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PERTURBATION OF A SIMPLE WAVE: FROM SIMULATION TO ASYMPTOTICS

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Abstract. We consider a problem on perturbation of a simple (travelling) wave at the example of a nonlinear partial differential equation that models domain wall dynamics in the weak ferromagnets. The main attention is focused on the case when, for fixed constants coefficients, there exist many exact solutions in the form of a simple wave. These solutions are determined by an ordinary differential equation with boundary conditions at infinity. The equation depends on the wave velocity as a parameter. Suitable solutions correspond to the phase trajectory connecting the equilibria. The main problem is that the wave velocity is not uniquely determined by the coefficients of the initial equations. For an equation with slowly varying coefficients, the asymptotics of the solution is constructed with respect to a small parameter. In the considered case, the well-known asymptotic construction turns out to be ambiguous due to the uncertainty of the perturbed wave velocity. For unique identification of the velocity, we propose an additional restriction on the structure of the asymptotic solution. This restriction is a stability of the wave front is formulated on the base of numerical simulation of the original equation.

Keywords: simple wave, perturbation, small parameter, asymptotics.

Mathematics Subject Classification: 35Q60, 35L20, 35A18

1. Introduction

1.1. Initial data. In this work, a simple wave is treated as a function with a specific dependence on variables: $\phi(x,t) = \Phi(x - vt)$. As $v = \text{const}$, the function $\Phi(s)$ of the variable $s = x - vt$ is interpreted as a wave traveling along the x axis with a velocity v. Finding solutions in this form is one of the methods of reducing to ordinary differential equations. The reductions based on others invariant solutions are not discussed here. For applications, simple waves are of interest, which on the phase plane $(\Phi, \dot{\Phi})$ correspond to trajectories connecting equilibria. They are related with the description of the dynamic transition from one equilibrium to another [1], [2]. The exact solutions obtained in this way can be used as approximations for analysing more complex problems by employing the perturbation theory and asymptotics in a small parameter as this was demonstrated in a number of publications [3]–[6]. Unfortunately, the implementation of the ideas used in [4], [5] for parabolic equations faces significant difficulties in the case of hyperbolic equations [6]. Problems arising in constructing asymptotics and ways for overcoming them are discussed at the example of the magnetodynamics equation derived in [7]:

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + \Omega^2\sin\phi\cos\phi + \omega^2\,\sin\phi + \alpha\frac{\partial\phi}{\partial t} = 0, \qquad t > 0, \qquad x \in \mathbb{R}. \tag{1.1}$$

The equation has trivial solutions, the equilibria $\phi \equiv 0$ and $\phi \equiv \pi$. For the magnetodynamics, the solutions with the following conditions at infinity are of interest [7], [8]:

$$\phi(x,t) \to 0 \quad \text{as} \quad x \to -\infty, \qquad \phi(x,t) \to \pi \quad \text{as} \quad x \to +\infty. \tag{1.2}$$

L.A. Kalyakin, Perturbation of a simple wave: from simulation to asymptotics.

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In the case of constant coefficients we can select such solutions as a simple wave $\phi = \Phi_0(x-vt)$. The finding of this wave is reduced to an ordinary differential equation:

$$[v^2 - c^2]\frac{d^2\Phi_0}{ds^2} + \Omega^2 \sin\Phi_0 \cos\Phi_0 + \omega^2 \sin\Phi_0 - \alpha\, v\frac{d\Phi_0}{ds} = 0, \quad s = x - vt. \tag{1.3}$$

This equation involves a parameter v, a velocity of wave, and the solutions exist for an arbitrary wave. However, the boundary conditions produces restrictions for v. The situation is similar to a spectral problem. Appropriate solutions correspond to phase trajectories, separatrices, which connect the equilibria $\Phi_0 \equiv 0$ and $\Phi_0 \equiv \pi$.

If $\omega^2 < \Omega^2$, then such separatrix exists for a unique v determined by the relation

$$\alpha\frac{v}{\sqrt{c^2 - v^2}}\,\Omega = \omega^2. \tag{1.4}$$

On the phase plane $(\Phi, \dot\Phi)$ this separatrix connects two saddles with the coordinates $(0,0)$ and $(\pi, 0)$. The corresponding solution is interpreted as a domain wall moving with the velocity v. For other values of v, the trajectory from $(0,0)$ enters other (stable) equilibria, so that the boundary conditions (1.2) are not satisfied. The corresponding solution is interpreted as a domain wall moving with the velocity v. For other values of v, the trajectory from $(0,0)$ approaches other (stable) equilibria and boundary conditions (1.2) are not satisfied.

If $\omega^2 > \Omega^2$, then for each velocity $0 < v < c$ the trajectory from the saddle $(0,0)$ approaches the equilibrium $(\pi, 0)$, which turns out to be either a node or a focus. That is, in this case, there are waves with conditions (1.2) travelling with different velocities.

Although equation (1.3) is not integrable, it has a specific feature discovered by Zvezdin in [7]. The wave with velocity satisfying (1.4), regardless of the relations between ω^2 and Ω^2, is written out in terms of the separatrix solution of the pendulum equation

$$\Phi_0(s) = 2\arctan\exp(s\,\Lambda_0), \qquad \Lambda_0 = \Omega/\sqrt{c^2 - v^2}. \tag{1.5}$$

It should be noted that a simple wave is an isolated solution of a partial differential equation. The problem of stabilization of other solutions to this one in the formulation of the Cauchy problem was studied for the parabolic equations of Kolmogorov-Petrovskii-Piskunov type [9], [10]. There are no similar general results for non-integrable hyperbolic equations.

1.2. Formulation of problem. If the coefficients depend on x and t, then in the general situation there is no simple wave. For an approximate analysis of the problem, either numerical [11] or asymptotic [3] methods are applied. In the case of slowly varying coefficients one can construct an asymptotic solution similar to a travelling wave [3]. Exactly this problem is studied in what follows.

To reveal the essence of the problem and not obscure the presentation of irrelevant details, the original problem is considered in its simplest form. The coefficients $c(\tau)$, $\Omega(\tau)$, $\alpha(\tau)$, $\omega(\tau)$ in equation (1.1) are assumed to be positive functions depending smoothly on the slow variable $\tau = \varepsilon t$. The small parameter $0 < \varepsilon \ll 1$ is involved only in τ. If the coefficients c, Ω are constant, then they are reduced to unity by scaling transformations. In the general case a renormalization of x, t leads to an equation perturbed by small terms of order $\mathcal{O}(\varepsilon)$. This approach gives no advantage in studying the problem and is not used here. In the magnetodynamics, the coefficient ω^2 corresponds to the amplitude of the external force, from the direction of which the sign at the corresponding term in (1.1) depends. Differences in sign at ω^2 are not significant due to the possibility of shifting the dependent variable $\phi \Rightarrow \phi + \pi$ which changes the sign. The boundary value problem additionally requires the change $x \Rightarrow -x$, which leads to a wave traveling in the opposite direction.

Differential equation (1.1) is complemented by an initial condition:

$$\phi(x,t)|_{t=0} = \Phi_0(x), \qquad \partial_t\phi(x,t)|_{t=0} = -v_0\Phi_0'(x), \qquad x \in \mathbb{R}. \tag{1.6}$$

The initial function is taken as a trace of the simple wave, that is, $\Phi_0(s)$ as $s = x - v_0 t$ satisfies equation (1.3) with constant (initial) coefficients and with boundary conditions (1.2). To identify the functions $\Phi_0(s)$ as the unique solution of an autonomous equation, it is necessary to fix the shift in s, for example, by the condition $\Phi_0(0) = \pi/2$. Restriction on the initial parameters at the initial moment

$$\alpha^2 v_0^2 - 4(\omega^2 - \Omega^2)(c^2 - v_0^2) > 0 \quad \text{as} \quad \tau = 0 \tag{1.7}$$

guarantees that for unperturbed equation (1.3) the equilibrium $\Phi_0 = \pi$ in the case of $\omega^2 > \Omega^2$ corresponds to a stable node.

The initial function, as a solution of differential equation with constant coefficients has the following asymptotics at the equilibria

$$\Phi_0(s) = \begin{cases} \exp(\lambda_-^0 s)[c_-^0 + \mathcal{O}\big(\exp(\lambda_-^0 s)\big)], & s \to -\infty, \\ \pi + \exp(-\lambda_+^0 s)[c_+^0 + \mathcal{O}\big(\exp(-\lambda_+^0 s)\big)], & s \to +\infty \end{cases}$$

with constants $c_\pm^0 \neq 0$. The exponents $\lambda_\pm^0 > 0$ satisfy corresponding characteristic equations:

$$(v_0^2 - c^2)(\lambda_\pm^0)^2 \pm \alpha\, v_0 \lambda_\pm^0 + \Omega^2 \mp \omega^2 = 0 \qquad \text{as} \qquad \tau = 0.$$

The aim of the present work is to construct an asymptotic as $\varepsilon \to 0$ solution for problem (1.1), (1.2), (1.6), which is applicable on some segment $0 < t \leqslant \tau_0 \varepsilon^{-1}$, ($\tau_0 = \text{const} > 0$), when the deformation of the equation becomes essential. By an asymptotic solution we mean a function $\phi_{as}(x, t; \varepsilon)$, which, being substituted into equation (1.1), gives a small error as $\varepsilon \to 0$ uniformly in x, t in a wide domain. This notion will be specified in Section 6. The main aim is an approximate description of the trajectory (of motion of the center) of the perturbed wave, the exact position of which is determined by the sought solution via the relation $\phi(x, t; \varepsilon) = \pi/2$.

2. Initial numerical simulations

When analyzing problems related to applications, numerical and analytical methods complement each other. In papers on construction of asymptotics, numerical calculations are often used for illustrations. Sometimes a comparison of numerical and analytical results is provided as an argument in favor of the formally obtained formulas instead of their rigorous justification. In the present paper, a similar comparison is made in order to choose an asymptotic ansatz. The point is that in the considered problem different constructions for the asymptotic solution are possible. Since the justification theorems are absent, this leads to the indeterminacy of the asymptotic solution and to the appearance of fictitious asymptotics known in different situations [12].

In this section we present the results of numerical simulations for equation (1.1) with coefficients $c^2 = \Omega^2 = \alpha = 1$. The perturbation consists in a slow variation in the coefficient $\omega^2(\tau) = (1 + \tau)\omega_0^2$, $\tau = \varepsilon t$ for small parameter $\varepsilon = 0.03$. The initial data corresponds to a simple wave in form (1.5). In Figures 1 and 2, the wave profiles as functions spatial coordinates x are given at distant times $t = 1/2\varepsilon$ for different values of the constant ω_0^2. The dotted line corresponds to the profile of the initial wave shifted for comparison by a suitable distance.

The main result of numerical experiments: for a perturbed wave we observe that the symmetry with respect to the center breaks with time. In this case, the structure of the wave front is preserved, while the wave tail is deformed. This effect is weakly expressed at $\omega^2 < \Omega^2$ and is clearly observed at $\omega^2 > 2\Omega^2$.

3. Ansatz for asymptotic solution

We choose an ansatz for the asymptotic solution as a partial sum of a series in powers of the small parameter:

$$\phi_{as}(x, t; \varepsilon) = \Phi(s; \tau) + \varepsilon\, \Phi_1(s; \tau) + \varepsilon^2 \Phi_2(s; \tau) + \ldots \tag{3.1}$$

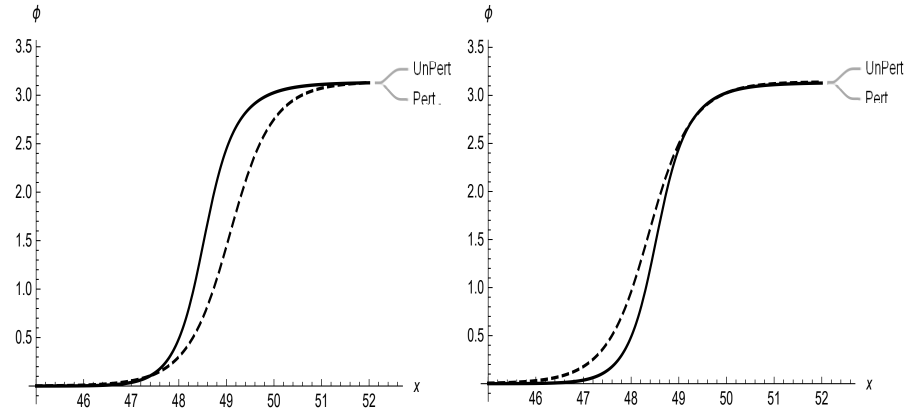

FIGURE 1. Comparison of the perturbed wave (solid line) with the profile of the initial function at $\Omega^2 < \omega^2 < 2\Omega^2$.

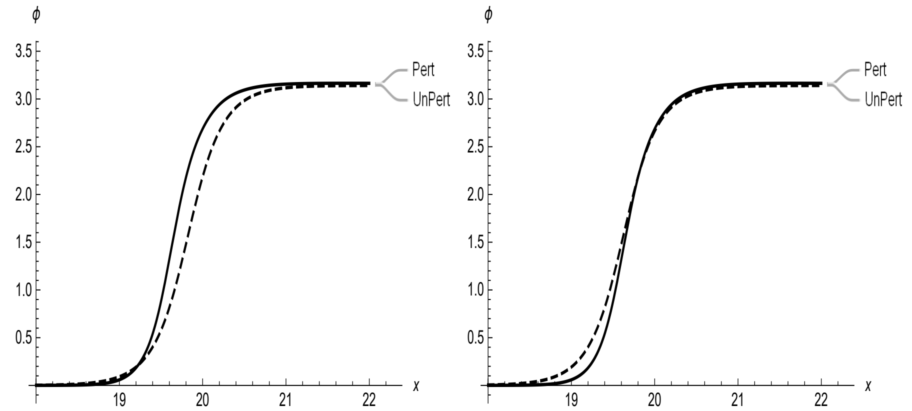

FIGURE 2. Comparison of the perturbed wave (solid line) with the profile of the initial function at $\omega^2 > 2\Omega^2$.

with a single fast variable

$$s = x - \varepsilon^{-1} S(\tau) - S_1(\tau) - \varepsilon S_2(\tau) - \dots, \qquad \tau = \varepsilon t.$$

The phase function $S(\tau)$, as well as the shift of the phase $S_1(\tau)$ and next correctors are to be determined. This approach correspond to the two-scale method.

Substituting ansatz (3.1) into original equation (1.1) and collecting terms of order $\mathcal{O}(1)$, $\varepsilon \to 0$, we arrive at a single equation for two functions $\Phi(s,\tau)$ and $V(\tau) = S'(\tau)$:

$$[V^2 - c^2]\frac{d^2\Phi}{ds^2} + \Omega^2 \sin\phi \cos\Phi + \omega^2 \sin\Phi - \alpha V \frac{d\Phi}{ds} = 0. \tag{3.2}$$

Additionally, we impose a boundary condition corresponding to the initial one:

$$\Phi(s;\tau) \to 0 \quad \text{as} \quad s \to -\infty, \qquad \Phi(s;\tau) \to \pi \quad \text{as} \quad s \to +\infty, \tag{3.3}$$

and an initial condition for the velocity: $V(0) = v_0$. Apart of this, in order to single out a unique solution to autonomous equation (3.2), we should fix the shift in the independent variable s. This can be done by an additional condition:

$$\Phi(0;\tau) = \frac{\pi}{2}. \tag{3.4}$$

If we choose the velocity $V = V(\tau)$ by the relation

$$\alpha \frac{V}{\sqrt{c^2 - V^2}} \Omega = \omega^2, \tag{3.5}$$

then the solution to ordinary differential equation (3.2) with conditions (3.3), (3.4) is written in elementary functions [7]:

$$\Phi(s;\tau) = 2\arctan\exp(s\,\Lambda), \qquad \Lambda = \Omega/\sqrt{c^2 - V^2}. \tag{3.6}$$

The asymptotics of this solution at infinity is described by the formulas

$$\Phi(s;\tau) = \begin{cases} \exp(\Lambda\,s)[2 + \mathcal{O}\big(\exp(\Lambda\,s)\big)], & s \to -\infty, \\ \pi + \exp(-\Lambda\,s)[2 + \mathcal{O}\big(\exp(-\Lambda\,s)\big)], & s \to +\infty. \end{cases}$$

We note that for variable coefficients the velocity $V(\tau)$ and the exponent $\Lambda = \Lambda(\tau)$ in the general case depend on the slow time τ. The analysis of the phase portrait of equation (3.2) implies that the found in this way a pair of functions Φ, V is a unique solution of problem (3.2), (3.3), (3.4) if $\omega^2 < \Omega^2$.

As $\omega^2 > \Omega^2$, there is no uniqueness and there exist other pairs Φ, V, for which explicit representations are absent. For $\Phi(s,\tau)$ we can write an asymptotics in the vicinity of the equilibria:

$$\Phi(s;\tau) = \begin{cases} \exp(\lambda_- s)[c_- + \mathcal{O}\big(\exp(\lambda_- s)\big)], & s \to -\infty, \\ \pi + \exp(-\lambda_+ s)[c_+ + \mathcal{O}\big(\exp(-\lambda_+ s)\big)], & s \to +\infty, \end{cases} \qquad c_\pm(\tau) \neq 0. \tag{3.7}$$

Here the functions of the slow time $\lambda_\pm(\tau) > 0$ and $c_\pm(\tau)$ depend on the choice of $V(\tau)$. There are no explicit expressions for the coefficients $c_\pm(\tau)$ as it happens for an non-integrable equation. The exponents $\lambda_\pm(\tau)$ satisfy algebraic equations

$$(V^2 - c^2)(\lambda_\pm)^2 \pm \alpha\,V\lambda_\pm + \Omega^2 \mp \omega^2 = 0. \tag{3.8}$$

Under Condition 1.7, the roots $\lambda_\pm(\tau)$ remain real and the property $c_\pm(\tau) \neq 0$ preserves depending on $V(\tau)$ in some neighbourhood of the initial point $0 < \tau < \tau_0$. However, the *evolution of the velocity $V(\tau)$ as $\tau > 0$ is not determined* and this remains a main problem in the case $\omega^2 > \Omega^2$.

Our approach is based on observation of the results of numerical simulations: as $\omega^2 > \Omega^2$, the wave front is deformed weakly. This is why it is assumed that the exponent λ_+ in the asymptotics at infinity remains constant and coincides with the value $\lambda_+(\tau) \equiv \lambda_+^0 = \text{const}$, which corresponds to the asymptotics of the initial (unperturbed) wave at velocity $V(0) = v_0$. In this case $V(\tau)$ for $\tau > 0$ is uniquely determined by (3.8):

$$V(\tau) = \frac{1}{2\lambda_+^0}\left[-\alpha + \sqrt{\alpha^2 + 4(c^2(\lambda_+^0)^2 + \omega^2 - \Omega^2)}\right]. \tag{3.9}$$

An opposite statement is true as well: if $V(\tau)$ is defined by formula (3.9), then corresponding root of equation (3.8) is constant $\lambda_+(\tau) \equiv \text{const}$. We call this property a *stability of the wave front*[1]. Other arguments supporting the stability of the wave front arise in the formal construction of the asymptotic solutions and are given in Section 5.

The second equation in (3.8) determines the root $\lambda_-(\tau)$, the dependence of which on τ reflects the deformation in time of the wave tail.

We note that when choosing the velocity $V(\tau)$ by relation (3.5), both equations (3.8) are satisfied, the roots are the same $\lambda_+ = \lambda_- = \Lambda(\tau)$ and are not constants. In this case, for the leading term of the asymptotics in form (3.6), the symmetry is preserved with deformation of the wave front and wave tail, which corresponds to numerical simulation at $\omega^2 < \Omega^2$. An alternative way to determine the velocity by the formula (3.9) is for the case $\omega^2 > \Omega^2$ when the symmetry is not is preserved, and the initial wave need not be symmetrical (3.6).

[1]For other equations, for instance, of Kolmogorov-Petrovskii-Piskunov type, the stability of wave front and wave tail was not discussed although it was in fact used in [5], [6].

Once the velocity is calculation, the phase function is recovered by the integral

$$S(\tau) = \int_0^\tau V(\eta)\, d\eta.$$

The corrector of the velocity $V_1(\tau) = S_1'(\tau)$ and the phase shift $S_1(\tau)$ at this step remain undetermined.

Since the function $\Phi(s;\tau)$ as a solution of problem (3.2), (3.3) fast stabilizes at infinity as $s \to \pm\infty$, then the zero of the phase $x - \varepsilon^{-1}S(\varepsilon t) + S_1(\varepsilon t) = 0$ can be identified with an approximate trajectory of the center of the perturbed wave. It is obvious that to find the trajectory at large time as $\varepsilon t \approx 1$, apart of the function $S(\tau)$ we need to determine the shift of the phase $S_1\tau)$, which can be found via the corrector of the velocity $V_1(\tau) = S_1'(\tau)$. As in other problems of the perturbation theory, the function $V_1(\tau)$ is determined at the next step by requiring the smallness of the first corrector in comparison with the leading term in the asymptotic solution.

4. First corrector

For the first corrector of asymptotic solution (3.1) we obtain a linear equation

$$[V^2 - c^2]\frac{d^2\Phi_1}{ds^2} + q(s;\tau)\Phi_1 - \alpha\, V\frac{d\Phi_1}{ds} = f(s;\tau) \tag{4.1}$$

with the coefficient

$$q(s;\tau) = \frac{d}{d\phi}\left[\Omega^2 \sin\phi\, \cos\phi + \omega^2\, \sin\phi\right]_{\phi=\Phi(s;\tau)}.$$

The right hand side f is written via the previous approximation. This function is extracted from the error term, which arises under the substitution of the leading term $\Phi(s;\tau)$ of the asymptotics into original equation (1.1):

$$f(s;\tau) = -2VV_1\Phi_{ss} + V'\Phi_s + 2V\Phi_{s\tau} + \alpha V_1\Phi_s - \alpha\Phi_\tau. \tag{4.2}$$

Comment. The formulation of problem and described construction of asymptotic solution are similar to the perturbation theory of solitons [13]–[16]. Main difference comes from the nonintegrability of the original unperturbed equation with constant coefficients (1.1). The lack of integrability makes it impossible to use an analogue of the Fourier expansion [17], [18] when solving the Cauchy problem for a linearized partial differential equation

$$\frac{\partial^2\phi_1}{\partial t^2} - c^2\frac{\partial^2\phi_1}{\partial x^2} + q\,\phi_1 + \alpha\frac{\partial\phi_1}{\partial t} = f, \qquad t > 0, \qquad x \in \mathbb{R}.$$

This is why for correctors the initial problem is not considered and the matter is restricted by partial solutions $\phi_1 = \Phi_1(s;\tau)$ determined by the ordinary differential equation[1].

An explicit representation for the corrector Φ_1 is written in terms of the function $\Phi(s;\tau)$ on the base of the fundamental system of solutions for a homogeneous linearized equation corresponding to (4.1). One of such solutions is given by the derivative $\Psi_1(s;\tau) = \partial_s\Phi(s;\tau)$ and possesses an exponential asymptotics at infinity:

$$\Psi_1(s;\tau) = \exp(\mp\lambda_\pm s)[\mp c_\pm\lambda_\pm + \mathcal{O}\big(\exp(\mp\lambda_\pm s)\big)], \qquad s \to \pm\infty. \tag{4.3}$$

By using the Wronskian $W(s;\tau) = \exp(-\beta(\tau)\, s)$, where

$$\beta(\tau) = \alpha V(\tau)/[c^2 - V(\tau)^2],$$

the second solution is determined by the Liouville formula:

$$\Psi_2(s;\tau) = \Phi_s(s;\tau)\int_0^s \frac{\exp(-\beta\eta)}{(\Phi_\eta(\eta;\tau))^2}\, d\eta. \tag{4.4}$$

[1]For nonintegrable equations the influence of small errors in initial data remains unclarified in all problems on perturbation of simple waves [3]; very often this issues is not discussed at all [16].

An exponential asymptotics is easily extracted from formula (4.4):

$$\Psi_2(s,\tau) = \exp((\lambda_- - \beta)s)[C_- + \mathcal{O}\big(\exp((\lambda_- - \beta)s)\big)], \quad s \to -\infty. \tag{4.5}$$

At the other infinity $s \to +\infty$ the structure of the asymptotics depends on the difference $\lambda_+ - \beta$:

$$\Psi_2(s,\tau) = \begin{cases} \exp((\lambda_+ - \beta)s)[C_+ + \mathcal{O}\big(\exp((\lambda_+ - \beta)s)\big)] & \text{if} \quad \lambda_+ - \beta > -\lambda_+, \\ \exp(-\lambda_+ s)[C_+ + \mathcal{O}\big(\exp(-\lambda_+ s)\big)] & \text{if} \quad \lambda_+ - \beta < -\lambda_+. \end{cases} \tag{4.6}$$

The coefficients $C_\pm(\tau) \neq 0$ are expressed in terms of β, $c_\pm$ and $\lambda_\pm$.

We note that the general solution of equation (4.1) involves a linear combinations of solutions from the basis Ψ_1, Ψ_2. The function $\Psi_1(s;\tau) = \partial_s \Phi(s;\tau)$ exponentially tends to zero as $s \to \pm\infty$ with the exponents $\lambda_\pm$. The adding of this function does not change the structure of the first corrector at infinity and can not be taken into consideration because the same effect is made by the corrector of the phase shift $\varepsilon S_1(\tau)$, the determination of which is moved to the next step. The function $\Psi_2(s,\tau)$ grows exponentially as $s \to -\infty$ and this is why it is not included into the corrector $\Phi_1(s;\tau)$. At the same time we should control the employed partial solution for Φ_1 in order it not to contain growing terms.

A particular solution of homogeneous equation can be written in various forms up to an additive term $\Psi_1(s;\tau) = \partial_s \Phi(s;\tau)$. For further calculations the following representation is convenient:

$$\Phi_1(s;\tau) = \Phi_s(s;\tau) \int_0^s \frac{\exp(-\beta\eta)}{(\Phi_\eta(\eta;\tau))^2} \int_{-\infty}^{\eta} f(\zeta;\tau)\Phi_\zeta(\zeta;\tau)\exp(\beta\zeta)\, d\zeta\, d\eta. \tag{4.7}$$

In what follows we analyze the first corrector $\Phi_1(s;\tau)$ in order to determine the correctors for the velocity $V_1(\tau)$. In order to do this, we construct an asymptotics for the function $\Phi_1(s;\tau)$ as $s \to \pm\infty$ aiming to select the terms growing slower than $\Psi_1(s;\tau)$ and because of which the asymptoticity is violated in sequence of approximations (3.1).

Lemma 4.1. *The right hand side in the equation for the first corrector* (4.1) *has the following asymptotics at infinity:*

$$f(s,\tau) = \begin{cases} c_- \exp(\lambda_- s)[s\, f^-(\tau) + f_0^-(\tau) + \mathcal{O}\big(\exp(\lambda_- s)\big)], & s \to -\infty, \\ c_+ \exp(-\lambda_+ s)[s\, f^+(\tau) + f_0^+(\tau) + \mathcal{O}\big(\exp(-\lambda_+ s)\big)], & s \to +\infty \end{cases} \tag{4.8}$$

with the coefficients at the leading part

$$\begin{aligned} f^\pm(\tau) &= \lambda'_\pm(\tau)[\pm\alpha + 2V(\tau)\lambda_\pm(\tau)], \\ f_0^\pm(\tau) &= -2VV_1\lambda_\pm^2 - [V' + \alpha V_1](\pm\lambda_\pm) - [2V(\pm\lambda_\pm) + \alpha]c'_\pm/c_\pm. \end{aligned} \tag{4.9}$$

The proof is carried out by substituting asymptotics (3.7) into formula (4.2). Growing in s factors arises while differentiating exponentials in τ.

Independently on the way of determining the velocity $V(\tau)$ the structure of the first corrector in the asymptotics as $s \to -\infty$ is determined by the function $\Psi_1(s;\tau) = \Phi_s(s;\tau)$ up to a power factor s^2.

Lemma 4.2. *The first corrector determined by formula* (4.7) *has the following asymptotics at the negative infinity*

$$\Phi_1(s,\tau) = \Psi_1(s;\tau)\frac{1}{2\lambda_-(\tau) + \beta(\tau)}\left[\frac{1}{2}s^2 f^-(\tau) + s\,\tilde{f}^-(\tau) + \mathcal{O}(1)\right], \quad s \to -\infty \tag{4.10}$$

with the coefficient

$$\tilde{f}^-(\tau) = f_0^-(\tau) - \frac{f^-(\tau)}{2\lambda_-(\tau) + \beta(\tau)}.$$

The proof is obtained by integrating the corresponding asymptotics in formula (4.7) after substituting (4.8).

The asymptotics at the other infinity $s \to +\infty$ differs from $\Psi_1(s;\tau)$ in exponential terms.

Lemma 4.3. *If $2\lambda_+(\tau) > \beta(\tau)$, then the first corrector determined by formula* (4.7) *has the following asymptotics as $s \to +\infty$:*

$$\Phi_1(s,\tau) = \Psi_2(s;\tau)J(\tau) + \Psi_1(s;\tau)\frac{1}{-2\lambda_+(\tau)+\beta(\tau)}\left[\frac{1}{2}s^2 f^+(\tau) + s\,\tilde{f}^+(\tau) + \mathcal{O}(1)\right]. \qquad (4.11)$$

The coefficient $J(\tau)$ is determined by the converging integral:

$$J(\tau) = \int_{-\infty}^{\infty} f(\zeta;\tau)\Phi_\zeta(\zeta;\tau)\exp(\beta\zeta)\,d\zeta; \qquad \tilde{f}^+(\tau) = f_0^+(\tau) - \frac{f^+(\tau)}{-2\lambda_+(\tau)+\beta(\tau)}. \qquad (4.12)$$

Proof. The leading term of the asymptotics of the inner integral in formula (4.7) as $\eta \to +\infty$ is determined by the expression $J(\tau)$ from (4.12). We note that the outer integral in formula (4.7) is a function $\Psi_2(s;\tau)$. Therefore, after separating the main term from the inner integral, we obtain the relation

$$\Phi_1(s;\tau) = \Psi_2(s;\tau)J(\tau) - \Psi_1(s;\tau)\int_0^s \frac{\exp(-\beta\eta)}{(\Phi_\eta(\eta;\tau))^2}\int_\eta^\infty f(\zeta;\tau)\Phi_\zeta(\zeta;\tau)\exp(\beta\zeta)\,d\zeta\,d\eta.$$

The asymptotics of the second term as $s \to +\infty$ is obtained by integration similar to Lemma 2. The proof is complete. □

Remark 4.1. *While using formula* (4.11) *we should have in mind that it has nothing to do with the a symptotics at minus infinity. The function $\Psi_2(s;\tau)$, which grows exponentially as $s \to -\infty$, is used here only for brevity in the asymptotics as $s \to +\infty$.*

In the case $2\lambda_+(\tau) < \beta(\tau)$ the integral in (4.12) diverges, so the asymptotics of the inner integral in (4.7) should be calculated by another way.

Lemma 4.4. *If $2\lambda_+(\tau) < \beta(\tau)$, then the first corrector determined by formula* (4.7) *possesses the following asymptotics at infinity*

$$\Phi_1(s,\tau) = \Psi_1(s;\tau)\frac{1}{-2\lambda_+(\tau)+\beta(\tau)}\left[\frac{1}{2}s^2 f^+(\tau) + s\tilde{f}^+(\tau) + \mathcal{O}(1)\right], \qquad s \to +\infty. \qquad (4.13)$$

The proof consists in selecting the leading terms in the exponentially growing asymptotics of the inner integral as $\eta \to +\infty$. Due to the conditions $2\lambda_+(\tau) < \beta(\tau)$, the function in outer integral (4.7) exponentially tends to zero as $\eta \to +\infty$. As a result, we arrive at required relation (4.13).

5. Phase shift

The following calculation of the velocity correction $V_1(\tau)$ and the corresponding phase shift $S_1(\tau)$ is not the main aim. These calculations are more focused on refinement of the first correction $\Phi_1(s;\tau)$ in the asymptotic solution and on justifying the formula for the leading term in the asymptotics of the velocity $V(\tau) + \varepsilon V_1(\tau)$.

The formulas for $V_1(\tau)$ are obtained from the condition that the leading terms of the asymptotics as $s \to +\infty$ should be excluded from the corrector $\Phi_1(s;\tau)$ of the principal terms of the asymptotics as $s \to +\infty$.

Lemma 5.1. *Let $2\lambda_+(\tau) > \beta(\tau)$. If the velocity corrector $V_1(\tau)$ is chosen by the relation*

$$a(\tau)V_1 + b(\tau) = 0, \qquad (5.1)$$

where

$$a(\tau) = [V(\tau)\beta(\tau) + \alpha] \int_{-\infty}^{\infty} \Phi_s^2(s;\tau) \exp(\beta(\tau)s)\, ds,$$

$$b(\tau) = \int_{-\infty}^{\infty} [\partial_\tau(V(\tau)\,\Phi_s^2(s;\tau)) - \alpha\Phi_\tau(s;\tau)\Phi_s(s;\tau)] \exp(\beta(\tau)s)\, ds,$$

then the first corrector of the asymptotic solution determined by formula (4.7) *has the following asymptotics at the positive infinity:*

$$\Phi_1(s;\tau) = \Psi_1(s;\tau)\frac{1}{-2\lambda_+(\tau) + \beta(\tau)}\left[\frac{1}{2}s^2 f^+(\tau) + s\,\tilde{f}^+(\tau) + \mathcal{O}(1)\right], \qquad s \to +\infty \tag{5.2}$$

independently on the way of determining the velocity $V(\tau)$.

Proof. If the condition $2\lambda_+(\tau) > \beta(\tau)$ is satisfied, then comparing formulas (4.3) and (4.6), we see that in expression (4.11) the leading term of the asymptotics as $s \to +\infty$ is contained in the term $\Psi_2(s;\tau)J(\tau)$. To exclude this term, we require the factor $J(\tau) = 0$ vanishes. In view of expressions (4.2), (4.12) such requirement is equivalent to relation (5.1). Therefore, under condition (5.1), relation (4.11) becomes (5.2). The uncertainty in the velocity $V(\tau)$ is preserved for the case $\omega^2 > \Omega^2$. The proof is complete. □

Corollary 5.1. *If the function* $V(\tau)$ *is chosen according to formula* (3.9) *and the velocity corrector* $V_1(\tau)$ *is determined by relation* (5.1)*, then the first corrector of the asymptotic solution given by formula* (4.7) *has the following asymptotics at infinity:*

$$\Phi_1(s;\tau) = \Psi_1(s;\tau)\frac{1}{-2\lambda_+(\tau) + \beta(\tau)}\left[s\, f_0^+(\tau) + \mathcal{O}(1)\right], \qquad s \to +\infty.$$

Proof. Since in this case $\lambda'_+ = 0$, by (4.9) the coefficient in the leading term of asymptotics (5.2) vanishes $f^+ = 0$, while the coefficient at s is equal to f_0^+. This result indicates that formula (3.9) is preferable in comparison with (3.5) while choosing the velocity $V(\tau)$ in the case $\omega^2 > \Omega^2$. □

If $2\lambda_+(\tau) < \beta(\tau)$, then formula (5.1) for $V_1(\tau)$ makes no sense since the integrals diverge. In this case the asymptotics for $\Phi_1(s;\tau)$ as $s \to +\infty$ has a different structure and the exclusion of secular terms leads us to another formula for the velocity corrector.

Lemma 5.2. *Let* $2\lambda_+(\tau) < \beta(\tau)$ *and the function* $V(\tau)$ *be chosen by formula* (3.9). *If the velocity corrector* $V_1(\tau)$ *is defined by the relation*

$$V_1 + \frac{V'}{2V\lambda_+ + \alpha} + \frac{c'_+}{c_+\lambda_+} = 0, \tag{5.3}$$

then the first corrector of the asymptotic solution determined by formula (4.7) *has the following asymptotics at the positive infinity*

$$\Phi_1(s,\tau) = \Psi_1(s;\tau)\cdot\mathcal{O}(1), \qquad s \to +\infty. \tag{5.4}$$

Proof. Under the condition $2\lambda_+(\tau) < \beta(\tau)$ the asymptotics of the function $\Phi_1(s,\tau)$ is represented in formula (4.13). The leading terms are determined by the terms with power factors s^2 and s. The exclusion of the coefficient $f_+(\tau)$ at s^2 with expression (4.9) taken into consideration leads to the identity $\lambda'_+(\tau) = 0$ and this corresponds to choosing $V(\tau)$ by formula (3.9). The exclusion of the remaining coefficient $f_+^0(\tau)$ at s with expression (4.9) taken into consideration gives the equation for V_1 in form (4.9). After that relation (4.13) becomes (5.4). □

Comment. Relation (5.1) involves both the velocity V (in the coefficients) and the corrector V_1. Such feature arises due to the presence of the dissipation with the coefficient $\alpha \neq 0$. Under perturbation of integrable equations[16] such relation dose not involve V_1 and is used to define the leading term of $V(\tau)$. In the considered problem, as $\omega^2 > \Omega^2$, requirement

(5.1) in the form of one equation for two functions V, V_1 reflects the essence of an asymptotic incorrectness phenomenon, which arises in the analysis of dissipative systems [4], [5]. As usual in mathematical problems, to eliminate an uncertainty, additional restrictions on the desired (asymptotic) solution are needed. One option is to require stability of the wave front in the form $\lambda'_+ = 0$. On one hand, this leads to a unique definition of V by formula (3.9). On the other hand, the relation $\lambda'_+ = 0$ leads to the extension the suitability domain of the asymptotic solution, which can be considered as another form of additional requirement.

6. Suitability domain of asymptotic solution

The construction of the asymptotic solution in the form of series (3.1) with coefficients depending on one fast variable s can be implemented up to an arbitrary order ε^n. The suitability domain of an asymptotic solution is understood as the set of points $(s,\tau) \in D \subset \mathbb{R}^2$ on the plane, on which the series (3.1) is uniformly asymptotic as $\varepsilon \to 0$ [19]. By the requirement of smallness of the subsequent correction $\varepsilon^{n+1}\Phi_{n+1}(s;\tau)$ compared to the previous $\varepsilon^n\Phi_n(s;\tau)$ uniformly in (s,τ) the constraints on D are extracted. These restrictions depend on the construction of the coefficients of the asymptotics $\Phi_n(s;\tau)$, $V_n(\tau)$. Since the functions $\Phi_n(s;\tau)$ are smooth, the series is asymptotic as $\varepsilon \to 0$ uniformly in s, τ in the strip $\{|s| \leqslant L,\ 0 < \tau \leqslant \tau_0\}$, of any width $L = \text{const} > 0$ independent of ε. This simplest result for the suitability region is independent of the way of calculating the velocity corrections and for $\omega^2 > \Omega^2$ it is also independent of the velocity $V(\tau)$.

The extension of the suitability region is possible by taking into account the structure of the coefficients $\Phi_n(s;\tau)$ at infinity. The feature of the considered problem manifests in the fact that the source of irregularities are the power factors s^k, $k > 0$, which appear at decreasing exponentials in the asymptotics of the functions $\Phi_n(s;\tau)$ for $s \to \pm\infty$. An extension of the suitability domain occurs when such (secular) terms are excluded and is possible under an appropriate choice of $V_n(\tau)$. In this way, the asymptotic solution is refined and velocity ambiguities are eliminated. These ideas are similar to those used in the theory of non-linear oscillations [20].

The simplest refinement of the suitability region can be obtained by taking into account the asymptotics at negative infinity. For the first correction, the asymptotics is given by the formula (4.10) and involves terms with factors s, s^2. In higher correctors, the powers of s increase by 2 at each step. The requirement of the asymptoticity of the sequence of corrections in the form $\varepsilon\, s^2 \leqslant \varepsilon^{2\delta}$ (for some $\delta > 0$) provide a description suitability regions at wave tail in the form

$$-\varepsilon^{-1/2+\delta} < s < L, \quad \text{for all} \quad \delta, L > 0.$$

At the wave front, similar expansion is obtained if in the construction of the asymptotic solution the velocity corrections are chosen appropriately, as it is done for the first correction in Lemma 5.

Theorem 6.1. *Let the initial wave have a special structure* (1.5)*, and let the velocity* $V(\tau)$ *be chosen according to formula* (3.5)*. If the first velocity correction is chosen by* (5.1) *and subsequent corrections are fixed by similar conditions in the leading terms of the asymptotics, then series* (3.1) *is an asymptotic solution of equation* (1.1) *in the strip*

$$-\varepsilon^{-1/2+\delta} < s < \varepsilon^{-1/2+\delta}, \quad \forall\, \delta > 0. \tag{6.1}$$

Proof. If the velocity $V(\tau)$ is determined by formula (3.5), then the following relations hold

$$\frac{\alpha V\Omega}{\sqrt{c^2 - V^2}} = \omega^2, \qquad \lambda_+ = \Lambda = \frac{\Omega}{\sqrt{c^2 - V^2}}, \qquad \beta = \frac{\alpha V\Omega}{c^2 - V^2}.$$

Therefore,

$$\lambda_+ - \beta = \frac{1}{\Omega\sqrt{c^2 - V^2}}(\Omega^2 - \omega^2).$$

In the case $\Omega^2 - \omega^2 > 0$ we obtain $\lambda_+ - \beta > 0 > -\lambda_+$, since, according to (4.5), the function $\Psi_2(s;\tau)$ grows exponentially as $s \to +\infty$. The corresponding term in the asymptotics of first corrector (4.11) should be excluded. This can be achieved by the identity $J(\tau) = 0$, which is reduced to equation (5.1) for the velocity corrector. Once V_1 is determined, asymptotics (4.11) still involve the terms with the factors s^2, s. They determine the boundary of the suitability region at wave front (6.1) by condition $\varepsilon s^2 < \varepsilon^{2\delta}$.

In the case $\Omega^2 - \omega^2 < 0$ we obtain $\lambda_+ - \beta < 0$ and here two options are possible:

1) If $-\lambda_+ < \lambda_+ - \beta$, then, according to (4.5), the leading term of the asymptotics of the function $\Psi_2(s;\tau)$ decays exponentially as $s \to +\infty$. However, it decays slower than the function $\Psi_1(s;\tau)(s;\tau) \approx \exp(-\lambda_+ s)$ and is treated as a secular term. This is why the term $\Psi_2(s;\tau)J(\tau)$ in formula (4.11) is to be excluded by the same condition $J(\tau) = 0$, which is reduced to equation (5.1) for the velocity corrector.

2) If $-\lambda_+ > \lambda_+ - \beta$, then asymtptotics (4.11) involves only the terms with the factors s^2, s, which are determined the suitability domain at the wave front in form (6.1). □

For the case $\omega^2 < \Omega^2$, the velocity $V(\tau)$, $\tau > 0$ and its corrections are uniquely determined, and the above construction shows no other possibilities for extending the suitability domain. The situation is different for $\omega^2 > \Omega^2$, when the choice of $V(\tau)$, $\tau > 0$, remains arbitrary. The requirement for the stability of the wave front makes it possible to expand the suitability region.

Theorem 6.2. *Let $\omega^2(\tau) > \Omega^2(\tau)$, $0 < \tau < \tau_0$, the initial parameters of the wave satisfy condition* (1.7) *and the velocity $V(\tau)$ is chosen by formula* (3.9). *If the first velocity corrector is determined by* (5.1) *an the next correctors are fixed by similar conditions in the leading terms of the asymptotics, then series* (3.1) *is an asymptotic solution of equation* (1.1) *in the strip*

$$-\varepsilon^{-1/2+\delta} < s < \varepsilon^{-1+\delta}, \quad \forall\, \delta > 0. \tag{6.2}$$

If, in addition, the relation for the initial parameters

$$2\lambda_+^0 < \alpha\, V/(c^2 - V^2) \equiv \beta(\tau), \qquad 0 < \tau < \tau_0,$$

hold and the first velocity corrector is determined by (5.3) *and the next correctors are fixed by similar conditions in the leading terms of the asymptotics, then the suitability domain at the wave front is extended to infinity:*

$$-\varepsilon^{-1/2+\delta} < s < \infty.$$

Proof. The choice of the velocity by (3.9) implies that the exponent is constant: $\lambda_+ = \lambda_+^0 =$ const. In this case the asymptotics of the first corrector (5.2) involves no term with the factor s^2 and it casts into the form

$$\Phi_1(s,\tau) = \Psi_1(s;\tau) \cdot \mathcal{O}(s), \qquad s \to +\infty.$$

This is way to ensure that the first corrector is small on the half-line $s > 0$, it is sufficient to impose the inequality $\varepsilon s < \varepsilon^\delta$, $\delta > 0$.

Under an additional condition, which means $2\lambda_+ < \beta$, the asymptotics of the first correction in form (4.13) contains no term with the factor s^2 due to $\lambda_+ = \lambda_+^0 =$ const. The term with the first power s is eliminated by choosing the velocity corrector by (5.3), as shown in Lemma 5.2. As a result, the asymptotics of the first corrector contains growing factors $\Phi_1(s,\tau) = \Psi_1(s;\tau) \cdot \mathcal{O}(1)$, $s \to +\infty$, and thus the asymptotic property holds regardless of $s > 0$. □

Comment. The domain described by formula (6.2) is obviously not symmetrical with respect to the center of the wave $s = 0$. Such asymmetry is due to the choice of the speed $V(\tau)$ from the requirement of stability of the wave front: $\lambda_+ =$ const. All formal constructions of (another) asymptotic solution in same form (3.1) can be obtained from from the wave tail stability requirement: $\lambda_- = \text{const}$. For such solution, the suitability domain is described by a formula similar to (6.2) with changed boundary:

$$-\varepsilon^{-1+\delta} < s < \varepsilon^{-1/2+\delta}. \tag{6.3}$$

As we see, the problem of the uniqueness of the asymptotic is not solved by extension of the suitability area. Since, in order to justify the asymptotics no prospects are visible at present, in this work the choice an asymptotic solution is proposed to be made from a comparison with a numerical simulation. This choice leads to the requirement stability of the wave front (rather than the wave tail): $\lambda_+ = \text{const}$.

7. Concluding numerical simulation

This section compares three methods for approximate calculation of the wave trajectories. In the first method, which we call numerical, the trajectory is found by the relation $\phi(x,t) = \pi/2$ based on the numerical solution of original equation (1.1). For numerical implementation, the problem is supplemented with initial conditions on a segment of large length $-l < x < L$, which correspond to special unperturbed solution (1.5). The boundary conditions $\phi(-l,t) = 0$, $\phi(-L,t) = \pi$, $t > 0$ on far edges of $l, L \approx \varepsilon^{-1}$ mimic conditions at infinity (1.2). The trajectory obtained in this way is close to the exact one. The error depends on the method of approximation of the equation and little depends on the relation between the parameters ω and Ω.

Two other approximations for the trajectory are defined on the base asymptotic formulas by the relation

$$x = \int_0^t V(\varepsilon\,\eta)\,d\eta$$

without using phase shift[1]. For the velocity $V(\tau)$ formulas (3.5) or (3.9) are used. The graphs of the corresponding three approximations are shown in the figures: bold dotted line, bold solid and weak solid lines. The weak dotted line corresponds to the unperturbed trajectory.

The numerical simulations were carried out for the coefficients $c^2 = \Omega^2 = \alpha = 1$. The perturbation is embedded in the slow coefficient $\omega^2(\tau) = (1 \pm \tau/2)\omega_0^2$, $\tau = \varepsilon t$ with the value of the small parameter $\varepsilon = 0.01 \div 0.03$. The direction of deformation of the trajectory under perturbation is determined by the sign of the derivative $(\omega^2)'(\tau) = \pm(\omega_0)^2/2$, see Figure 3.

The closeness of asymptotic trajectories to numerical ones depends on a relation between the parameters ω and Ω. If $\omega^2 \leqslant \Omega^2$, then calculations on the base of formula (3.5) are unalternative since the speed V is unique. In Figure 4 the corresponding asymptotic trajectory practically coincides with the numerical one. If $\omega^2 > \Omega^2$, then as the difference $\omega^2 - \Omega^2$ increases, formula (3.9) becomes more suitable, cf. Figure 5; this is especially noticeable for $\omega^2 > 2\Omega^2$ in Figure 6.

Comment. On the considered wave with special initial profile (1.5), for which $\lambda_+ = \Lambda(0)$, the following relation holds:

$$2\Lambda(0) - \beta(0) = \frac{\lambda}{\Omega^2}(2\Omega^2 - \omega^2)|_{\tau=0}.$$

Therefore, for $\omega^2 > 2\Omega^2$ the inequality $2\lambda_0 < \beta(\tau)$ is satisfied on some interval $0 < \tau < \tau_0$. Then, by virtue of Theorem 6.2, formula (3.9) implied by the requirement of stability ensures the suitability domain of the asymptotic solution at the wave front. The presence of a wide suitability region indicates the closeness of the corresponding asymptotic trajectory to the exact one. Nevertheless, the *extension of the suitability domain cannot serve as a criterion for selecting a formula for the velocity.* For example, the requirement of the stability of the wave

[1]The efficiency of equations (5.3) for calculating the velocity correction and phase shift is not great due to the lack of explicit expressions for solving the ordinary differential equation for $\Phi(s;\tau)$ and for the asymptotic coefficients $c_\pm(\tau)$.

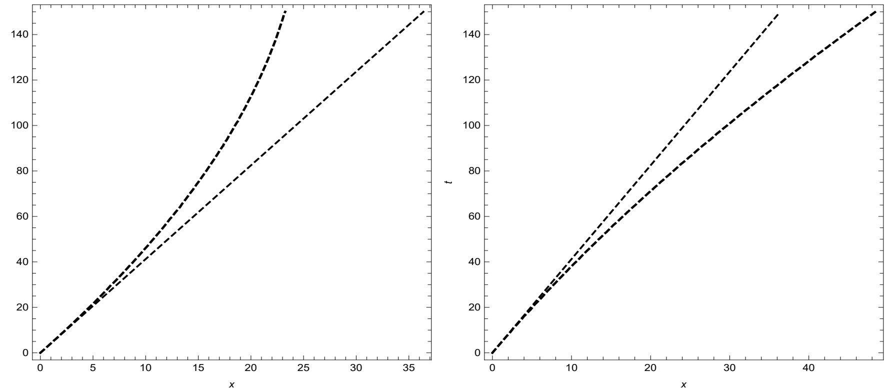

FIGURE 3. The trajectory of the perturbed wave (thick dotted line) in comparison with the trajectory of the unperturbed wave for different directions of deformation of the coefficient $\omega^2(\tau)$.

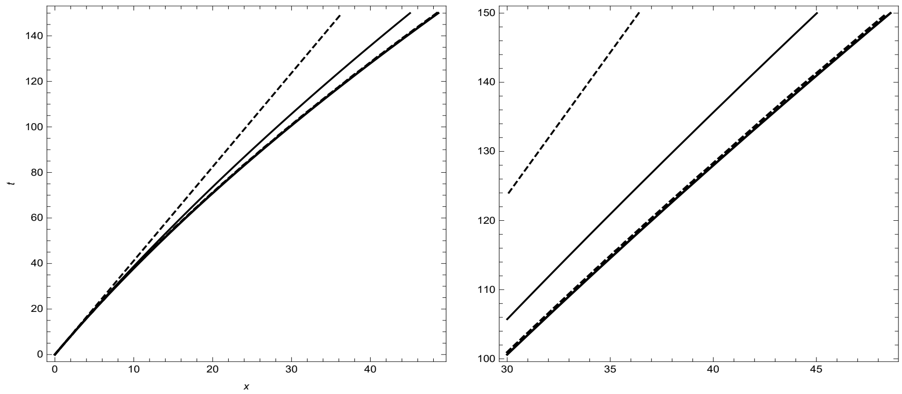

FIGURE 4. Approximate trajectories of the perturbed wave on different scales at $\omega^2 < \Omega^2$.

tail $\lambda_- = \text{const}$ also expands the suitability domain of (6.3). This requirement, applied to the second equation in (3.8), gives the formula for the velocity

$$V(\tau) = \frac{1}{2\lambda_-^0}\left[\alpha + \sqrt{\alpha^2 + 4(c^2(\lambda_-^0)^2 - \omega^2 - \Omega^2)}\right]$$

different from (3.9). The unsuitability of this result is found when compared with a numerical simulation. The asymptotic trajectory obtained in this way is not close to the numerical trajectory for all parameters ω and Ω, as seen in Figure 7.

A similar situation occurs for formula (3.5). In the problem with special initial condition (1.5), the estimate of suitability region (6.1) is independent on the relation between the parameters ω^2, Ω^2. However, for $\omega^2 > \Omega^2$, the use of formula (3.5) produces large errors, which is revealed when compared with a numerical simulation. However, for initial data other than (1.5), formula (3.5) is definitely not suitable.

8. CONCLUSION

When a simple wave is perturbed, the asymptotic solution depends on the choice of the leading term in the asymptotics of the slowly deforming velocity $V(\varepsilon t) + \mathcal{O}(\varepsilon)$, $\varepsilon \to 0$. For the problem for equation (1.1) we propose a construction based on the requirement for the stability of the wave front. It leads to an algebraic equation for the velocity[1] and to a unique definition of $V(\tau)$ by the formula (3.9). The choice of an asymptotic solution with a stable front is indicated

[1]In a more general problem, this becomes the Hamilton-Jacobi equation for the phase function [5], [6].

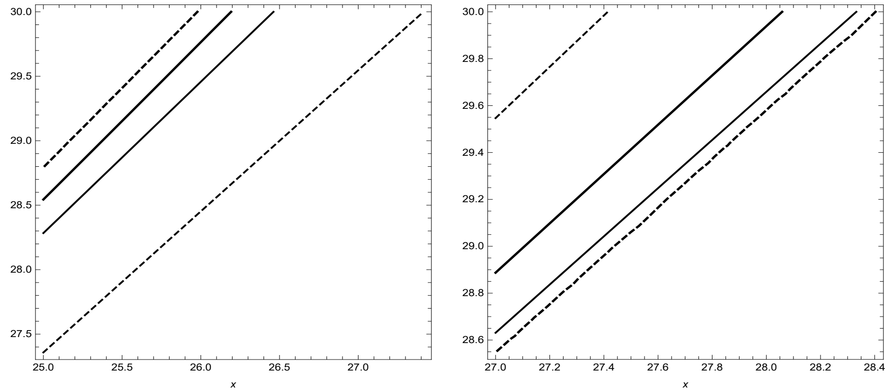

FIGURE 5. Approximate trajectories of the perturbed wave for different directions of deformation of the coefficient $\omega^2(\tau)$ in the case $\Omega^2 < \omega^2 < 2\Omega^2$.

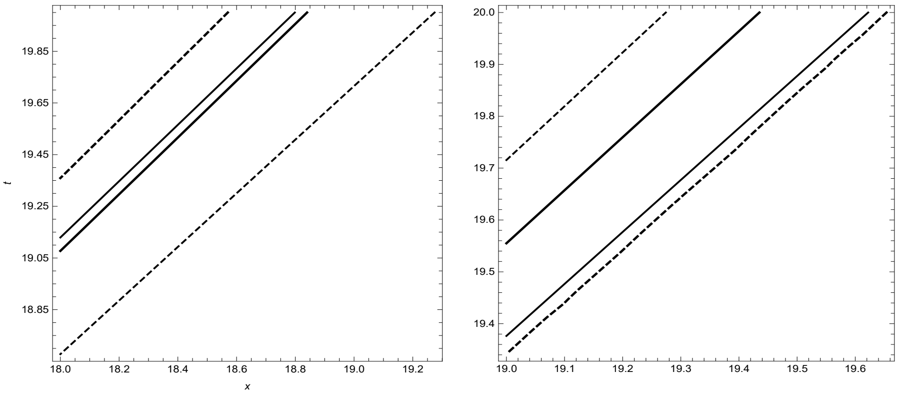

FIGURE 6. Approximate trajectories of the perturbed wave for different directions of deformation of the coefficient $\omega^2(\tau)$ in the case $\omega^2 > 2\Omega^2$.

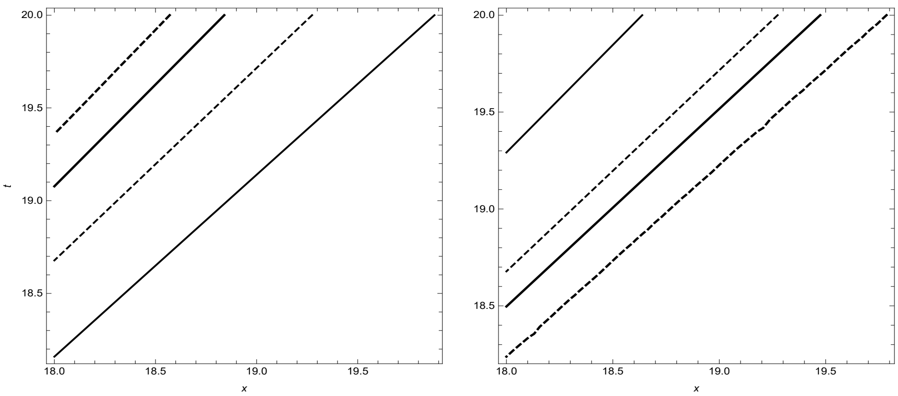

FIGURE 7. Approximate trajectories of the perturbed wave for different directions of deformation of the coefficient $\omega^2(\tau)$ calculated from the condition of the wave tail stability. The parameters ω^2, Ω^2 correspond to Figure 5.

by numerical simulations. The role of the wave front in determining the wave velocity was discussed in the first work by Fischer [21], see also [2]. There is no rigorous justification of the asymptotics presented here with a proof of the existence theorem and with an estimate for the remainder. This applies to all known results on the perturbation of simple waves.

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