseudo-Banach algebra. Then for $x \in A$, $x = (x_\alpha)$, $(x_\alpha \in \hat{A}_\alpha)$, we have*

$$r_A(x) = \sup_\alpha v_\alpha(x)^{\frac{1}{k_\alpha}}, \quad \textit{where} \quad v_\alpha(x) = \lim_{n\to\infty} p_\alpha(x^n)^{\frac{1}{n}}.$$

Theorem 3.3. *Let $(A, (p_\alpha)_{\alpha\in I}))$ be a pseudo-Michael algebra with projective limit decomposition $A = \varprojlim \hat{A}_\alpha$. If A is semisimple and semicommutative, then it is commutative.*

Proof. From semicommutativity of A, we have

$$(xy)^2 = (xy)(xy) = ((xy)x)y = x(yx)y = xy^2x = x^2y^2$$

for all $x, y \in A$. By induction we get

$$(xy)^n = x^n y^n \quad \text{for any } n \in \mathbb{N} \quad \text{and} \quad x, y \in A.$$

Let $x, y \in A$ and $n \in \mathbb{N}$. Then

$$p_\alpha((xy)^n)^{\frac{1}{nk_\alpha}} = p_\alpha(x^n y^n)^{\frac{1}{nk_\alpha}} \leqslant p_\alpha(x^n)^{\frac{1}{nk_\alpha}} p_\alpha(y^n)^{\frac{1}{nk_\alpha}}.$$

It follows from Theorem 3.2 that

$$r_A(xy) \leqslant r_A(x) r_A(y).$$

Since A is semicommutative, we obtain $(xy-yx)^2 = 0$, for all $x, y \in A$. This implies that

$$r_A(xy-yx) = \sup_{\alpha\in I} \lim_{n\to\infty} p_\alpha((xy-yx)^n)^{\frac{1}{nk_\alpha}} = 0 \quad \text{for all} \quad x, y \in A.$$

On the other hand,

$$r_A((xy-yx)z) \leqslant r_A(xy-yx) r_A(z) = 0 \quad \text{for all} \quad x, y, z \in A.$$

By Lemma 2.1, $xy - yx \in \mathrm{Rad}(A)$. Since A is semisimple, then $xy = yx$ for all $x, y \in A$. □

Theorem 3.4. *Let $(A, (p_\alpha)_{\alpha\in I})$ be an ample pseudo-Michael algebra, where $(p_\alpha)_{\alpha\in I}$ is saturated. If for any $\alpha \in I$ there exists, $M_\alpha > 0$ such that*

$$p_\alpha(xy) \leqslant M_\alpha p_\alpha(yx) \quad \textit{for all} \quad x, y \in A,$$

then A is commutative.

Proof. Let the function f be as defined in Lemma 3.2. For all $\alpha \in I$ and $\lambda \in \mathbb{C}$, we have

$$p_\alpha(f(\lambda)) = p_\alpha(\exp(\lambda x) y \exp(-\lambda x)) \leqslant M_\alpha p_\alpha(y).$$

By Lemma 3.2, the function f is entire. Also it is bounded on $\mathbb{C}$. Let $\varphi \in A^*$. Then $\varphi \circ f(\lambda) = \varphi(f(\lambda))$ is a scalar entire function. This function is also bounded. Indeed, by [1, 4.3.11], there exists a $p_\alpha \in \mathcal{P}$ and a constant $C = C_\alpha > 0$ such that

$$|\varphi \circ f(\lambda)| = |\varphi(f(\lambda))| \leqslant C p_\alpha(f(\lambda))^{\frac{1}{k_\alpha}} \leqslant C(M_\alpha p_\alpha(y))^{\frac{1}{k_\alpha}}.$$

Then by the Liouville theorem $\varphi \circ f$ is constant and hence,

$$\varphi(f(\lambda)) = \varphi(f(0)) = \varphi(y).$$

Since A is ample, $f(\lambda) = y$ for all $\lambda \in \mathbb{C}$. This implies the identity $f'(\lambda) = 0$ for each $\lambda \in \mathbb{C}$ and therefore,

$$x \exp(\lambda x) y \exp(-\lambda x) - \exp(\lambda x) y x \exp(-\lambda x) = 0 \quad \text{for each} \quad \lambda \in \mathbb{C}.$$

For $\lambda = 0$ we get $xy = yx$. □

Theorem 3.5. *Let $(A, (p_\alpha)_{\alpha\in I})$ be an ample pseudo-Michael Q-algebra, where $(p_\alpha)_{\alpha\in I}$ is saturated. Suppose that for each $\alpha \in I$ there exists $M_\alpha > 0$ such that*

$$p_\alpha(x) \leqslant M_\alpha r_A(x) \quad \textit{for any } x \in A.$$

Then A is commutative.

Proof. Let the function f be as defined in Lemma 3.2. For all $\alpha \in I$ and $\lambda \in \mathbb{C}$ we have

$$p_\alpha(f(\lambda)) = p_\alpha(\exp(\lambda x) y \exp(-\lambda x)) \leqslant M_\alpha r_A(\exp(\lambda x) y \exp(-\lambda x)) = M_\alpha r_A(y).$$

Since the spectral radius satisfies the identity $r_A(xy) = r_A(yx)$ for all $x, y \in A$, (see [1, 1.8.12]), the function f is entire and bounded on $\mathbb{C}$. Let $\varphi \in A^*$. Then $\varphi \circ f(\lambda) = \varphi(f(\lambda))$ is a scalar entire function, which is bounded. Now the rest of the proof is the same as that for Theorem 3.4. □

Theorem 3.6. *Let $(A, (p_\alpha)_{\alpha\in I})$ be an ample pseudo-Michael algebra with projective limit decomposition $A = \varprojlim \hat{A}_\alpha$, where $(p_\alpha)_{\alpha\in I}$ is saturated. Suppose that for any $\alpha \in I$ there exists $M_\alpha > 0$ such that*

$$p_\alpha^2(x) \leqslant M_\alpha p_\alpha(x^2), \quad \textit{for any } x \in A.$$

Then A is commutative.

Proof. Let $\alpha \in I$ and $x \in A$. By induction we get

$$p_\alpha(x) \leqslant M_\alpha^{1-\frac{1}{2^n}} \left(p_\alpha(x^{2^n})\right)^{\frac{1}{2^n}}.$$

Since $v_\alpha(x) = \lim_{n\to\infty} p_\alpha(x^{2^n})^{\frac{1}{2^n}}$, letting $n \to \infty$, we have

$$p_\alpha(x) \leqslant M_\alpha v_\alpha(x).$$

On the other hand, $v_\alpha(xy) = v_\alpha(yx)$ for all $x, y \in A$ [1, Sect. 3.3.7]. By this identity and submultiplicativity of p_α we find:

$$p_\alpha(xy) \leqslant M_\alpha v_\alpha(xy) = M_\alpha v_\alpha(yx) \leqslant M_\alpha p_\alpha(yx),$$

for any $x, y \in A$. Using Theorem 3.4, we obtain that A is commutative. $\square$

Definition 3.1. *Let A be a pseudo-Michael Q-algebra. We say that a linear continuous functional $\psi : A \to \mathbb{C}$ is a spectral state on A if it is satifies the following properties*

$$\psi(e) = 1 \quad \textit{and} \quad |\psi(x)| \leqslant r_A(x) \qquad \textit{for each} \quad x \in A.$$

Theorem 3.7. *Let A be a pseudo-Michael Q-algebra. If ψ is an injective spectral state on A, then A is commutative.*

Proof. Let the function f be as defined in Lemma 3.2. Now, we have

$$\psi \circ f(\lambda) = \psi(f(\lambda)) = \psi(\exp(\lambda x) y \exp(-\lambda x)).$$

Since ψ is a spectral state, we get

$$|\psi \circ f(\lambda)| = |\psi(\exp(\lambda x) y \exp(-\lambda x))| \leqslant r_A(y)$$

because $r_A(xy) = r_A(yx)$, for any $x, y \in A$. So $\psi \circ f : \mathbb{C} \to \mathbb{C}$ is bounded. On the other hand, $\psi \circ f$ is an entire function. Now the rest of the proof is the same as that for Theorem 3.4. $\square$

Theorem 3.8. *Let A and B be two pseudo-Michael Q-algebras. Suppose that $T : A \to B$ is a unital injective continuous linear map satisfying*

$$r_B(Tx) \leqslant r_A(x), \qquad \textit{for any } x \in A.$$

If f is an injective spectral state on B, then A is commutative.

Proof. Let $\psi = f \circ T$. Then $\psi(e) = 1$ and we have

$$|\psi(x)| = |f(Tx)| \leqslant r_B(Tx) \leqslant r_A(x) \quad \text{for each} \quad x \in A.$$

Thus, ψ is an injective spectral state on A. The result follows from Theorem 3.7. $\square$

Remark 3.2. *In Theorem 3.8, if T is an injective continuous homomorphism, then the inequality*

$$r_B(Tx) \leqslant r_A(x), \qquad \textit{for any} \quad x \in A,$$

can be omitted since in this case we have $\mathrm{sp}_B(Tx) \subseteq \mathrm{sp}_A(x)$ and hence $r_B(Tx) \leqslant r_A(x)$, see [1, Sect. 1.7.19].

Remark 3.3. [6, 3.10] *Let H be the quaternion field. Then the subset*

$$\big\{x(\exp \beta y) - (\exp \beta y)x : x \in H, \beta \in \mathbb{R}\big\}$$

can not be bounded in H.

Theorem 3.9. *Let $(A, (p_\alpha)_{\alpha \in I})$ be a pseudo-Michael algebra. If the set*

$$\big\{x(\exp \beta y) - (\exp \beta y)x : x \in A, \beta \in \mathbb{R}\big\}$$

is bounded, then A is commutative.

Proof. By the assumptions, there exists $M_\alpha > 0$ such that

$$p_\alpha(x(\exp \beta y) - (\exp \beta y)x) \leqslant M_\alpha \quad \text{for each} \quad \beta \in \mathbb{R}.$$

Then

$$p_\alpha(x(\exp \beta y) - (\exp \beta y)x) \leqslant \frac{M_\alpha}{n^{k_\alpha}} \quad \text{for all} \quad x \in A, n \in \mathbb{N}.$$

This yields

$$p_\alpha(x(\exp \beta y) - (\exp \beta y)x) = 0 \quad \text{as} \quad n \to \infty.$$

Since A is Hausdorff,

$$x(\exp \beta y) - (\exp \beta y)x = 0 \quad \text{for each} \quad \beta \in \mathbb{R}.$$

Differentiating this identity, we get

$$xy = yx.$$

□

Theorem 3.10. *Let $(A, (p_\alpha)_{\alpha \in I})$ be a pseudo-Michael Q-algebra. If for each x in* Inv(A) *the subset*

$$\{xy - yx : y \in A\}$$

is bounded, then A is commutative.

Proof. We take an arbitrary x in $\text{Inv}(A)$. By the assumptions there exists $M_\alpha > 0$ such that

$$p_\alpha(xy - yx) \leqslant M_\alpha \quad \text{for each} \quad y \in A.$$

Then we have

$$p_\alpha(xy - yx) \leqslant \frac{M_\alpha}{n^{k_\alpha}} \quad \text{for each} y \in A \quad \text{and} \quad n \in \mathbb{N}.$$

Hence,

$$p_\alpha(xy - yx) = 0 \quad \text{as} \quad n \to \infty.$$

Since A is Hausdorff, we get

$$xy = yx \quad \text{for each} \quad y \in A.$$

Assume that $x \notin \text{Inv}(A)$. Since $\text{Inv}(A)$ is open, then $e - \text{Inv}(A)$ is a neighborhood of zero. So there exists $\lambda > 0$ such that $\lambda x \in e - \text{Inv}(A)$ or $e - \lambda x \in \text{Inv}(A)$. Therefore, as in the first part, we have

$$(e - \lambda x)y = y(e - \lambda x),$$

and hence

$$xy = yx.$$

Thus, A is commutative. □

Theorem 3.11. *Let $(A, (p_\alpha)_{\alpha \in I})$ be an ample pseudo-Michael algebra, where $(p_\alpha)_{\alpha \in I}$ is saturated. If*

$$\{xy - yx : x \in A, y \in \text{Inv}(A)\}$$

is bounded. Then A is commutative.

Proof. By the assumptions, the subset

$$\{x \exp(\lambda y) - \exp(\lambda y)x : \lambda \in \mathbb{C}\}$$

is bounded in A. Then the function $g : \mathbb{C} \to A$ defined by

$$g(\lambda) = x \exp(\lambda y) - \exp(\lambda y)x$$

is entire and bounded. Hence, there exists $M_\alpha > 0$ such that $p_\alpha(g(\lambda)) \leqslant M_\alpha$.

Let $\varphi \in A^*$. Then there exists a $p_\alpha \in \mathcal{P}$ and a constant $C = C_\alpha > 0$ such that

$$|\varphi \circ g(\lambda)| = |\varphi(g(\lambda)| \leqslant C p_\alpha(g(\lambda))^{\frac{1}{k_\alpha}} \leqslant C M_\alpha^{\frac{1}{k_\alpha}}.$$

Hence, $\varphi \circ g$ is a bounded entire function. By the Liouville theorem, the function $\varphi \circ g$ is constant and hence,

$$\varphi(g(\lambda)) = \varphi(g(0)) = 0.$$

Since A is ample, $g(\lambda) = 0$ for any $\lambda \in \mathbb{C}$. This implies that

$$x \exp(\lambda y) - \exp(\lambda y)x = 0, \qquad \text{for any } \lambda \in \mathbb{C}.$$

Differentiating this identity, we get

$$xy = yx.$$

□

Remark 3.4. *Every k-Banach algebra is a pseudo-Michael algebra* [1]*. Thus, all the above theorems and results which are true for pseudo-Michael algebras, also hold for k-Banach algebras.*

Acknowledgements

The author would like to thank the referee for helpful suggestions and other valuable remarks.

BIBLIOGRAPHY

1. V. K. Balachandran. *Topological algebras.* Elsevier, Amsterdam, (2000).
2. H.G. Dales. *Banach Algebras and Automatic Continuity.* Clarendon Press, Oxford (2000).
3. M. Fragoulopoulou. *Topological algebras with involution.* Elsevier, Amsterdam (2005).
4. A. Mallios. *Topological Algebras, Selected Topics.* North-Holland, Amsterdam, (1986).
5. E. Michael. *Locally multiplicatively convex topological algebras.* Amer. Math. Soc., Providence, RI. (1952).
6. M. Oudadess. *A general commutativity condition in topological and bornological algebras* // Bull. Greek Math. Soc. 58, 29–42 (2015).
7. W. Rudin. *Principles of mathematical analysis.* McGraw-Hill, New York (1976).
8. A. Toma. *Commutativity criterions in Locally M-Convex Algebras* // Extracta Mathematicae. **18**:1, 81–89 (2003).

Ali Naziri-Kordkandi,
Department of Mathematics,
Payame Noor University,
P.O.Box 19395-3697,
Tehran, Iran
E-mail: ali_naziri@pnu.ac.ir