ISSN 2304-0122 Ufa Mathematical Journal. Vol. 15. No 2 (2023). P. 119-134.

517.958
doi:10.13108/2023-15-2-119

INVERSE PROBLEM ON DETERMINING TWO KERNELS IN INTEGRO-DIFFERENTIAL EQUATION OF HEAT FLOW

D.K. DURDIEV, J.J. JUMAEV, D.D. ATOEV

Abstract. We study the inverse problem on determining the energy-temperature relation $\chi(t)$ and the heat conduction relation $k(t)$ functions in the one-dimensional integro-differential heat equation. The direct problem is an initial-boundary value problem for this equation with the Dirichlet boundary conditions. The integral terms involve the time convolution of unknown kernels and a direct problem solution. As an additional information for solving inverse problem, the solution of the direct problem for $x = x_0$ and $x = x_1$ is given. We first introduce an auxiliary problem equivalent to the original one. Then the auxiliary problem is reduced to an equivalent closed system of Volterra-type integral equations with respect to the unknown functions. Applying the method of contraction mappings to this system in the continuous class of functions, we prove the main result of the article, which a local existence and uniqueness theorem for the inverse problem.

Keywords: Banach principle, resolvent, Volterra equation, operator equation, initial-boundary problem, inverse problem, Green function.

Mathematics Subject Classification: 35A01; 35A02; 35L02; 35L03; 35R03.

1. Introduction

Integro-differential equations with integral term of convolution type arise in many fields of physics and applied mathematics for modeling the processes of heat and mass transfer with finite propagation speed, systems with thermal memory, viscoelasticity problems and acoustic waves in composite media. In [1] Gurtin and Pipkin derived the integro-differential equation

$$u_{tt} = \triangle u(x,\tau) + \int\limits_0^t K'(t-\tau)\triangle u(x,\tau)d\tau + q(x,t), \tag{1.1}$$

describing the heat propagation in a media with memory at a finite speed. Here Δ is the Laplace operator in the variables $x = (x_1, \dots, x_n)$. Apart of equation (1.1), in the literature the following equation

$$u_t(x,t) = \int\limits_0^t K(t-\tau)\triangle u(x,\tau)d\tau + g(x,t) \tag{1.2}$$

was considered, it is of the first order in the time variable t. Nowadays, equations (1.1) and (1.2) are referred to as the Gurtin-Pipkin equations. It can readily be seen that equation (1.1) is derived from (1.2) by differentiating with respect to the variable t if we let $K(0) = 1$ and $q(x,t) = g_t(x,t)$.

D.K. Durdiev, J.J. Jumaev, D.D. Atoev, Inverse problem of determining two kernels in the integro-differential equation of heat flow.

Submitted April 14, 2022.

In the linear theory of heat conduction in media with memory the constitutive equations between heat flux and gradient of temperature contain integral terms over the past history of the material involving time-dependent convolution. In [2] Miller studied the existence, uniqueness, and continuous dependence on parameters for solutions of the certain initial boundary value problem for following system of integro-ifferential equations:

$$\begin{aligned} e(t,x) &= e_0 + \chi(0)\theta(t,x) + \int_0^t \chi'(t-\tau)\theta(\tau,x)d\tau, \\ q(t,x) &= -k(0)\theta_x(t,x) - \int_0^t k'(t-\tau)\theta_x(\tau,x)d\tau, \\ e_t(t,x) &= -q_x(t,x) + r(t,x), \end{aligned} \tag{1.3}$$

where $t \in (0,T]$, $x \in (0;l)$, $e_t = \big(\partial/\partial t\big)e$, $q_x = \big(\partial/\partial x\big)q$. In (1.3) $\chi(t)$ and $k(t)$ are relaxation functions of internal energy and heat flow, respectively. Moreover, $\theta(t,x)$ is a function of temperature, $r(t,x)$ is an external heat source function.

The first and second equations in (1.3) are linearized (with respect to certain constant e_0 energy) constitutive equations for internal energy and heat flow, respectively. The third equation in (1.3) expresses the fundamental law of thermal conductivity, the Fourier law. For $k(0) = 0$ these equations represent the linearized theory for heat flow in a rigid, isotropic, homogeneous material as proposed by Gurtin and Pipkin, see, for instance, [1], [3]. For $k(0) > 0$ the equations represent an alternate linearized theory proposed by Coleman and Gurtin [4]. For the direct problem consisting in determining the distribution of heat from some initial-boundary value problem for equation (1.3) Grabmueller [5] gave a very general uniqueness proof for generalized solutions in a Sobolev space and proved existence theorems in certain special situations.

The determination of the integral operator from the observable information about the solutions of the corresponding equations is a new class of inverse problems that has not yet been studied in details. In view of a wide range of applications, the theory of inverse problems for integro-differential equations is one of the most urgent and rapidly developing direction in mathematics.

The problem of determining the kernel $K(t)$ of the integral term in equation (1.1) was studied in many publications [6]–[14], see also the references therein, in which for one-dimensional inverse problems the issues on well-posedness were investigated. Inverse problems to determine time- and space-dependent kernels for initial, initial-boundary problem in hyperbolic integro-differential equations with several additional conditions were studied in [15]-[26] and there were proved existence, uniqueness and stability theorems.

In papers [27]-[31] inverse problems on determining the coefficients and kernel of parabolic and pseudo-parabolic equation with several overdetermination conditions were investigated. Solvability of these inverse problems in the classical and generalized sense were studied.

In the present paper, we study the inverse problem on determining the kernels of an integral convolution-type terms in the system of integro-differential equations (1.3) by the single observations at the points $x = x_0$ and $x = x_1$.

Among the works close to our problem we mention [32]–[35]. In [32] the uniqueness theorem for solution of kernel determination problem for one-dimensional heat conduction equation was proven. Papers [33]–[35] dealt with the inverse problems of determining the kernel depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1,\ldots,x_{n-1})$. The main part of the considered integro-differential equation was a n-dimensional heat conduction operator and the integral term had a convolution type form with respect to unknown functions, which the solutions of direct and inverse problem. In these works the theorems of existence and uniqueness of problems solutions were obtained.

It should also be noted that the statement of the problem and the technique used in this paper differ from those in the above cited papers and the conditions in the theorems differ essentially from those in them. A distinctive feature of this article is the inverse problem on determining two unknown functions, we determine the energy-temperature relation $\chi(t)$ and the heat conduction relation $k(t)$ functions in the integro-differential heat equation.

2. Problem and auxiliary constructions

It is supposed a rigid body occupies a fixed interval $(0,l)$ (one dimensional case). We also suppose that the functions $\chi(t)$ and $k(t)$ are sufficiently continuously differentiable functions.

If follows from (1.3) that

$$\begin{aligned}\theta_t(t,x) = &-\frac{\chi'(0)}{\chi(0)}\theta(t,x) + \frac{k(0)}{\chi(0)}\theta_{xx}(t,x)\\ &+\int_0^t \left(\frac{k'(t-\tau)}{\chi(0)}\theta_{xx}(\tau,x) - \frac{\chi''(t-\tau)}{\chi(0)}\theta(\tau,x)\right) d\tau + \frac{r(t,x)}{\chi(0)}.\end{aligned} \tag{2.1}$$

Throughout the paper $\chi(0)$ and $k(0)$ are given numbers such that $k(0) > 0$, $\chi(0) > 0$.

We rewrite equation (2.1) in a compact form:

$$\theta_t(t,x) = f(t,x) + C\theta_{xx}(t,x) - a(0)\theta(t,x) + \int_0^t \left(Cb(t-\tau)\theta_{xx}(\tau,x) - a'(t-\tau)\theta(\tau,x)\right) d\tau \tag{2.2}$$

for all $t \in (0,T]$, $x \in (0,l)$ and for this equation we consider a problem with an initial condition

$$\theta(0,x) = \theta_0(x), \tag{2.3}$$

and a boundary condition

$$\theta(t,0) = \mu_1(t), \qquad \theta(t,l) = \mu_2(t); \qquad \theta_0(0) = \mu_1(0), \qquad \theta_0(l) = \mu_2(0), \tag{2.4}$$

where

$$C := \frac{k(0)}{\chi(0)}, \qquad a(t) := \frac{\chi'(t)}{\chi(0)}, \qquad b(t) := \frac{k'(t)}{k(0)}, \qquad f(t,x) := \frac{r(t,x)}{\chi(0)}.$$

In identities (2.3) and (2.4), by $\theta_0(x)$, $\mu_1(t)$ and $\mu_1(t)$ we denote some given functions. If $r(t,x)$, $\theta_0(x)$, $a(t)$, $b(t)$, $\mu_1(t)$, $\mu_2(t)$ are given functions, then problem on finding the function $\theta(t,x)$ from (2.2), (2.3), (2.4) is called *direct problem.* This direct problem was investigated in paper [32].

We pose an *inverse problem.* For given functions $r(t,x)$, $\theta_0(x)$, $\mu_1(t)$, $\mu_2(t)$ and numbers $k(0) > 0$, $\chi(0) > 0$ it is required to determine the kernels $k(t)$, $\chi(t)$ (these functions are included in the definitions of a, b of the integral terms in (2.2)) using additional conditions about the solution of the direct problem (2.2), (2.3), (2.4):

$$\theta\,|_{x=x_0} = \psi_0(t), \qquad \theta\,|_{x=x_1} = \psi_1(t), \qquad x_0, x_1 \in (0,l), \quad t > 0. \tag{2.5}$$

Here $\psi_0(t)$, $\psi_1(t)$ are also assumed to be given functions.

Since the method for studying the inverse problem allow us to find simultaneously the solution to the inverse problem and the solution to the direct problem, in what follows we regard the inverse problem as a problem on determining functions $\theta(t,x)$, $k(t)$, $\chi(t)$ from equations (2.2), (2.3), (2.4), (2.5).

Let $C^m(0;l)$ be the class of m times continuously differentiable on $(0;l)$ functions. In the case $m=0$ this space coincides with the class of continuous functions. By $C^{m,n}(D_T)$ we denote the class of m times continuously differentiable with respect to t and n times continuously differentiable with respect to x in the domain D_T functions,

$$D_T := \{(t,x) : 0 < t \leqslant T, 0 < x < l\}.$$

We need the following lemma.

Lemma 2.1. *Suppose that $\chi(t) \in C^2[0,T]$, $k(t) \in C^1[0,T]$, $T>0$ is an arbitrary fixed number, and $\chi(0)>0, k(0)>0$. Then problem (2.2), (2.3), (2.4), (2.5) is equivalent to the auxiliary problem on determining the functions $\vartheta(x,t)$, $a(t)$, $b(t)$:*

$$\begin{aligned}\vartheta_t(t,x) =& C\vartheta_{xx}(t,x) + f_t(t,x) - a(0)\vartheta(t,x) - a'(t)\theta_0(x) - \int_0^t a'(\tau)\vartheta(t-\tau,x)d\tau + \\ &+ b(t)C\theta_0''(x) + \int_0^t R(t-\tau)F\big(\vartheta(\tau,x), \vartheta_\tau(\tau,x), a'(\tau), b(\tau)\big)\,d\tau,\end{aligned} \tag{2.6}$$

$$\vartheta|_{t=0} = f(0,x) + C\theta_0''(x) - a(0)\theta_0(x), \tag{2.7}$$

$$\vartheta|_{x=0} = \mu_1'(t), \qquad \vartheta|_{x=l} = \mu_2'(t), \tag{2.8}$$

$$\vartheta\,|_{x=x_0} = \psi_0'(t), \qquad \vartheta\,|_{x=x_1} = \psi_1'(t), \qquad x_0, x_1 \in (0,l), \quad t>0, \tag{2.9}$$

where

$$\vartheta(t,x) = \theta_t(t,x),$$

F is defined as

$$\begin{aligned}F\big(\vartheta(t,x), \vartheta_t(t,x), a'(t), b(t)\big) :=& \vartheta_t(t,x) - f_t(t,x) + a(0)\vartheta(t,x) + a'(t)\theta_0(x) \\ &+ \int_0^t a'(\tau)\vartheta(t-\tau,x)d\tau - b(t)C\theta_0''(x)\end{aligned}$$

and $R(t)$ is the resolvent of kernel $b(t)$ and they are related by the identity

$$R(t) = -b(t) - \int_0^t R(t-\tau)b(\tau)d\tau. \tag{2.10}$$

Proof. The proof consists of several steps. At the first step, we find $\theta(t,x)$ from the equation $\theta_t(t,x) = \vartheta(t,x)$:

$$\theta(x,t) = \int_0^t \vartheta(x,\tau)d\tau + \theta_0(x).$$

By ϑ we denote the function $\theta_t(t,x) := \vartheta(t,x)$. Differentiating (2.2), (2.3), (2.4) with respect to t, we obtain the following problem:

$$\begin{aligned}\theta_{tt}(t,x) =& f_t(t,x) + C\theta_{txx}(t,x) - a(0)\theta_t(t,x) - \int_0^t a'(\tau)\theta_t(t-\tau,x)\,d\tau \\ &+ Cb(t)\theta_0''(x) + \int_0^t Cb(\tau)\theta_{txx}(t-\tau,x)\,d\tau.\end{aligned} \tag{2.11}$$

The following statement was proved in [36], [37], see Proposition 2.1 in the cited works.

Lemma 2.2. *If $\phi(t), \nu(t) \in L_1[0,T]$ for a fixed $T > 0$ and $\phi(t)$, $\nu(t)$ satisfy the integral equation*

$$\nu(t) = \phi(t) + \int_0^t \phi(t-\tau)\nu(\tau)\,d\tau, \quad t \in [0,T],$$

then the solution of the integral equation

$$\varphi(t) = p(t) + \int_0^t \phi(t-\tau)\varphi(\tau)\,d\tau, \quad p(t) \in L_1[0,T],$$

is expressed by formula

$$\varphi(t) = p(t) + \int_0^t \nu(t-\tau)p(\tau)\,d\tau.$$

We observe that equation (2.11) can be treated as integral Volterra equation of the second kind with respect to $C\vartheta_{xx}(t,x)$ ($\theta_{txx} = \vartheta_{xx}$) with the kernel $b(t)$,

$$\begin{aligned} C\vartheta_{xx}(t,x) =& \vartheta_t(t,x) - f_t(t,x) + a(0)\vartheta(t,x) + \int_0^t a'(\tau)\vartheta(t-\tau,x)\,d\tau \\ &- Cb(t)\theta_0''(x) - \int_0^t b(\tau)C\vartheta_{xx}(t-\tau,x)\,d\tau. \end{aligned}$$

It follows from Lemma 2.2 that the solution of this equation is expressed by the formula

$$\begin{aligned} C\vartheta_{xx}(t,x) =& \vartheta_t(t,x) - f_t(t,x) + a(0)\vartheta(t,x) + \int_0^t a'(\tau)\vartheta(t-\tau,x)d\tau - Cb(t)\theta_0''(x) \\ &- \int_0^t R(t-\tau)\Bigg(\vartheta_\tau(\tau,x) - f_\tau(\tau,x) + a(0)\vartheta(\tau,x) \\ &+ \int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,x)d\alpha - Cb(\tau)\theta_0''(x)\Bigg)\,d\tau. \end{aligned}$$

Using the notation $\theta_t(t,x) = \vartheta(t,x)$, we derive equation (2.7) from equations (2.2) and (2.3), boundary condition (2.8) from identity (2.4) and also additional condition (2.9) from equation (2.5). The proof is complete. □

Theorem 2.1. *Let*

$$\theta_0(x) \in C^2[0,l], \qquad f(t,x) \in C^{1,0}(D_T),$$
$$a(t),\ \mu_1(t),\ \mu_2(t) \in C^1[0,T], \qquad R(t),\ b(t) \in C[0,T], \qquad C > 0$$

and the matching conditions

$$f(0,0) + C\theta_0''(0) - a(0)\theta_0(0) = \mu_1'(0), \qquad f(0,l) + C\theta_0''(l) - a(0)\theta_0(l) = \mu_2(0)$$

are satisfied. Then there exists a unique classical solution $\vartheta(t,x)$ to problem (2.6), (2.7), (2.8) in the class $C^{1,2}(D_T)$.

This theorem can be proved similarly to Theorem 1 in [32].

3. Reduction of problem (2.6), (2.7), (2.8), (2.9) to integral equations

In this section, we reduce problem (2.6), (2.7), (2.8), (2.9) to a closed system of nonlinear integral equations with respect to $\vartheta(t,x)$, $a(t)$, $b(t)$ and some of their combinations.

The solution of initial boundary problem (2.6),(2.7), (2.8) satisfies the integral equation, see [39]:

$$\begin{aligned}\vartheta(t,x) =&\Psi(t,x) + \int_0^t\int_0^l G(t-\tau,x,\xi)\Big(Cb(\tau)\theta_0''(\xi) - a(0)\vartheta(\tau,\xi) - a'(\tau)\theta_0(\xi)\Big)d\xi d\tau\\ &- \int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,\xi)\,d\alpha d\xi d\tau\\ &+ \int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau R(\tau-\alpha)\Bigg(\vartheta_\alpha(\alpha,\xi) - f_\alpha(\alpha,\xi) + a(0)\vartheta(\alpha,\xi)\\ &+ a'(\alpha)\theta_0(\xi) + \int_0^\alpha a'(\beta)\vartheta(\alpha-\beta,\xi)\,d\beta - b(\alpha)C\theta_0''(\xi)\Bigg)\,d\alpha d\xi d\tau,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\Psi(t,x) = &\int_0^l G(t,x,\xi)\big(f(0,\xi) + C\theta_0(\xi) - a(0)\theta_0(\xi)\big)\,d\xi\\ &+ \int_0^t\int_0^l G(t-\tau,x,\xi)f_\tau(\tau,\xi)\,d\xi d\tau\\ &+ \sum_{n=1}^\infty\int_0^t \frac{2\pi n}{l^2}\big(\mu_1'(\tau) - (-1)^n\mu_2'(\tau)\big)e^{-\frac{\pi^2n^2}{l^2}C(t-\tau)}\sin\frac{\pi n}{l}x\,d\tau,\end{aligned}$$

$$G(t-\tau,x,\xi) = \frac{2}{l}\sum_{n=1}^\infty e^{-\frac{\pi^2n^2}{l^2}C(t-\tau)}\sin\frac{\pi n}{l}\xi\sin\frac{\pi n}{l}x$$

is the Green function of the first initial-boundary problem for heat equation.

We differentiate equation (3.1) with respect to t and taking into account the relations

$$\lim_{t\to 0} G(t,x,\xi) = \delta(x-\xi),$$

$$\lim_{t\to 0}\int_0^l G(t,x,\xi)\theta_0(\xi)\,d\xi = \theta_0(x),$$

where $\delta(\cdot)$ is the Dirac delta function, we rewrite the result as

$$\begin{aligned}
\vartheta_t(t,x) =& \Psi_t(t,x) + Cb(t)\theta_0''(x) - a(0)\vartheta(t,x) - a'(t)\theta_0(x) \\
&+ \int_0^t \int_0^l G_t(t-\tau,x,\xi)\big(Cb(\tau)\theta_0''(\xi) - a(0)\vartheta(\tau,\xi) - a'(\tau)\theta_0(\xi)\big)\, d\xi d\tau \\
&- \int_0^t a'(\tau)\vartheta(t-\tau,x)d\tau - \int_0^t \int_0^l G_t(t-\tau,x,\xi) \int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,\xi)\, d\alpha d\xi d\tau \\
&- \int_0^t R(t-\tau)\Bigg(\vartheta_\tau(\tau,x) - f_\tau(\tau,x) + a(0)\vartheta(\tau,x) + a'(\tau)\theta_0(x) - b(\tau)C\theta_0''(x) \\
&+ \int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,x)\, d\alpha\Bigg)\, d\tau - \int_0^t \int_0^l G_t(t-\tau,x,\xi) \int_0^\tau R(\tau-\alpha)\Bigg(\vartheta_\alpha(\alpha,\xi) - f_\alpha(\alpha,\xi) \\
&+ a(0)\vartheta(\alpha,\xi) + a'(\alpha)\theta_0(\xi) + \int_0^\alpha a'(\beta)\vartheta(\alpha-\beta,\xi)d\beta - b(\alpha)C\theta_0''(\xi)\Bigg)\, d\alpha d\xi d\tau.
\end{aligned} \tag{3.2}$$

Using conditions (2.9), from the above equation we obtain the integral equations of the second order with respect to unknown functions $a'(t)$, $b(t)$:

$$\begin{aligned}
a'(t) =& \frac{1}{\Delta}\Big(\theta_0''(x_1)(\Psi_t(t,x_0) - a(0)\psi_0'(t) - \psi_0''(t)) - \theta_0''(x_0)(\Psi_t(t,x_1) - a(0)\psi_1'(t) - \psi_1''(t))\Big) \\
&+ \frac{1}{\Delta}\int_0^t \int_0^l \Big(\theta_0''(x_1)G_t(t-\tau,x_0,\xi) - \theta_0''(x_0)G_t(t-\tau,x_1,\xi)\Big)\Big(b(\tau)\theta_0''(\xi) - a(0)\vartheta(\tau,\xi) \\
&- a'(\tau)\theta_0(\xi)\Big)\, d\xi d\tau - \frac{1}{\Delta}\Bigg(\theta_0''(x_1)\int_0^t a'(\tau)\psi_1'(t-\tau)\, d\tau - \theta_0''(x_0)\int_0^t a'(\tau)\psi_0'(t-\tau)\, d\tau\Bigg) \\
&- \frac{1}{\Delta}\int_0^t \int_0^l \Big(\theta_0''(x_1)G_t(t-\tau,x_0,\xi) - \theta_0''(x_0)G_t(t-\tau,x_1,\xi)\Big)\int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,\xi)\, d\alpha d\xi d\tau \\
&+ \frac{1}{\Delta}\int_0^t R(t-\tau)\Big(\theta_0''(x_1)F(\vartheta(\tau,x_0),\vartheta_\tau(\tau,x_0),a'(\tau),b(\tau)) \\
&- \theta_0''(x_0)F(\vartheta(\tau,x_1),\vartheta_\tau(\tau,x_1),a'(\tau),b(\tau))\Big)\, d\tau \\
&+ \frac{1}{\Delta}\int_0^t \int_0^l \Big(\theta_0''(x_1)G_t(t-\tau,x_0,\xi) - \theta_0''(x_0)G_t(t-\tau,x_1,\xi)\Big) \\
&\cdot \int_0^\tau R(\tau-\alpha)F(\vartheta(\alpha,\xi),\vartheta_\alpha(\alpha,\xi),a'(\alpha),b(\alpha))\, d\alpha d\xi d\tau,
\end{aligned} \tag{3.3}$$

where $\Delta = \theta_0(x_0)\theta_0''(x_1) - \theta_0(x_1)\theta_0''(x_0)$,

$$\begin{aligned}
b(t) =& \frac{1}{C\Delta}\Big(\theta_0(x_1)(\Psi_t(t,x_0) - a(0)\psi_0'(t) - \psi_0''(t)) - \theta_0(x_0)(\Psi_t(t,x_1) - a(0)\psi_1'(t) - \psi_1''(t))\Big) \\
&+ \frac{1}{C\Delta}\int_0^t\int_0^l \Big(\theta_0(x_1)G_t(t-\tau,x_0,\xi) - \theta_0(x_0)G_t(t-\tau,x_1,\xi)\Big)\Big(b(\tau)\theta_0''(\xi) - a(0)\vartheta(\tau,\xi) \\
&- a'(\tau)\theta_0(\xi)\Big)d\xi d\tau - \frac{1}{C\Delta}\left(\theta_0(x_1)\int_0^t a'(\tau)\psi_1'(t-\tau)d\tau - \theta_0(x_0)\int_0^t a'(\tau)\psi_0'(t-\tau)\,d\tau\right) \\
&- \frac{1}{C\Delta}\int_0^t\int_0^l \Big(\theta_0(x_1)G_t(t-\tau,x_0,\xi) - \theta_0(x_0)G_t(t-\tau,x_1,\xi)\Big)\int_0^\tau a'(\alpha)\vartheta(\tau-\alpha,\xi)\,d\alpha d\xi d\tau \\
&- \frac{1}{C\Delta}\int_0^t R(t-\tau)\big(\theta_0(x_1)F(\vartheta(\tau,x_0),\vartheta_\tau(\tau,x_0),a'(\tau),b(\tau)) \\
&- \theta_0(x_0)F(\vartheta(\tau,x_1),\vartheta_\tau(\tau,x_1),a'(\tau),b(\tau))\Big)\,d\tau \\
&+ \frac{1}{C\Delta}\int_0^t\int_0^l \Big(\theta_0(x_1)G_t(t-\tau,x_0,\xi) - \theta_0(x_0)G_t(t-\tau,x_1,\xi)\Big) \\
&\cdot\int_0^\tau R(\tau-\alpha)F(\vartheta(\alpha,\xi),\vartheta_\alpha(\alpha,\xi),a'(\alpha),b(\alpha))\,d\alpha d\xi d\tau.
\end{aligned} \tag{3.4}$$

4. Main result and its proof

The main result of this work is a theorem on existence and uniqueness of the solution to integral equations (2.10), (3.1), (3.2), (3.3), (3.4).

Theorem 4.1. *Assume that*

$$\begin{aligned}
&\theta_0(x) \in C^2[0,l], \qquad \psi_0(t),\,\psi_1(t) \in C^2[0;T], \qquad \mu_i(t) \in C^2[0,T], \quad i=1,2, \\
&\theta_0(x_0) = \psi_1(0), \qquad \theta_0(x_1) = \psi_2(0), \qquad \Delta \neq 0, \qquad \theta_0(0) = \mu_1(0), \qquad \theta_0(l) = \mu_2(0).
\end{aligned}$$

Then there exists a sufficiently small number $T^* \in (0,T)$ that integral equations (2.10), (3.1), (3.2), (3.3), (3.4) are uniquely solvable in the class of functions $\vartheta(t,x) \in C^{1,2}(D_{T^*})$, $a(t) \in C^2[0,T^*]$, $b(t) \in C^1[0;T^*]$, $D_{T^*} = \Big\{(x,t)|x \in (0,l), t \in (0,T^*]\Big\}$.

Proof. We represent the system of equations (2.10)and (3.1)-(3.4) in the form

$$Ah = h, \tag{4.1}$$

where

$$\begin{aligned}
h =& \Big(h_1,h_2,h_3,h_4,h_5\Big) \\
=& \Big(\vartheta(t,x), \vartheta_t(t,x) - C\theta_0''(x)b(t) + a(0)\vartheta(t,x) + a'(t)\theta_0(x), a'(t),\; b(t), R(t) + b(t)\Big)
\end{aligned}$$

is the vector-function and the unknown functions are represented by functions h_1, h_2, h_3, h_4, h_5 as follows:

$$\vartheta(t,x)=h_1(t),\qquad \vartheta_t(t,x)=h_2(t,x)+C\theta_0''(x)h_4(t)-a(0)h_1(t,x)-\theta_0(x)h_3(t),$$
$$a'(t)=h_3(t),\qquad b(t)=h_4(t),\qquad R(t)=h_5(t)-h_4(t).$$

The operator $A=(A_1,A_2,A_3,A_4,A_5)$ is defined as

$$\begin{aligned}
A_1h=&h_{01}+\int_0^t\int_0^l G(t-\tau,x,\xi)\Big(Ch_4(\tau)\theta_0''(\xi)-a(0)h_1(\tau,\xi)-h_3(\tau)\theta_0(\xi)\Big)d\xi d\tau\\
&-\int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau h_3(\alpha)h_1(\tau-\alpha,\xi)d\alpha d\xi d\tau+\int_0^t\int_0^l G(t-\tau,x,\xi)\\
&\cdot\int_0^\tau(h_5(\tau-\alpha)-h_4(\tau-\alpha))\left(h_2(\alpha,\xi)-f_\alpha(\alpha,\xi)+\int_0^t h_3(\beta)h_1(\alpha-\beta,\xi)d\beta\right)d\alpha d\xi d\tau,
\end{aligned}$$

$$\begin{aligned}
A_2h=&h_{02}+\int_0^t\int_0^l G_t(t-\tau,x,\xi)\Big(Ch_4(\tau)\theta_0''(\xi)-a(0)h_1(\tau,\xi)-h_3(\tau)\theta_0(\xi)\Big)\,d\xi d\tau\\
&-\int_0^t h_3(\tau)h_1(t-\tau,x)d\tau-\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau h_3(\alpha)h_1(\tau-\alpha,\xi)d\alpha d\xi d\tau\\
&-\int_0^t[h_5(t-\tau)-h_4(t-\tau)]\Big(h_2(\tau,x)-f_\tau(\tau,x)+\int_0^\tau h_3(\alpha)h_1(\tau-\alpha,x)d\alpha\Big)\,d\tau\\
&-\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau\Big(h_5(\tau-\alpha)-h_4(\tau-\alpha)\Big)\\
&\cdot\left(h_2(\alpha,\xi)-f_\alpha(\alpha,\xi)+\int_0^\alpha h_3(\beta)h_1(\alpha-\beta,\xi)d\beta\right)d\alpha d\xi d\tau,
\end{aligned}$$

$$\begin{aligned}
A_jh=&h_{0j}+\frac{1}{C^{j-3}\Delta}\int_0^t\int_0^l\Big(\theta_0^{(8-2j)}(x_1)G_t(t-\tau,x_0,\xi)-\theta_0^{(8-2j)}(x_0)G_t(t-\tau,x_1,\xi)\Big)\\
&\cdot\Big(h_4(\tau)\theta_0''(\xi)-a(0)h_1(\tau,\xi)-h_3(\tau)\theta_0(\xi)\Big)\,d\xi d\tau\\
&-\frac{1}{C^{j-3}\Delta}(\theta_0^{(8-2j)}(x_1)\int_0^t h_3(\tau)\psi_1'(t-\tau)d\tau-\theta_0^{(8-2j)}(x_0)\int_0^t h_3(\tau)\psi_0'(t-\tau)d\tau)\\
&-\frac{1}{C^{j-3}\Delta}\int_0^t\int_0^l\Big(\theta_0^{(8-2j)}(x_1)G_t(t-\tau,x_0,\xi)-\theta_0^{(8-2j)}(x_0)G_t(t-\tau,x_1,\xi)\Big)\\
&\times\int_0^\tau h_3(\alpha)h_1(\tau-\alpha,\xi)\,d\alpha d\xi d\tau+\frac{\theta_0^{(8-2j)}(x_1)}{\Delta}\int_0^t\big(h_5(t-\tau)-h_4(t-\tau)\big)
\end{aligned}$$

$$\cdot\left(\psi_0''(\tau)-f_\tau(\tau,x_0)+a(0)\psi_0'(\tau)+h_3(\tau)\theta_0^{(8-2j)}(x_0)+\int\limits_0^\tau h_3(\alpha)\psi_0'(\tau-\alpha)\,d\alpha\right)d\tau$$

$$-\frac{\theta_0^{(8-2j)}(x_0)}{C^{j-3}\Delta}\int\limits_0^t\big(h_5(t-\tau)-h_4(t-\tau)\big)\Big(\psi_1''(\tau)-f_\tau(\tau,x_1)+a(0)\psi_1'(\tau)$$

$$+h_3(\tau)\theta_0^{(8-2j)}(x_1)+\int\limits_0^\tau h_3(\alpha)\psi_1'(\tau-\alpha)\,d\alpha\Big)d\tau$$

$$+\frac{1}{C^{j-3}\Delta}\int\limits_0^t\int\limits_0^l\big(\theta_0^{(8-2j)}(x_1)G_t(t-\tau,x_0,\xi)-\theta_0^{(8-2j)}(x_0)G_t(t-\tau,x_1,\xi)\big)$$

$$\cdot\int\limits_0^\tau\big(h_5(\tau-\alpha)-h_4(\tau-\alpha)\big)\left(h_2(\alpha,\xi)-f_\alpha(\alpha,\xi)+\int\limits_0^\alpha h_3(\beta)h_1(\alpha-\beta,\xi)\,d\beta\right)d\alpha d\xi d\tau,$$

here $\theta_0^{(8-2j)}(x_i)$ is the value of the $(8-2j)$th derivative of the function $\theta_0(x)$ at the points $x=x_i$, $j=3,4$, $i=0,1$;

$$A_5h=h_{05}+\int\limits_0^t h_5(t-\tau)h_4(\tau)\,d\tau+\int\limits_0^t h_4(t-\tau)h_4(\tau)\,d\tau.$$

Denote

$$\begin{aligned}h_0(t,x):=&\Big(h_{01}(t,x),h_{02}(t,x),h_{03}(t),h_{04}(t),h_{05}(t)\Big)\\ :=&\Big(\Psi(t,x),\Psi_t(t,x),\frac{1}{C^{j-3}\Delta}\big(\theta_0^{(8-2j)}(x_1)(\Psi_t(t,x_0)-a(0)\psi_0'(t)-\psi_0''(t))\\ &-\theta_0^{(8-2j)}(x_0)(\Psi_t(t,x_1)-a(0)\psi_1'(t)-\psi_1''(t))\big),(j=3,4),0\Big).\end{aligned}$$

We also introduce the class of all real-valued vector functions continuous in the domain $\overline{D}_T$ with values in $\mathbb{R}^5$; we denote this space by $C\big(\overline{D}_T,\mathbb{R}^5\big)$. The norm on this space is introduced as

$$\|h\|=\max\Big\{\max_{(x,t)\in\overline{D}_T}|h_i(x,t)|,\ i=1,2;\ \ \max_{t\in[0,T]}|h_j(t)|,\ j=3,4,5\Big\},$$

It is clear that the operator A acts from the space $C\big(\overline{D}_T,\ \mathbb{R}^5\big)$ into itself.

We observe that

$$\|h_0\|=\max\Big\{\max_{(x,t)\in\overline{D}_T}|h_{0i}(x,t)|,\ i=1,2;\ \ \max_{t\in[0,T]}|h_{0j}(t)|,\ j=3,4,5\Big\}.$$

By $S(h_0,\|h_0\|)$ we denote the ball of vector-functions $h\in C\big(\overline{D}_T,\mathbb{R}^5\big)$ with center at the point h_0 and radius $\|h_0\|$, that is,

$$S(h_0,\|h_0\|)=\Big\{h:\|h-h_0\|\leqslant\|h_0\|\Big\}\subset C\big(\overline{D}_T,\mathbb{R}^5\big).$$

It is clear that $\|h\|\leqslant 2\|h_0\|$ for $h(x,t)\in S(h_0,\|h_0\|)$. We are going to prove that the operator A is contracting in the Banach space $S(h_0,\|h_0\|)$ if the number T is chosen in a suitable way.

We begin by checking the first condition of contractive mapping, see [38], for the operator A. We let

$$M_0:=\|\theta_0\|_{C^2[0,l]},\qquad M_1:=\|f\|_{C^1(\overline{D}_T)},\qquad M_2:=\{\|\psi_i\|_{C^2[0,T]},\,i=0,1\}.$$

Let $h(x,t)$ be an element in $S(h_0,\|h_0\|)$, that is, $h(x,t)\in S(h_0,\|h_0\|)$. Then for $(x,t)\in D_T$ we have the estimates

$$
\begin{aligned}
\|A_1h-h_{01}\| &= \max_{(x,t)\in\overline{D}_T}|(A_1h-h_{01})| \\
&\leqslant \max_{(x,t)\in\overline{D}_T}\left|\int_0^t\int_0^l G(t-\tau,x,\xi)\Big(Ch_4(\tau)\theta_0''(\xi)-a(0)h_1(\tau,\xi)-h_3(\tau)\theta_0(\xi)\Big)d\xi d\tau\right| \\
&\quad+\max_{(x,t)\in\overline{D}_T}\left|\int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau h_3(\alpha)h_1(\tau-\alpha,\xi)d\alpha d\xi d\tau\right| \\
&\quad+\max_{(x,t)\in\overline{D}_T}\left|\int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau (h_5(\tau-\alpha)-h_4(\tau-\alpha))[h_2(\alpha,\xi)-f_\alpha(\alpha,\xi)\right. \\
&\quad\left.+\int_0^\alpha h_3(\beta)h_1(\alpha-\beta,\xi)d\beta]d\alpha d\xi d\tau\right| \\
&\leqslant\Big(2T(CM_0+a(0)+M_0)+(6\|h_0\|+4M_1)T^2+\frac{4\|h_0\|^2T^3}{3}\Big)\|h_0\|, \\
\|A_2h-h_{02}\| &= \max_{(x,t)\in\overline{D}_T}\big|(A_2h-h_{02})\big|\leqslant\left(\frac{16\|h_0\|^2T^3}{3l}+\left(8\|h_0\|^2+\frac{4M_1+12\|h_0\|}{l}\right)T^2\right. \\
&\quad\left.+\left(\frac{4CM_0+4a(0)+4C}{l}+12\|h_0\|+4M_1\right)T\right)\|h_0\|, \\
\|A_jh-h_{0j}\| &\leqslant\frac{8M_0}{C^{j-3}l\Delta}\Big(\Big(2M_0+a(0)+1,5lM_2+lM_1+la(0)M_2+2M_0\|h_0\|\Big)T \\
&\quad+\Big(3\|h_0\|+M_1+2lM_2\|h_0\|\Big)T^2+4\|h_0\|^2T^3\Big)\|h_0\|,\qquad j=3,4, \\
\|A_5h-h_{05}\| &\leqslant 8\|h_0\|^2T.
\end{aligned}
$$

As a result we conclude that if T satisfies the inequalities

$$
\begin{aligned}
&2T(CM_0+a(0)+M_0)+(6\|h_0\|+4M_1)T^2+\frac{4\|h_0\|^2T^3}{3}\leqslant 1, \\
&\frac{16\|h_0\|^2T^3}{3l}+\left(8\|h_0\|^2+\frac{4M_1+12\|h_0\|}{l}\right)T^2 \\
&\qquad+\left(\frac{4CM_0+4a(0)+4C}{l}+12\|h_0\|+4M_1\right)T\leqslant 1, \\
&\frac{8M_0}{C^{j-3}l\Delta}\Big(\big(2M_0+a(0)+1,5lM_2+lM_1+la(0)M_2+2M_0\|h_0\|\big)T \\
&\qquad+\Big(3\|h_0\|+M_1+2lM_2\|h_0\|\Big)T^2+4\|h_0\|^2T^3\Big)\leqslant 1,\quad j=3,4, \\
&8\|h_0\|T\leqslant 1,
\end{aligned}
\tag{4.2}
$$

then operator A maps $S(h_0,\|h_0\|)$ into itself, that is, $Ah\in S(h_0,\|h_0\|)$.

We proceed to checking the second condition of contractive mapping. In accordance with (3.1) for the first component of operator A we have:

$$\|(Ah^1-Ah^2)_1\| = \max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)C\Big[h_4^1(\tau)-h_4^2(\tau)\Big]\theta_0''(\xi)d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)a(0)\big(h_1^1(\tau,\xi)-h_1^2(\tau,\xi)\big)\,d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|g\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\big(h_3^1(\tau)-h_3^2(\tau)\big)\theta_0(\xi)\,d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\int\limits_0^\tau\big(h_3^1(\alpha)h_1^1(\tau-\alpha,\xi)-h_3^2(\alpha)h_1^2(\tau-\alpha,\xi)\big)\,d\alpha d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\int\limits_0^\tau\big(h_5^1(\tau-\alpha)h_2^1(\alpha,\xi)-h_5^2(\tau-\alpha)h_2^2(\alpha,\xi)\big)\,d\alpha d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\int\limits_0^\tau\big(h_5^1(\tau-\alpha)-h_5^2(\tau-\alpha)\big)f_\alpha(\alpha,\xi)\,d\alpha d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\Bigg(\int\limits_0^\tau h_5^1(\tau-\alpha)\int\limits_0^\alpha h_3^1(\beta)h_1^1(\alpha-\beta,\xi)\,d\beta d\alpha\right.$$

$$\left.-\int\limits_0^\tau h_5^2(\tau-\alpha)\int\limits_0^\alpha h_3^2(\beta)h_1^2(\alpha-\beta,\xi)\,d\beta d\alpha]d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\int\limits_0^\tau\big(h_4^1(\tau-\alpha)h_2^1(\alpha,\xi)-h_4^2(\tau-\alpha)h_2^2(\alpha,\xi)\big)\,d\alpha d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\int\limits_0^\tau\big(h_4^1(\tau-\alpha)-h_4^2(\tau-\alpha)\big)f_\alpha(\alpha,\xi)\,d\alpha d\xi d\tau\right|$$

$$+\max_{(x,t)\in\overline{D}_T}\left|\int\limits_0^t\int\limits_0^l G(t-\tau,x,\xi)\Bigg(\int\limits_0^\tau h_4^1(\tau-\alpha)\int\limits_0^\alpha h_3^1(\beta)h_1^1(\alpha-\beta,\xi)\,d\beta d\alpha\right.$$

$$\left.-\int\limits_0^\tau h_4^2(\tau-\alpha)\int\limits_0^\alpha h_3^2(\beta)h_1^2(\alpha-\beta,\xi)d\beta d\alpha\Bigg)\,d\xi d\tau\right|.$$

To estimate the integrands, we use the following inequalities:

$$\|h_2^1h_1^1-h_2^2h_1^2\| = \|(h_2^1-h_2^2)h_1^1+h_2^2(h_1^1-h_1^2)\|$$
$$\leqslant 2\|h^1-h^2\|\max\big(\|h_1^1\|,\|h_2^2\|\big)\leqslant 4\|h_0\|\|h^1-h^2\|,$$
$$\|h_1^1h_2^1h_3^1-h_1^2h_2^2h_3^2\| = \|(h_1^1-h_1^2)h_2^1h_3^1+(h_2^1-h_2^2)h_1^2h_3^1+(h_3^1-h_3^2)h_1^2h_2^2\|$$

$$\leqslant 3\|h^1-h^2\|\max\left(\|h_3^1h_2^1\|,\|h_1^2h_3^1\|,\|h_1^2h_2^2\|\right)\leqslant 12\|h_0\|^2\|h^1-h^2\|.$$

Therefore,

$$\begin{aligned}\|(Ah^1-Ah^2)_1\|\leqslant&\big(CM_0+a(0)+M_0\big)T\|h^1-h^2\|\\&+(10\|h_0\|+M_1)T^2\|h^1-h^2\|+4\|h_0\|^2T^3\|h^1-h^2\|.\end{aligned}$$

The other components can be estimated in a similar way:

$$\begin{aligned}\|(Ah^1-Ah^2)_2\|\leqslant&\left(\left(\frac{2}{l}CM_0+\frac{2}{l}a(0)+\frac{2}{l}M_0+12\|h_0\|+2f_0\right)T\right.\\&\left.+\left(\frac{12\|h_0\|}{l}+6\|h_0\|^2+\frac{2M_1}{l}\right)T^2+4\|h_0\|^2T^3\right)\|h^1-h^2\|,\\\|(Ah^1-Ah^2)_j\|\leqslant&\frac{4M_0}{C^{j-3}\Delta}\Big((2M_0+a(0)+M_1+a(0)M_2+4M_0\|h_0\|)T\\&+(6\|h_0\|+2M_2\|h_0\|+M_1)T^2+8\|h_0\|^2T^3\Big)\|h^1-h^2\|,\qquad j=3,4,\\\|(Ah^1-Ah^2)_5\|\leqslant&\,8\|h_0\|T\|h^1-h^2\|.\end{aligned}$$

If T satisfies the inequalities

$$\begin{aligned}&\Big(CM_0+a(0)+M_0\Big)T+\Big(10\|h_0\|+M_1\Big)T^2+4\|h_0\|^2T^3<1,\\&\left(\frac{2}{l}CM_0+\frac{2}{l}a(0)+\frac{2}{l}M_0+12\|h_0\|+2f_0\right)T\\&+\left(\frac{12\|h_0\|}{l}+6\|h_0\|^2+\frac{2M_1}{l}\right)T^2+4\|h_0\|^2T^3<1,\\&\frac{4M_0}{C^{j-3}\Delta}\Big((2M_0+a(0)+M_1+a(0)M_2+4M_0\|h_0\|)T\\&+(6\|h_0\|+2M_2\|h_0\|+M_1)T^2+8\|h_0\|^2T^3\Big)<1,\\&8\|h_0\|T\leqslant 1,\end{aligned}\tag{4.3}$$

then the operator A satisfies the second condition of contracting mapping.

Therefore, if the number T is small enough to ensure inequalities (4.2) and (4.3), then A is a contraction operator on $S(h_0,\|h_0\|)$ and by the Banach principle, the equation $h=Ah$ has a unique solution in $S(h_0,\|h_0\|)$.

Since h involves ϑ and ϑ_t, this implies that ϑ is differentiable in t. By the proved theorem this implies that all functions in identity (2.6) are continuous except for ϑ_{xx}. Since system of equations (2.6), (2.7), (2.8), (2.9) is equivalent to operator equation (4.1), then ϑ_{xx} is continuous, i.e. ϑ is twice continuously differentiable in x. Thus, $\vartheta(t,x)\in C^{1,2}(D_T)$. The proof is complete. □

By the known functions $a(t)$, $b(t)$, solving the differential equations

$$a(t)=\frac{\chi'(t)}{\chi(0)},\qquad b(t)=\frac{k'(t)}{k(0)},$$

we find the functions

$$\chi(t)=\chi(0)+\chi(0)\int_0^t a(\tau)d\tau,\qquad k(t)=k(0)+k(0)\int_0^t b(\tau)d\tau$$

as solutions to inverse problem (2.2-(2.5).

5. Conclusion

In this work, inverse problem is considered for determining the kernels $\chi(t)$ and $k(t)$ included in the system of equations (1.3) by simple observation (2.5) at the points $x_0, x_1 \in (0, l)$ of the solution of this system with initial and boundary conditions (2.2), (2.3). We obtain conditions for given functions, under which the inverse problem has unique solutions for a sufficiently small time interval. We note that global solvability of this kind of problems is an open issue.

Acknowledgements

The authors express their profound thanks to the anonymous reviewers for their insightful comments and helpful suggestions.

BIBLIOGRAPHY

1. M.E. Gurtin, A.C. Pipkin. *A general theory of heat conduction with finite wave speeds* // Arch. Rational Mech. Anal. **31**:2, 113–126 (1968).
2. R.K. Miller. *An integro-differential equation for rigid heat conductors with memory* // J. Math. Anal. Appl. **66**:2, 313–332 (1978).
3. M.E. Gurtin. *On the thermodynamics of materials with memory* // Arch. Rational Mech. Anal. **28**:1, 40–50 (1968).
4. B.D. Coleman , M.E. Gurtin. *Equipresense and constitutive equation for rigid heat conductors* // Z. Angew. Math. Phys. **18**:2, 199–208 (1967).
5. H. Grabmueller. *Linear Theorie der Waermeleitung in Medium mit Gedaechtnis; Existenz und Eindeutigkeit von Loesungen sum Inversen Problem* // Technische Hochschule Darmstdadt, Preprint 226 (1975).
6. G.V. Dyatlov. *Determination for the memory kernel from boundary measurements on a finite time interval* // J. Inverse Ill-Posed Probl. **11**:1, 59–66 (2003).
7. D.K. Durdiev, Zh.D. Totieva. *The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type* // J. Inverse Ill-Posed Probl. **28**:1, 43–52 (2020).
8. D.K. Durdiev, Zh.Sh. Safarov. *Inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded domain* // Math. Notes. **97**:6, 867–877 (2015).
9. D.K. Durdiev, Zh.D. Totieva. *Problem of determining one-dimensional kernel of viscoelasticity equation* // Sib. Zh. Ind. Mat. **16**:2, 72–82 (2013).
10. . Zh.D. Totieva. *The problem of determining the piezoelectric module of electro visco-elasticity equation* // Math. Meth. Appl. Sci. **41**:17, 6409–6321 (2018).
11. A. Lorenzi, E. Paparoni. *Direct and inverse problems in the theory of materials with memory* // Rend. Semin. Math. Univ. Padova. **87**, 105–138 (1992).
12. A. Lorenzi, V. Priymenko. *A duality approach for solving identification problems related to integro-differential Maxwell's equations* // Rend. Semin. Math. Univ. Padova. **94**, 31–51 (1994).
13. A. Lorenzi, F. Messina, V.G. Romanov. *Recovering a Lame kernel in a viscoelastic system* // Appl. Anal. **86**:11, 1375–1395 (2007).
14. D.K. Durdiev, A.A. Rakhmonov. *Inverse problem for a system of integro-differential equations for SH waves in a visco-elastic porous medium: global solvability* // Theor. Math. Phys. **195**:3, 923–937 (2018).
15. V.G. Romanov. *Stability estimates for the solution to the problem of determining the kernel of a viscoelastic equation* // J. Appl. Ind. Math. **6**:4, 360–370 (2012).
16. V.G. Romanov, A. Lorenzi. *Stability estimates for an inverse problem related to viscoelasticmedia* // J. Inverse Ill-Posed Probl. **14**:1, 57–82 (2006).
17. V.G. Romanov, A. Lorenzi. *Identification of an electromagnetic coefficient connected with deformation currents* // Inverse Probl. **9**:2, 301–319 (1993).

18. V.G. Romanov. *Problem of determining the permittivity in the stationary system of Maxwell equations* // Dokl. Math. **95**:3, 230–234 (2017).
19. A.L. Bukhgein, G.V. Dyatlov. *Uhlmann, Unique continuation for hyperbolic equations with memory* // J. Inverse Ill-Posed Probl. **15**:6, 587–598 (2007).
20. D.K. Durdiev, A.A. Rahmonov. *The problem of determining the 2D-kernel in a system of integro-differential equations of a viscoelastic porous medium* // J. Appl. Ind. Math. **14**:2, 281–295 (2020).
21. V.G. Romanov. *Inverse problems for equation with a memory* // Eurasian J Math. Comput. Appl. **2**:4, 51–80 (2014).
22. A. Lorenzi. *An identification problem related to a nonlinear hyperbolic integro-differential equation* // Nonl. Anal. Theory, Meth. Appl. **22**:1, 21–44 (1994).
23. D.K.Durdiev. *Global solvability of an inverse problem for an integro-differential equation of electrodynamics* // Diff. Equat. **44**:2, 893–899 (2008).
24. D.K. Durdiev , A.A. Rahmonov. *A 2D kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity* // Math. Meth. Appl. Sci. **43**:15, 8776–8796 (2020).
25. D.K. Durdiev, Zh.D. Totieva. *Problem of determining the multidimensional kernel of viscoelasticity equation* // Vladikavkaz. Mat. Zh. **17**:4, 18–43(2015).
26. U.D.Durdiev , Z.D.Totieva. *A problem of determining a special spatial part of 3D memory kernel in an integro-differential hyperbolic equation* // Math. Meth. Appl. Sci. **42**:18, 7440–7451 (2019).
27. J. Janno, L.V. Wolfersdorf. *Inverse problems for identification of memory kernels in heat flow* // J. Inverse Ill-Posed Probl. **4**:1, 39–66 (1996).
28. J. Janno, A. Lorenzi. *Recovering memory kernels in parabolic transmission problems* // J. Inverse Ill-Posed Probl. **16**:3, 239–265 (2008).
29. F. Colombo. *A inverse problem for a parabolic integro-differential model in the theory of combustion* // Physica D. **236**:2, 81–89 (2007).
30. D. Serikbaev. *Inverse problem for fractional order pseudo-parabolic equation with involution* // Ufa Math. J. **12**:4, 119–135 (2020).
31. A. Gladkov, M. Guedda. *Influence of variable coeffcients on global existence of solutions of semilinear heat equations with nonlinear boundary conditions* // Elect. J. Qualitative Theory Diff. Equat. **2020**, id 63 (2020).
32. D.K. Durdiev, Zh.Zh. Zhumaev. *One-dimensional inverse problems of finding the kernel of the integro-differential heat equation in a bounded domain* // Ukrains'kyi Matematychnyi Zhurnal. **73**:11, 1492–1506 (2021).
33. D.K. Durdiev, A.Sh. Rashidov. *Inverse problem of determining the kernel in an integro-differential equation of parabolic type* // Diff. Equat. **50**:1, 110–116 (2014).
34. D.K. Durdiev, Zh.Zh. Zhumaev. *Problem of determining a multidimensional thermal memory in a heat conductivity equation* // Meth. Funct. Anal. Topology. **25**:3, 219–226 (2019).
35. D.K. Durdiev , Zh.Zh. Zhumaev. *Problem of Determining the Thermal Memory of a Conducting Medium* // Diff. Equat. **56**:6, 785–796 (2020).
36. D. Durdiev, E. Shishkina, S. Sitnik. *The Explicit Formula for Solution of Anomalous Diffusion Equation in the Multi-Dimensional Space* // Lobachevskii J. Math. **42**:6, 1264–1273 (2021).
37. A.A. Kilbas. *Integral equations: course of lectures.* Belorus State Univ., Minsk (2005) (in Russian).
38. A.N. Kolmogorov, S.V. Fomin. *Introductory real analysis.* Nauka, Moscow (1972). [Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1970).]
39. A.N. Tikhonov, A.A. Samarsky. *Equations of mathematical physics.* Nauka, Moscow (1977). [Pergamon Press, Oxford (1963).]

Durdimurod Kalandarovich Durdiev,
Bukhara branch of the Institute of Mathematics
named after V.I. Romanovskiy,
Academy of Sciences of the Republic of Uzbekistan,
M. Ikbal Str. 11,
200100, Bukhara, Uzbekistan,
Bukhara State University,
M. Ikbal Str. 11 ,
200100, Bukhara, Uzbekistan
E-mail: d.durdiev@mathinst.ru

Jonibek Jamolovich Jumaev,
Bukhara branch of the Institute of Mathematics
named after V.I. Romanovskiy,
Academy of Sciences of the Republic of Uzbekistan,
M. Ikbal Str. 11,
200100, Bukhara, Uzbekistan,
Bukhara State University,
M. Ikbal Str. 11,
200100, Bukhara, Uzbekistan
E-mail: jonibekjj@mail.ru

Dilshod Dilmurodovich Atoev,
Bukhara State University,
M. Ikbal Str. 11 ,
200100, Bukhara, Uzbekistan
E-mail: atoevdd@mail.ru