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ON SPACE OF HOLOMORPHIC FUNCTIONS WITH BOUNDARY SMOOTHNESS AND ITS DUAL

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Abstract. We consider a Fréchet-Schwartz space $A_{\mathcal{H}}(\Omega)$ of functions holomorphic in a bounded convex domain Ω in a multidimensional complex space and smooth up to the boundary with the topology defined by means of a countable family of norms. These norms are constructed via some family $\mathcal{H}$ of convex separately radial weight functions in $\mathbb{R}^n$. We study the problem on describing a strong dual space for this space in terms of the Laplace transforms of functionals. An interest to such problem is motivated by the researches by B.A. Derzhavets devoted to classical problems of theory of linear differential operators with constant coefficients and the researches by A.V. Abanin, S.V. Petrov and K.P. Isaev of modern problems of the theory of absolutely representing systems in various spaces of holomorphic functions with given boundary smoothness in convex domains in complex space; these problems were solved by Paley-Wiener-Schwartz type theorems. Our main result states that the Laplace transform is an isomorphism between the strong dual of our functional space and some space of entire functions of exponential type in $\mathbb{C}^n$, which is an inductive limit of weighted Banach spaces of entire functions. This result generalizes the corresponding result of the second author in 2020. To prove this theorem, we apply the scheme proposed by M. Neymark and B.A. Taylor. On the base of results from monograph by L. Hörmander (L. Hörmander. *An Introduction to complex analysis in several variables*, North Holland, Amsterdam (1990)), a problem of solvability of systems of partial differential equations in $A^m_{\mathcal{H}}(\Omega)$ is considered. An analogue of a similar result from monograph by L. Hörmander is obtained. In this case we employ essentially the properties of the Young-Fenchel transform of functions in the family $\mathcal{H}$.

Keywords: Laplace transform, entire functions.

1. Introduction

1. Problem. Let Ω be a bounded convex domain in $\mathbb{C}^n$, $A_c(\Omega)$ be the space of functions holomorphic in Ω and continuous on the closure $\overline{\Omega}$ of the domain Ω. For each $m \in \mathbb{Z}_+$ by $A_c^{(m)}(\Omega)$ we denote the space of holomorphic in Ω functions f, the partial derivatives of which $(D^\alpha f)(z) := \dfrac{\partial^{|\alpha|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$ (if $\alpha = (0,0,\ldots,0)$, then $(D^\alpha f)(z) := f(z)$) up to the order m can be continuously continued on $\overline{\Omega}$. Thus, $A_c^{(0)}(\Omega) = A_c(\Omega)$. We equip the space $A_c^{(m)}(\Omega)$ with the norm

$$q_m(f) = \sup_{z\in\Omega, |\alpha|\leqslant m} |(D^\alpha f)(z)|.$$

Let $A^\infty(\Omega) = \lim\limits_{m\to\infty} \mathrm{pr}\ A_c^{(m)}(\Omega)$.

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Let $\mathcal{H} = \{h_m\}_{m=1}^{\infty}$ be the family of convex functions $h_m : \mathbb{R}^n \to [0, \infty)$ with $h_m(0) = 0$ such that for each $m \in \mathbb{N}$:

i_1) $h_m(x) = h_m(|x_1|, \dots, |x_n|), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$;

i_2) there exist numbers $a_m > 0$ such that

$$h_m(x) \geqslant \|x\| \ln(1 + \|x\|) - a_m\|x\| - a_m, \qquad x \in \mathbb{R}^n;$$

i_3) $\lim\limits_{x\to\infty} (h_m(x) - h_{m+1}(x)) = +\infty$;

i_4) $\sup\limits_{\alpha\in\mathbb{Z}_+^n} (h_{m+1}(\alpha + \beta) - h_m(\alpha)) < \infty$ for $\beta \in \mathbb{Z}_+^n$ with $|\beta| = 1$;

i_5) for each $p \in \mathbb{N}$ there exists a number $l = l(m, p) \in \mathbb{N}$ such that

$$\sum_{\alpha\in\mathbb{Z}_+^n} \exp(\max_{|\beta|\leqslant p} h_{m+l}(\alpha + \beta) - h_m(\alpha)) < \infty.$$

We define the space $A_{\mathcal{H}}(\Omega)$ of functions on Ω as a projective limit of normed spaces

$$A_m(\Omega) = \{f \in A^\infty(\Omega) : p_m(f) = \sup_{z\in\Omega,\alpha\in\mathbb{Z}_+^n} \frac{|(D^\alpha f)(z)|}{e^{h_m(\alpha)}} < \infty\}, \quad m \in \mathbb{N}.$$

In view of Condition i_3), the space $A_{m+1}(\Omega)$ is continuously embedded into $A_m(\Omega)$ for each $m \in \mathbb{N}$. It is clear that $A_{\mathcal{H}}(\Omega)$ is a Frechét space continuously embedded into $A^\infty(\Omega)$, and by Condition i_4), it is invariant with respect to the differentiation.

The spaces of holomorphic functions with a boundary smoothness naturally arise in studying many problems in complex analysis, operator theory, approximation theory [1]–[9]. It is clear that for each $z \in \mathbb{C}^n$ the function $f_z(\lambda) = e^{\langle\lambda,z\rangle}$ belongs to $A^\infty(\Omega)$. We also have $f_z \in A_{\mathcal{H}}(\Omega)$, see Lemma 2.1. This is why for each linear continuous functional Φ on $A^\infty(\Omega)$ $(A_{\mathcal{H}}(\Omega))$ in $\mathbb{C}^n$ the function $\hat{\Phi}(z) = \Phi(e^{\langle\lambda,z\rangle})$ is well-defined. It is called the Laplace transform of the functional Φ. In this work we study the issue on describing a strong dual space $A^*_{\mathcal{H}}(\Omega)$ for the space $A_{\mathcal{H}}(\Omega)$ in terms of the Laplace transform of the functionals. The resolving of this problem will allow us to proceed to studying classical problems in theory of linear differential operators with constant coefficients in the considered space as well as modern problems in the theory of absolutely representing systems. Our main Theorem 1.1 states that the Laplace transform of the linear continuous functionals is an isomorphism between the strong dual space to the considered functional space and some space of entire functions of exponential type in $\mathbb{C}^n$ being an internal inductive limit of weighted Banach spaces of entire functions.

We note that the problem on describing a strong dual space for various spaces of holomorphic functions with a boundary smoothness were considered by B.A. Derzhavets [2], A.V. Abanin and S.V. Petrov [10], S.V. Pertrov [11], K.P. Isaev [12] and by the second co-author of this work in [5], [13]. In particular, in [13] this problem was studied for the case, when the family $\mathcal{H}$ consists of the functions h_m defined by the rule:

$$h_m : x = (x_1, \dots, x_n) \in \mathbb{R}^n \to h\left(\sum_{j=1}^n |x_j| - m\right) \quad \text{under the condition} \quad \sum_{j=1}^n |x_j| > m;$$

$h_m(x) = 0$ if $\sum\limits_{j=1}^n |x_j| \leqslant m$, $m \in \mathbb{N}$, where $h : \mathbb{R} \to [0, \infty)$ is a convex function satisfying $h(0) = 0$ and such that

1) $h(t) = h(|t|),\ t \in \mathbb{R}$;
2) h is non-decreasing on $[0, \infty)$;
3) there exists a number $a > 0$ such that $h(t) \geqslant t \ln(t + 1) - at - a$, $t \geqslant 0$.

It is easy to confirm that in this particular case Conditions $i_1) - i_4)$ are satisfied. Condition i_5) is satisfied, too. Indeed, we introduce a logarithmically convex sequence $(M_k)_{k=0}^{\infty}$ by letting

$M_k = e^{h(k)}$. It is known [13] that for each natural number $s \geqslant n+1$ the series $\sum\limits_{|\alpha|\geqslant 0} \frac{M_{|\alpha|}}{M_{|\alpha|+s}}$ converges. Then for $m, p, l \in \mathbb{N}$ and $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geqslant m+l$ we have

$$\max_{|\beta|\leqslant p} h_{m+l}(\alpha+\beta) - h_m(\alpha) = h(|\alpha| + p - m - l) - h(|\alpha| - m).$$

Therefore,

$$\exp(\max_{|\beta|\leqslant p} h_{m+l}(\alpha+\beta) - h_m(\alpha)) = \frac{M_{|\alpha|+p-m-l}}{M_{|\alpha|-m}}.$$

Thus, if $l \geqslant p+n+1$, then for each $m \in \mathbb{N}$ the series

$$\sum_{\alpha\in\mathbb{Z}_+^n} \exp(\max_{|\beta|\leqslant p} h_{m+l}(\alpha+\beta) - h_m(\alpha))$$

converges.

We also note that if for each $m \in \mathbb{N}$ the restriction of h_m to $[0,\infty)^n$ is non-decreasing in each variable, then Condition i_5) can be replaced by the following one:

i_5') for each $m, \nu \in \mathbb{N}$ there exists a number $l = l(m,\nu) \in \mathbb{N}$ such that

$$\sum_{\alpha\in\mathbb{Z}_+^n} \exp(h_{m+l}(\alpha+\nu\gamma) - h_m(\alpha)) < \infty,$$

where $\gamma = (1,\ldots,1) \in \mathbb{Z}_+^n$.

Thus, the problem we consider here is more general than that in [13].

2. Notation and definitions. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}_+^n$, $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ we let $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}$.

For $u = (u_1,\ldots,u_n)$, $v = (v_1,\ldots,v_n) \in \mathbb{C}^n$, we denote

$$\langle u,v\rangle = u_1v_1 + \ldots + u_nv_n, \qquad \|u\| = \sqrt{|u_1|^2 + \cdots + |u_n|^2}.$$

By λ_m we denote the Lebesgue measure in $\mathbb{C}^m$.

The space of functions holomorphic in a domain $\mathcal{O} \subseteq \mathbb{C}^n$ with the topology of uniform convergence on compact subsets in $\mathcal{O}$ is denoted by $A(\mathcal{O})$.

The symbol $A'_{\mathcal{H}}(\Omega)$ stands for the space of linear continuous functionals on $A_{\mathcal{H}}(\Omega)$, while $A^\infty(\Omega))^*$ and $A_c^*(\Omega)$ are respectively strongly dual spaces for the spaces $A^\infty(\Omega)$ and $A_c(\Omega)$.

By $H_\Omega(z) = \sup\limits_{\lambda\in\Omega} Re\,\langle\lambda, z\rangle$, $z \in \mathbb{C}^n$ we denote a support function function of the domain Ω.

We let $\ln^+ t = \ln t$ for $t > 1$ and $\ln^+ t = 0$ for $0 \leqslant t \leqslant 1$.

We denote $\mathbb{R}_+^n = \{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_1 \geqslant 0, \ldots, x_n \geqslant 0\}$.

The Young-Fenchel transform of the function $g : \mathbb{R}^n \to [-\infty,+\infty]$ is a function $g^* : \mathbb{R}^n \to [-\infty,+\infty]$ defined by the formula

$$g^*(x) = \sup_{y\in\mathbb{R}^n} (\langle x,y\rangle - g(y)), \qquad x \in \mathbb{R}^n.$$

3. Main result and structure of work. For each $m \in \mathbb{N}$ we define a function φ_m in $\mathbb{C}^n$ by the rule

$$\varphi_m(z) = h_m^*(\ln^+|z_1|,\ldots,\ln^+|z_n|), \quad z = (z_1,\ldots,z_n) \in \mathbb{C}^n.$$

Since h_m^* is a convex function in $\mathbb{R}^n$ with finite values, then the function h_m^* is continuous in $\mathbb{R}^n$. Therefore, φ_m is a continuous pluri-subharmonic function in $\mathbb{C}^n$.

For each $m \in \mathbb{N}$ we define a normed space

$$P_m = \left\{ F \in H(\mathbb{C}^n) : \|F\|_m = \sup_{z\in\mathbb{C}^n} \frac{|F(z)|}{\exp(H_\Omega(z) + \varphi_m(z))} < \infty \right\}.$$

It is clear that the Banach space P_m is continuously embedded into P_{m+1} for each $m \in \mathbb{N}$. By $P_{\mathcal{H}}$ we denote an inductive limit of the spaces P_m.

Theorem 1.1. *The mapping $L : T \in A^*_{\mathcal{H}}(\Omega) \to \hat{T}$ is a topological isomorphism between the spaces $A^*_{\mathcal{H}}(\Omega)$ and $P_{\mathcal{H}}$.*

The proof of the theorem is based on the ideas by M. Neymark [14] and B.A. Taylor [15] and it uses also a series of results from [5] and [16], which are provided in Section 3. In Section 2 we establish some useful properties of the spaces $A_{\mathcal{H}}(\Omega)$, $A^*_{\mathcal{H}}(\Omega)$, $P_{\mathcal{H}}$ and of the family of the functions φ_m. In Section 4 we give one more application of Theorem 1.1 concerning the theory of partial differential equations with constant coefficients. Namely, we establish an analogue of Theorem 7.6.13 from [23].

2. Auxiliary statements

Proposition 2.1. *Let a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying $g(0) = 0$ be such that for some $a > 0$*

$$g(x) \geqslant \|x\| \ln(\|x\| + 1) - a\|x\| - a, \quad x \in \mathbb{R}^n.$$

Then for each $x \in \mathbb{R}^n$ the supremum of the function $g_x(\xi) = \langle \xi, x\rangle - g(\xi)$ in $\mathbb{R}^n$ is attained at some point $\xi^* = \xi^*(x)$ such that $\|\xi^*\| \leqslant e^{2a}e^{\|x\|}$.

Proof. We first mention that by the condition for g we have

$$0 \leqslant g^*(x) \leqslant \sup_{\xi\in\mathbb{R}^n} (\|\xi\|(\|x\| - \ln(1 + \|\xi\|) + a) + a), \quad x \in \mathbb{R}^n.$$

This implies that for each $x \in \mathbb{R}^n$ the supremum of the function g_x is attained at some point $\xi^* = \xi^*(x)$ such that $\|\xi^*\| \leqslant e^{2a}e^{\|x\|}$. Indeed, if we assume that $\|\xi^*\| > e^{2a}e^{\|x\|}$, then

$$\sup_{\xi\in\mathbb{R}^n} (\|\xi\|(\|x\| - \ln(1 + \|\xi\|) + a) + a) = \|\xi^*\|(\|x\| - \ln(1 + \|\xi^*\|) + a) + a$$
$$< -a\|\xi^*\| + a < 0.$$

But this contradicts to the inequality $g^*(x) \geqslant 0$ for all $x \in \mathbb{R}^n$. □

Proposition 2.2. *Let a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying $g(0) = 0$ be such that for some $a > 0$*

$$g(x) \geqslant \|x\| \ln(\|x\| + 1) - a\|x\| - a, \qquad x \in \mathbb{R}^n.$$

Let $b > 0$.

Then for all $x, y \in \mathbb{R}^n$ such that $\|y - x\| \leqslant be^{-\|x\|}$ we have

$$|g^*(y) - g^*(x)| \leqslant be^{2a+b}.$$

Proof. Employing Proposition 2.1 and its notation, for all $x, y \in \mathbb{R}^n$ such that $\|y - x\| \leqslant be^{-\|x\|}$ we have:

$$\begin{aligned} g^*(y) - g^*(x) &= \sup_{\xi\in\mathbb{R}^n} (\langle y, \xi\rangle - g(\xi)) - \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - g(\xi)) \\ &\leqslant \langle y, \xi^*(y)\rangle - g(\xi^*(y))) - \langle x, \xi^*(y)\rangle + g(\xi^*(y))) = \langle y - x, \xi^*(y)\rangle \\ &\leqslant \|y - x\|\|\xi^*(y)\| \leqslant be^{-\|x\|}e^{2a}e^{\|y\|} \leqslant be^{2a+b}. \end{aligned}$$

On the other hand,

$$\begin{aligned} g^*(x) - g^*(y) &= \sup_{\xi\in\mathbb{R}^n} (\langle x, \xi\rangle - g(\xi)) - \sup_{\xi\in\mathbb{R}^n} (\langle y, \xi\rangle - g(\xi)) \\ &\leqslant \langle x, \xi^*(x)\rangle - g(\xi^*(x)) - \langle y, \xi^*(x)\rangle + g(\xi^*(x)) = \langle x - y, \xi^*(x)\rangle \\ &\leqslant \|x - y\|\|\xi^*(x)\| \leqslant be^{-\|x\|}e^{2a}e^{\|x\|} \leqslant be^{2a+b}. \end{aligned}$$

□

Proposition 2.2 implies the following corollary.

Corollary 2.1. *Let a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying $g(0) = 0$ be such that for some $a > 0$*

$$g(x) \geqslant \|x\| \ln(\|x\| + 1) - a\|x\| - a, \quad x \in \mathbb{R}^n.$$

Let $b > 0$.

Then for all $x = (x_1, \dots, x_n)$, $y \in \mathbb{R}^n$, such that $\|y - x\| \leqslant be^{-(|x_1|+\cdots+|x_n|)}$ we have

$$|g^*(y) - g^*(x)| \leqslant be^{2a+b}.$$

Proposition 2.3. *Let a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying $g(0) = 0$ be such that for some $a > 0$*

$$g(x) \geqslant \|x\| \ln(\|x\| + 1) - a\|x\| - a, \quad x \in \mathbb{R}^n.$$

Then for all $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ such that

$$|y_j - x_j| \leqslant \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)}, \qquad j = 1, \dots, n,$$

we have:

$$|g^*(\ln^+ |y_1|, \dots, \ln^+ |y_n|) - g^*(\ln^+ |x_1|, \dots, \ln^+ |x_n|)| \leqslant 2ne^{2a+2n}.$$

Proof. Let the points $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ satisfy the assumptions of our statement. We first consider the case when $|x_j| \geqslant 1$, $|y_j| \geqslant 1$ for $j \in \{1, \dots, n\}$. Then

$$|\ln^+ |y_j| - \ln^+ |x_j|| = \left|\ln \frac{|y_j|}{|x_j|}\right| .$$

If $\frac{|y_j|}{|x_j|} \geqslant 1$, then

$$\left|\ln \frac{|y_j|}{|x_j|}\right| = \ln \frac{|y_j|}{|x_j|} = \ln \left(1 + \frac{|y_j| - |x_j|}{|x_j|}\right) \leqslant \frac{|y_j| - |x_j|}{|x_j|} \leqslant \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)}. \tag{2.1}$$

If $\frac{|y_j|}{|x_j|} < 1$ and since $|y_j| \geqslant |x_j| - \frac{1}{\prod_{k=1}^{n} (1+|x_k|)}$, we hence obtain that

$$\ln \frac{|y_j|}{|x_j|} = \ln \left(1 + \frac{|y_j| - |x_j|}{|x_j|}\right) \geqslant \ln \left(1 - \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)}\right).$$

Therefore,

$$\left|\ln \frac{|y_j|}{|x_j|}\right| \leqslant \left|\ln \left(1 - \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)}\right)\right| \leqslant \frac{2}{\prod\limits_{k=1}^{n} (1 + |x_k|)}. \tag{2.2}$$

Thus, if $|x_j| \geqslant 1$, $|y_j| \geqslant 1$ for $j \in \{1, \dots, n\}$, then owing to inequalities (2.1) and (2.2) we have

$$|\ln^+ |y_j| - \ln^+ |x_j|| \leqslant \frac{2}{\prod\limits_{k=1}^{n} (1 + |x_k|)}. \tag{2.3}$$

We proceed to the second case: $|x_j| < 1$ or $|y_j| < 1$ for $j \in \{1, \dots, n\}$. For the sake of definiteness we suppose that $|y_j| < 1$. Then

$$|\ln^+ |y_j| - \ln^+ |x_j|| \leqslant \ln \left(1 + \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)} \right) \leqslant \frac{1}{\prod\limits_{k=1}^{n} (1 + |x_k|)} . \tag{2.4}$$

Employing inequalities (2.3) and (2.4), we obtain that

$$\begin{aligned} \|(\ln^+ |y_1|, \dots, \ln^+ |y_n|) - (\ln^+ |x_1|, \dots, \ln^+ |x_n|)\| &\leqslant \sum_{j=1}^{n} |\ln^+ |y_j| - \ln^+ |x_j|| \\ &\leqslant \frac{2n}{\prod\limits_{k=1}^{n} (1 + |x_k|)} \\ &= 2ne^{-(\ln(1+|x_1|)+\dots+\ln(1+|x_n|))} \\ &\leqslant 2ne^{-(\ln^+ |x_1|+\dots+\ln^+ |x_n|)}. \end{aligned}$$

Now by Corollary 2.1 we obtain that

$$|g^*(\ln^+ |y_1|, \dots, \ln^+ |y_n|) - g^*(\ln^+ |x_1|, \dots, \ln^+ |x_n|)| \leqslant 2ne^{2a+2n}.$$

□

By Proposition 2.3 we obtain the following corollaries.

Corollary 2.2. *Let a function $g : \mathbb{R}^n \to \mathbb{R}$ satisfies the assumptions of Proposition 2.3. Let $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ be such that*

$$|\zeta_j - z_j| \leqslant \frac{1}{\prod\limits_{k=1}^{n} (1 + |z_k|)} , \qquad j = 1, \dots, n.$$

Then

$$|g^*(\ln^+ |\zeta_1|, \dots, \ln^+ |\zeta_n|) - g^*(\ln^+ |z_1|, \dots, \ln^+ |z_n|)| \leqslant 2ne^{2a+2n}.$$

Corollary 2.3. *Let $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ be such that*

$$|\zeta_j - z_j| \leqslant \frac{1}{\prod\limits_{k=1}^{n} (1 + |z_k|)}, \quad j = 1, \dots, n.$$

Then

$$|\varphi_m(\zeta) - \varphi_m(z)| \leqslant 2ne^{2a_m+2n}$$

for each $m \in \mathbb{N}$.

Corollary 2.4. *Let $z, \zeta \in \mathbb{C}^n$ be such that*

$$\|\zeta - z\| \leqslant \frac{1}{(1 + \|z\|)^n}.$$

Then

$$|\varphi_m(\zeta) - \varphi_m(z)| \leqslant 2ne^{2a_m+2n}$$

for each $m \in \mathbb{N}$.

Proposition 2.4. *For each $m \in \mathbb{N}$ there exists a constant $l_m > 0$ such that*

$$h^*_{m+n}(x) \geqslant h^*_m(x) + \sum_{j=1}^{n} x_j - l_m, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n_+.$$

Proof. Let

$$\gamma = (1,\dots,1) \in \mathbb{Z}_+^n, \qquad Y_\gamma = \{y = (y_1,\dots,y_n) \in \mathbb{R}^n : |y_1| \geqslant 1,\dots,|y_n| \geqslant 1\}.$$

Then for each $x \in \mathbb{R}_+^n$

$$\begin{aligned} h^*_{m+n}(x) &= \sup_{y\in\mathbb{R}^n} (\langle x,y\rangle - h_{m+n}(y)) = \sup_{y\in\mathbb{R}_+^n} (\langle x,y\rangle - h_{m+n}(y)) \\ &\geqslant \sup_{y\in Y_\gamma} (\langle x,y\rangle - h_{m+n}(y)) = \sup_{y\in\mathbb{R}_+^n} (\langle x,y+\gamma\rangle - h_{m+n}(y+\gamma)) \\ &= \langle x,\gamma\rangle + \sup_{y\in\mathbb{R}_+^n} (\langle x,y\rangle - h_m(y) + h_m(y) - h_{m+n}(y+\gamma)). \end{aligned}$$

Now, employing Condition i_4) on $\mathcal{H}$, for some $l_m > 0$ we have:

$$\begin{aligned} h^*_{m+n}(x) &\geqslant \langle x,\gamma\rangle + \sup_{y\in\mathbb{R}_+^n} (\langle x,y\rangle - h_m(y)) - l_m \\ &= \langle x,\gamma\rangle + \sup_{y\in\mathbb{R}^n} (\langle x,y\rangle - h_m(y)) - l_m = h^*_m(x) + \langle x,\gamma\rangle - l_m. \end{aligned}$$

□

Corollary 2.5. *For each $m \in \mathbb{N}$*

$$\varphi_{m+n}(z) \geqslant \varphi_m(z) + \sum_{j=1}^n \ln^+ |z_j| - l_m, \quad z = (z_1,\dots,z_n) \in \mathbb{C}^n,$$

where l_m is a constant from Proposition 2.4.

Proposition 2.5. *Let numbers $m \in \mathbb{N}$, $c > 0$ be such that*

$$|S(f)| \leqslant c p_m(f), \quad f \in A_{\mathcal{H}}(\Omega)$$

for $S \in A^*_{\mathcal{H}}(\Omega)$

Then the functional S can be represented as

$$S(f) = \sum_{|\alpha|\geqslant 0} S_\alpha(D^\alpha f), \quad f \in A_{\mathcal{H}}(\Omega),$$

where $S_\alpha \in A^*_c(\Omega)$, and the norms $\|S_\alpha\|_{A^*_c(\Omega)}$ of the functionals S_α obey the inequality

$$\|S_\alpha\|_{A^*_c(\Omega)} \leqslant \frac{c}{e^{h_m(\alpha)}}, \ \alpha \in \mathbb{Z}_+^n.$$

The proof of this proposition follows a standard scheme [15, Prop. 2.11, Cor. 2.12] with using of Condition i_3).

Lemma 2.1. *For each $z \in \mathbb{C}^n$ the function $f_z(\lambda) = \exp(\langle\lambda,z\rangle)$ belongs to $A_{\mathcal{H}}(\Omega)$ and*

$$p_m(f_z) \leqslant \exp(H_\Omega(z) + \varphi_m(z)) < \infty \tag{2.5}$$

for each $m \in \mathbb{N}$.

Proof. Let $z \in \mathbb{C}^n$. Then, for each $m \in \mathbb{N}$,

$$\begin{aligned} p_m(f_z) &= \sup_{\lambda\in\Omega,\alpha\in\mathbb{Z}_+^n} \frac{|z^\alpha e^{\langle\lambda,z\rangle}|}{e^{h_m(\alpha)}} = \sup_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n} \frac{|z_1|^{\alpha_1}\cdots|z_n|^{\alpha_n}}{e^{h_m(\alpha)}} \cdot e^{H_\Omega(z)} \\ &\leqslant \exp(H_\Omega(z) + \sup_{\alpha\in\mathbb{Z}_+^n} (\alpha_1 \ln^+|z_1| + \cdots + \alpha_n \ln^+|z_n| - h_m(\alpha))) = \exp(H_\Omega(z) + \varphi_m(z)). \end{aligned}$$

Thus, $f_z \in A_{\mathcal{H}}(\Omega)$ and we arrive at the desired inequality. □

Lemma 2.2. *For each $S \in A^*_{\mathcal{H}}(\Omega)$ we have $\hat{S} \in P_{\mathcal{H}}$.*

Proof. For each functional $S \in A^*_{\mathcal{H}}(\Omega)$ the function $\hat{S}(z) = S(e^{\langle \lambda z \rangle})$ is well-defined in $\mathbb{C}^n$. Employing Proposition 2.1 and Condition i_5), it is easy to show that $\hat{S}$ is an entire function in $\mathbb{C}^n$. Since there exist numbers $m \in \mathbb{N}$ and $c > 0$ such that $|S(f)| \leqslant c p_m(f)$ for all $f \in A_{\mathcal{H}}(\Omega)$, then, employing inequality (2.5), we have:

$$|\hat{S}(z)| \leqslant c \exp(H_\Omega(z) + \varphi_m(z)), \quad z \in \mathbb{C}^n.$$

Therefore, $\hat{S} \in P_{\mathcal{H}}$. □

3. Description of space $A^*_{\mathcal{H}}(\Omega)$

3.1. Three auxiliary theorems. For each $m \in \mathbb{Z}_+$ we let

$$E_m = \left\{ F \in H(\mathbb{C}^n) : N_m(F) = \sup_{z \in \mathbb{C}^n} \frac{|F(z)|}{(1 + \|z\|)^m \exp(H_\Omega(z))} < \infty \right\}.$$

By E we denote an inductive limit of the spaces E_m.

Theorem 3.1. *The mapping $\mathcal{L} : S \in (A^\infty(\Omega))^* \to \hat{S}$ is a topological isomorphism between the spaces $(A^\infty(\Omega))^*$ and E.*

Theorem 3.1 was proved in [5, Thm. 1]. Under the assumption that the boundary of the domain Ω is C^2-smooth, Theorem 3.1 was proved by B.A. Derzhavets [2].

The next theorem will be employed in the proof that the mapping L in Theorem 1.1 is surjective. It was proved in [13, Lm. 6.2].

Theorem 3.2. *Let $\mathcal{O}$ be a domain of the holomorphy in $\mathbb{C}^n$, h be a pluri-subgarmonic function in $\mathcal{O}$ and φ be a pluri-subharmonic function in $\mathbb{C}^n$ such that for some $c_\varphi > 0$ and $\nu > 0$*

$$|\varphi(z) - \varphi(t)| \leqslant c_\varphi \quad \text{if} \quad \|z - t\| \leqslant \frac{1}{(1 + \|t\|)^\nu}.$$

Let a function $f \in H(\mathcal{O})$ satisfy the condition

$$\int_{\mathcal{O}} |f(\zeta)|^2 e^{-2(\varphi(\zeta)+h(\zeta))} \, d\lambda_n(\zeta) < \infty.$$

Then there exists a function $F \in H(\mathbb{C}^n \times \mathcal{O})$ such that $F(\zeta, \zeta) = f(\zeta)$ for $\zeta \in \mathcal{O}$ and

$$\int_{\mathbb{C}^n \times \mathcal{O}} \frac{|F(z,\zeta)|^2 e^{-2(\varphi(z)+h(\zeta))}}{(1 + \|(z,\zeta)\|)^{2n(\nu+3)}} \, d\lambda_{2n}(z,\zeta) \leqslant C \int_{\mathcal{O}} |f(\zeta)|^2 e^{-2(\varphi(\zeta)+h(\zeta))} \, d\lambda_n(\zeta),$$

where a positive constant C depends only on n, ν and φ.

The next result was proved in [13, Lm. 6.4]. It will be employed in the proof of the injectivity of the mapping L in Theorem 1.1.

Theorem 3.3. *Let $\mathcal{O}$ be a domain of holomorphy in $\mathbb{C}^n$, h be a pluri-subharmonic function in $\mathcal{O}$ and φ be a pluri-subharmonic function in $\mathbb{C}^n$ such that for some $c_\varphi > 0$ and $\nu > 0$*

$$|\varphi(z) - \varphi(t)| \leqslant c_\varphi \quad \text{if} \quad \|z - t\| \leqslant \frac{1}{(1 + \|t\|)^\nu}.$$

Let a function $S \in H(\mathbb{C}^n \times \mathcal{O})$ satisfies the inequality

$$|S(z,\zeta)| \leqslant e^{\varphi(z)+h(\zeta)}, \quad z \in \mathbb{C}^n, \quad \zeta \in \mathcal{O},$$

and $S(\zeta,\zeta) = 0$ for $\zeta \in \mathcal{O}$. Then there exist functions $S_1, \ldots, S_n \in H(\mathbb{C}^n \times \mathcal{O})$ such that

a) $S(z,\zeta) = \sum_{j=1}^{n} S_j(z,\zeta)(z_j - \zeta_j), \quad (z,\zeta) \in \mathbb{C}^n \times \mathcal{O};$

b) for some $m > 0$ independent of S,

$$\int\limits_{\mathbb{C}^n \times \mathcal{O}} \frac{|S_j(z,\zeta)|^2}{e^{2(\varphi(z)+h(\zeta)+m\ln(1+\|(z,\zeta)\|))}}\, d\lambda_{2n}(z,\zeta) < \infty, \quad j = 1, \ldots, n.$$

3.2. Properties of spaces $A_{\mathcal{H}}(\Omega)$, $A^*_{\mathcal{H}}(\Omega)$ and $P_{\mathcal{H}}$.

Definition 3.1. *([17]) A space, which can be represented as a projective limit of a sequence of normed spaces S_n, $n \in \mathbb{N}$, with respect to linear continuous mappings $g_{mn} : S_n \to S_m$, $m < n$, such that $g_{n,n+1}$ is completely continuous for each n is called space (M^*).*

Proposition 3.1. *The space $A_{\mathcal{H}}(\Omega)$ is space (M^*).*

The proof Proposition 3.1 is the same as that of Lemma 6 in [5]. The only change is that at an appropriate place one needs to employ Condition i_3) instead of Condition i_4) in [5]. Thus, the Frechét space $A_{\mathcal{H}}(\Omega)$ is a Frechét-Schwarz space [18].

Definition 3.2. *([18], [19]) Let $(E_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces such that E_m is continuously embedded into E_{m+1} for each $m \in \mathbb{N}$ and $E = \cup_{m\in\mathbb{N}} E_m$. If for each $m \in \mathbb{N}$ there exists $k > m$ such that the embedding of E_m into E_k is compact, then the inductive limit of the spaces $E := \varinjlim E_m$ is called (DFS)-space.*

By means of Montel theorem and Corollary 2.5 one can show easily that the embeddings

$$j_m : P_m \to P_{m+n}(T_C)$$

are compact for each $m \in \mathbb{N}$. Therefore, $P_{\mathcal{H}}$ is a (DFS)-space.

The space $A^*_{\mathcal{H}}(\Omega)$, as a strong dual space to the Frechét-Schwartz space $A_{\mathcal{H}}(\Omega)$ is (DFS)-space [18], [19].

3.3. Proof of Theorem 1.1. According to Lemma 2.2, a linear mapping $L : T \in A^*_{\mathcal{H}}(\Omega) \to \hat{T}$ acts from $A^*_{\mathcal{H}}(\Omega)$ into $P_{\mathcal{H}}$.

The mapping L is continuous. This can be shown exactly in the same way as in [5].

Let us show that the mapping L is surjective. Let an entire function F satisfies $F \in P_{\mathcal{H}}$. Then $F \in P_m$ for some $m \in \mathbb{N}$. Therefore,

$$\int\limits_{\mathbb{C}^n} \frac{|F(\zeta)|^2 e^{-2H_\Omega(\zeta)+\varphi_m(\zeta))}}{(1+\|\zeta\|)^{2n+1}}\, d\lambda_n(\zeta) < \infty.$$

Hence, by Corollary 2.5, we obtain that

$$\int\limits_{\mathbb{C}^n} |F(\zeta)|^2 e^{-2(H_\Omega(\zeta)+\varphi_{m+n(n+1)}(\zeta))}\, d\lambda_n(\zeta) < \infty.$$

We note that the functions H_Ω and $\varphi_{m+n(n+1)}$ are pluri-subharomnic in $\mathbb{C}^n$ and for some $C_\Omega > 0$ we have:

$$|H_\Omega(u) - H_\Omega(v)| \leqslant C_\Omega, \qquad u, v \in \mathbb{C}^n : \|u - v\| \leqslant 1. \tag{3.1}$$

This is why, applying Theorem 3.2 with $\nu = 1$, we find a function $\Phi \in H(\mathbb{C}^{2n})$ such that $\Phi(z,z) = F(z)$ for $z \in \mathbb{C}^n$ and, for some $c > 0$ independent of F,

$$\int\limits_{\mathbb{C}^{2n}} \frac{|\Phi(z,\zeta)|^2 e^{-2(H_K(Im\, z)+\varphi_{m+n(n+1)}(\zeta))}}{(1+\|(z,\zeta)\|)^{8n}}\, d\lambda_{2n}(z,\zeta) \leqslant c \int\limits_{\mathbb{C}^n} |F(\zeta)|^2 e^{-2(H_\Omega(\zeta)+\varphi_{m+n(n+1)}(\zeta))}\, d\lambda_n(\zeta).$$

Since $|\Phi|^2 \in psh(\mathbb{C}^{2n})$, then

$$|\Phi(z,\zeta)|^2 \leqslant \frac{1}{\lambda_{2n}(R)} \int\limits_{B_R(z,\zeta)} |\Phi(t,u)|^2\, d\lambda_{2n}(t,u), \quad z, \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n,$$

for each $R > 0$, where $B_R(z,\zeta)$ is a closed ball in $\mathbb{C}^{2n}$ of radius R centered at the point (z,ζ), $\lambda_{2n}(R)$ is the volume of the ball $B_R(z,\zeta)$. Letting $R = \frac{1}{\prod\limits_{k=1}^{n}(1+|\zeta_k|)}$, by this inequality, (3.1) and Corollary 2.3, in a standard way we obtain an uniform estimate for $|\Phi(z,\zeta)|$ and for some $c_1 > 0$ we obtain:

$$|\Phi(z,\zeta)| \leqslant c_1(1+\|z\|)^{4n}\prod_{k=1}^{n}(1+|\zeta_k|)^{6n}e^{H_\Omega(z)+\varphi_{m+n(n+1)}(\zeta)}, \quad (z,\zeta)\in\mathbb{C}^{2n},$$

where $c_1 > 0$ is some constant. Then, using Corollary 2.5, we get that, for some $c_2 > 0$,

$$|\Phi(z,\zeta)| \leqslant c_2(1+\|z\|)^{4n}e^{H_\Omega(z)+\varphi_{m+7n^2+n}(\zeta)}, \quad (z,\zeta)\in\mathbb{C}^{2n}. \tag{3.2}$$

We expand $\Phi(z,\zeta)$ into a series in powers of ζ: $\Phi(z,\zeta) = \sum\limits_{|\alpha|\geqslant 0}\Phi_\alpha(z)\zeta^\alpha$. By the Cauchy formula for all $\alpha\in\mathbb{Z}_+^n$, $r_1 > 0,\dots,r_n > 0$ we have:

$$C_\alpha(z) = \frac{1}{(2\pi i)^n}\int\limits_{|\zeta_1|=r_1}\cdots\int\limits_{|\zeta_n|=r_n}\frac{\Phi(z,\zeta)}{\zeta_1^{\alpha_1+1}\dots\zeta_n^{\alpha_n+1}}\,d\zeta_1\dots d\zeta_n, \quad z\in\mathbb{C}^n.$$

This implies that $C_\alpha\in H(\mathbb{C}^n)$. Letting $\nu = 7n^2+n$, employing inequality (3.2) and the non-decreasing of the function $\varphi_{m+\nu}$ in each variable, we get:

$$|C_\alpha(z)| \leqslant \frac{c_2(1+\|z\|)^{4n}e^{H_\Omega(z)+\varphi_{m+\nu}(r)}}{r^\alpha}, \qquad z\in\mathbb{C}^n,$$

where all components $r=(r_1,\dots,r_n)\in\mathbb{R}^n$ are positive. Therefore,

$$|C_\alpha(z)| \leqslant c_2(1+\|z\|)^{4n}e^{H_\Omega(z)}\left(\inf_{r\in\Pi_1}\frac{e^{\varphi_{m+\nu}(r)}}{r^\alpha}\right), \quad z\in\mathbb{C}^n,$$

where $\Pi_1 = \{(r_1,\dots,r_n)\in\mathbb{R}^n : r_1\geqslant 1,\dots,r_n\geqslant 1\}$. We note that

$$\begin{aligned}\inf_{r\in\Pi_1}\frac{e^{\varphi_{m+\nu}(r)}}{r^\alpha} &= e^{\inf\limits_{r\in\Pi_1}(\varphi_{m+\nu}(r)-\sum\limits_{j=1}^{n}\alpha_j\ln r_j)}\\ &= e^{-\sup\limits_{\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}_+^n}(\langle\alpha,\xi\rangle-h^*_{m+\nu}(\ln^+(e^{\xi_1}),\dots,\ln^+(e^{\xi_n})))}\\ &= e^{-\sup\limits_{\xi\in\mathbb{R}^n}(\langle\alpha,\xi\rangle-h^*_{m+\nu}(\xi))} = e^{-h_{m+\nu}(\alpha)}.\end{aligned}$$

Thus, for all $\alpha\in\mathbb{Z}_+^n$ the estimate holds:

$$|C_\alpha(z)| \leqslant \frac{c_2(1+\|z\|)^{4n}e^{H_\Omega(z))}}{e^{h_{m+\nu}(\alpha)}}, \qquad z\in\mathbb{C}^n.$$

Therefore, the set $\left\{e^{h_{m+\nu}(\alpha)}C_\alpha\right\}_{\alpha\in\mathbb{Z}_+^n}$ is bounded in E_{4n}, and hence, in E. Since the spaces $(A^\infty(\Omega))^*$ and E are isomorphic by Theorem 3.1, there exist functionals $S_\alpha\in(A^\infty(\Omega))^*$ such that $\hat{S_\alpha} = C_\alpha$ and the set $\mathcal{A} = \left\{e^{h_{m+\nu}(\alpha)}S_\alpha\right\}_{\alpha\in\mathbb{Z}_+^n}$ is bounded in $(A^\infty(\Omega))^*$. As in [13], this implies that there exist numbers $l\in\mathbb{Z}_+$ and $c_3 > 0$ such that

$$|S_\alpha(f)| \leqslant \frac{c_3 q_l(f)}{e^{h_{m+\nu}(\alpha)}}, \quad f\in A^\infty(\Omega), \tag{3.3}$$

for each $\alpha\in\mathbb{Z}_+^n$. We define a functional T on $A_{\mathcal{H}}(\Omega)$ by the law

$$T(f) = \sum_{|\alpha|\geqslant 0}S_\alpha(D^\alpha f), \quad f\in A_{\mathcal{H}}(\Omega). \tag{3.4}$$

Let us show that T is a linear continuous functional on $A_{\mathcal{H}}(\Omega)$. Employing inequality (3.3), for each $f \in A_{\mathcal{H}}(\Omega)$, $\alpha \in \mathbb{Z}_+^n$ and $s \in \mathbb{N}$ we have:

$$|S_\alpha(D^\alpha f)| \leqslant \frac{c_3 \sup\limits_{z\in\Omega, |\beta|\leqslant l} |(D^{\alpha+\beta} f)(z)|}{e^{h_{m+\nu}(\alpha)}} \leqslant c_3 p_{m+s}(f) e^{\max\limits_{|\beta|\leqslant l} h_{m+s}(\alpha+\beta) - h_{m+\nu}(\alpha)}.$$

Using Condition i_5), we choose a natural number $s > \nu$ such that the series

$$\sum_{|\alpha|\geqslant 0} e^{\max\limits_{|\beta|\leqslant l} h_{m+s}(\alpha+\beta) - h_{m+\nu}(\alpha)}$$

converges. Then for each $f \in A_{\mathcal{H}}(\Omega)$ the series in the right hand side in (3.4) converges absolutely. And for a chosen s there exists a constant $c_4 > 0$ independent of $f \in A_{\mathcal{H}}(\Omega)$ such that $|T(f)| \leqslant c_4 p_{m+s}(f)$. Therefore, a linear functional T is well-defined and is continuous. It is obvious that $\hat{T} = F$. Thus, the mapping L is surjective.

Following the scheme in [15], let us show that mapping L is bijective. Indeed, let for $T \in A'_{\mathcal{H}}(\Omega)$ we have $\hat{T} \equiv 0$. We are going to show that $T(f) = 0$ for each $f \in A_{\mathcal{H}}(\Omega)$. Since T is a linear continuous functional on $A_{\mathcal{H}}(\Omega)$, then there exist numbers $m \in \mathbb{N}$, $c_5 > 0$ such that

$$|T(f)| \leqslant c_5 p_m(f), \qquad f \in A_{\mathcal{H}}(\Omega).$$

Employing Proposition 2.5, we represent the functional T in the form:

$$T(f) = \sum_{|\alpha|\geqslant 0} T_\alpha(D^\alpha f), \qquad f \in A_{\mathcal{H}}(\Omega),$$

where $T_\alpha \in A_c^*(\Omega)$, and for the norms $\|T_\alpha\|_{A_c^*(\Omega)}$ of the functionals T_α we have:

$$\|T_\alpha\|_{A_c^*(\Omega)} \leqslant \frac{c_5}{e^{h_m(\alpha)}}, \qquad \alpha \in \mathbb{Z}_+^n.$$

This yields: $\hat{T}(z) = \sum_{\alpha\in\mathbb{Z}_+^n} \hat{T}_\alpha(z) z^\alpha$ for each $z \in \mathbb{C}^n$, and entire functions $\hat{T}_\alpha$ satisfy the estimate

$$|\hat{T}_\alpha(z)| \leqslant \frac{c_5 e^{H_\Omega(z)}}{e^{h_m(\alpha)}}, \qquad \alpha \in \mathbb{Z}_+^n, \qquad z \in \mathbb{C}^n. \tag{3.5}$$

We consider an entire function

$$S(u, z) = \sum_{|\alpha|\geqslant 0} \hat{T}_\alpha(z) u^\alpha, \qquad z, u \in \mathbb{C}^n.$$

We note that $S \in H(\mathbb{C}^{2n})$. Employing inequality (3.5) and Condition i_5), we get:

$$|S(u, z)| \leqslant c_5 e^{H_\Omega(z))} \sum_{|\alpha|\geqslant 0} \frac{|u^\alpha|}{e^{h_m(\alpha)}} \leqslant c_6 e^{H_\Omega(z)) + \varphi_{m+l}(u)}$$

where $c_6 = c_5 \sum_{|\alpha|\geqslant 0} e^{h_{m+l}(\alpha) - h_m(\alpha)}$. At that, $S(z, z) = 0$ for each $z \in \mathbb{C}^n$. Then, taking into consideration inequality (3.1), by Theorem 3.3 there exist functions $S_1, \ldots, S_n \in H(\mathbb{C}^{2n})$ such that

$$S(z, \zeta) = \sum_{j=1}^n S_j(z, \zeta)(z_j - \zeta_j), \quad z, \zeta \in \mathbb{C}^n,$$

and for some $k \in \mathbb{N}$

$$\int\limits_{\mathbb{C}^{2n}} \frac{|S_j(z, \zeta)|^2}{e^{2(H_\Omega(z) + \varphi_{m+l}(\zeta) + k \ln(1 + \|(z,\zeta)\|))}}\, d\lambda_{2n}(z, \zeta) < \infty, \qquad j = 1, \ldots, n.$$

As above in obtaining the uniform estimate for $|\Phi(z,\zeta)|$, by the latter integral inequality, for all $j=1,2,\dots,n$ we get:

$$|S_j(z,\zeta)| \leqslant c_7(1+\|z\|)^k \prod_{k=1}^{n}(1+|\zeta_k|)^{k+2n}\exp(H_\Omega(z)+\varphi_{m+l}(\zeta)), \quad z,\zeta\in\mathbb{C}^n,$$

where $c_7>0$ is some constant. By means of Corollary 2.5 this implies that for some $c_8>0$, for all $j=1,2,\dots,n$ we have:

$$|S_j(z,\zeta)| \leqslant c_8(1+\|z\|)^k \exp(H_\Omega(z)+\varphi_{m+l+n(k+2n)}(\zeta)), \quad z,\zeta\in\mathbb{C}^n. \tag{3.6}$$

We denote $l+n(k+2n)$ by q. We expand S_j into a series in powers of ζ:

$$S_j(z,\zeta)=\sum_{|\alpha|\geqslant 0} S_{j,\alpha}(z)\zeta^\alpha, \qquad z,\zeta\in\mathbb{C}^n, \qquad j=1,\dots,n.$$

Proceeding as in estimating the functions C_α, by inequality (3.6) we obtain that for all $\alpha\in\mathbb{Z}_+^n$, $j=1,\dots,n$,

$$|S_{j,\alpha}(z)| \leqslant \frac{c_8(1+\|z\|)^k\exp(H_\Omega(z))}{e^{h_{m+q}(\alpha)}}.$$

By Theorem 3.1 there exist functionals $\psi_{j,\alpha}\in(A^\infty(\Omega))^*$ such that $\hat{\psi}_{j,\alpha}=S_{j,\alpha}$. It follows from the latter estimate that the set $\{S_{j,\alpha}e^{h_{m+q}(\alpha)}\}_{\alpha\in\mathbb{Z}_+^n,j=1,\dots,n}$ is bounded in E. Therefore, the set $\Psi=\{e^{h_{m+q}(\alpha)}\psi_{j,\alpha}\}_{\alpha\in\mathbb{Z}_+^n,j=1,\dots,n}$ is bounded in $(A^\infty(\Omega))^*$. Then there exist numbers $c_9>0$ and $p\in\mathbb{N}$ such that

$$|F(f)|\leqslant c_9 q_p(f), \quad F\in\Psi, \quad f\in A^\infty(\Omega).$$

Hence, for each $f\in A^\infty(\Omega)$,

$$|\Psi_{j,\alpha}(f)| \leqslant \frac{c_9}{e^{h_{m+q}(\alpha)}}q_p(f), \qquad \alpha\in\mathbb{Z}_+^n, \qquad j=1,\dots,n. \tag{3.7}$$

Then, for $\alpha\in\mathbb{Z}^n$ with negative components, $j\in\{1,\dots,n\}$, let $\Psi_{j,\alpha}$ be the zero functional in $(A^\infty(\Omega))^*$, $S_{j,\alpha}$ be a function vanishing identically in $\mathbb{C}^n$. Then

$$S(z,\zeta)=\sum_{j=1}^{n}\sum_{\alpha\in\mathbb{Z}_+^n}(S_{j,\alpha}(z)z_j - S_{j,(\alpha_1,\dots,\alpha_{j-1},\dots,\alpha_n)})\zeta^\alpha, \qquad z,\zeta\in\mathbb{C}^n.$$

Thus,

$$\hat{T}_\alpha(z)=\sum_{j=1}^{n}(S_{j,\alpha}(z)z_j - S_{j,(\alpha_1,\dots,\alpha_{j-1},\dots,\alpha_n)}(z)), \qquad \alpha\in\mathbb{Z}_+^n.$$

In other words,

$$\hat{T}_\alpha(z)=\sum_{j=1}^{n}(\Psi_{j,\alpha}(\frac{\partial}{\partial\lambda_j}(e^{\langle\lambda,z\rangle})) - \hat{\Psi}_{j,(\alpha_1,\dots,\alpha_j-1,\dots,\alpha_n)}(z)), \qquad z\in\mathbb{C}^n.$$

Employing Theorem 3.1, we hence have:

$$T_\alpha(f)=\sum_{j=1}^{n}(\Psi_{j,\alpha}(\frac{\partial}{\partial\lambda_j}f) - \Psi_{j,(\alpha_1,\dots,\alpha_j-1,\dots,\alpha_n)}(f)), \qquad f\in A^\infty(\Omega).$$

This is why

$$T(f)=\sum_{|\alpha|\geqslant 0}\sum_{j=1}^{n}(\Psi_{j,\alpha}(\frac{\partial}{\partial\lambda_j}(D^\alpha f)) - \Psi_{j,(\alpha_1,\dots,\alpha_j-1,\dots,\alpha_n)}(D^\alpha f)), \qquad f\in A_{\mathcal{H}}(\Omega).$$

For $N \in \mathbb{N}$ and $j = 1, \ldots, n$ we define the sets

$$B_N = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \leqslant N, \ldots, \alpha_n \leqslant N\},$$
$$R_{N,j} = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \leqslant N, \ldots, \alpha_j = N, \ldots, \alpha_n \leqslant N\},$$

and we introduce a functional T_N on $A_{\mathcal{H}}(\Omega)$ by the law

$$T_N(f) = \sum_{\alpha \in B_N} \sum_{j=1}^{n} \left(\Psi_{j,\alpha} \left(\frac{\partial}{\partial \lambda_j} (D^\alpha f) \right) - \Psi_{j,(\alpha_1,\ldots,\alpha_j-1,\ldots,\alpha_n)}(D^\alpha f) \right).$$

Then $T(f) = \lim\limits_{N \to \infty} T_N(f)$, $f \in A_{\mathcal{H}}(\Omega)$. We note that

$$T_N(f) = \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} \Psi_{j,\alpha} \left(\frac{\partial}{\partial \lambda_j} (D^\alpha f) \right), \qquad f \in A_{\mathcal{H}}(\Omega).$$

Employing inequality (3.7), for each $f \in A_{\mathcal{H}}(\Omega)$ we have:

$$|T_N(f)| \leqslant \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} |\Psi_{j,\alpha}(\frac{\partial}{\partial \lambda_j}(D^\alpha f))|$$
$$\leqslant c_9 \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} \frac{\sup\limits_{\lambda \in \Omega, |\beta| \leqslant p} |(D^\beta(\frac{\partial}{\partial \lambda_j}(D^\alpha f)))(\lambda)|}{e^{h_{m+q}(\alpha)}}.$$

This yields that for all $\mu \in \mathbb{N}$ and $f \in A_{\mathcal{H}}(\Omega)$ the estimate holds:

$$|T_N(f)| \leqslant c_9 p_\mu(f) \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} e^{\sup\limits_{|\beta| \leqslant p+1} h_\mu(\alpha+\beta) - h_{m+q}(\alpha)}.$$

Now, using Condition i_5), we choose μ so that the series

$$\sum_{|\alpha| \geqslant 0} e^{\sup\limits_{|\beta| \leqslant p+1} h_\mu(\alpha+\beta) - h_{m+q}(\alpha)}$$

converges. Then

$$|T_N(f)| \leqslant n c_9 p_\mu(f) \sum_{|\alpha| \geqslant N} e^{\sup\limits_{|\beta| \leqslant p+1} h_\mu(\alpha+\beta) - h_{m+q}(\alpha)}.$$

Hence, $T_N(f) \to 0$ as $N \to \infty$. Thus, $T(f) = 0$ for each $f \in A_{\mathcal{H}}(\Omega)$. This means that the mapping L is injective.

By the open mapping theorem [20], [21], [22, Thm. 24.30] the mapping L^{-1} is continuous. Thus, L is an injective linear continuous mapping of the space $A^*_{\mathcal{H}}(\Omega)$ onto $P_{\mathcal{H}}$. The proof of Theorem 1.1 is complete.

4. Application of Theorem 1.1 to systems of partial differential equations

Let P_{ij} be polynomials, $i = 1, \ldots, m$, $j = 1, \ldots, r$, and $P = (P_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,r}}}$. We define an operator $\vec{P}(D)$ acting from $A^m_{\mathcal{H}}(\Omega)$ into $A^r_{\mathcal{H}}(\Omega)$ by the rule:

$$\vec{P}(D) \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{m} P_{1j}(D) f_j \\ \vdots \\ \sum\limits_{j=1}^{m} P_{rj}(D) f_j \end{pmatrix}.$$

We consider the set of all vectors $Q = (Q_1, \ldots, Q_r)$ with polynomial components such that

$$P_{1j}(z)Q_1(z) + \cdots P_{rj}(z)Q_r(z) = 0, \qquad j = 1, \ldots, m.$$

It is known that this set, being a module over the ring of polynomials, has finitely many generators. Let $Q^{(l)} = (Q_1^{(l)}, \ldots, Q_r^{(l)})$, $l = 1, \ldots, s$, be its generators.

Theorem 4.1. *The equation $\vec{P}(D)\vec{f} = \vec{g}$ is solvable in $A^m_{\mathcal{H}}(\Omega)$ for $\vec{g} \in A^r_{\mathcal{H}}(\Omega)$ if and only if*

$$\sum_{i=1}^{r} Q_i^{(l)}(D) g_i = 0, \qquad l = 1, \ldots, s.$$

The proof of Theorem 4.1 is based on a theorem by L. Hörmander [23] and is proved by a standard scheme, see, for instance, [24]. We denote by $M[p \times q]$ the set of matrices with p rows and q columns, the entries of which are polynomials.

Theorem 4.2. *For a given system $P \in M[p \times q]$ there exists an integer number N such that for pluri-subharmonic functions φ in $\mathbb{C}^n$ and $-\ln d$ such that*

$$0 < d \leqslant 1, \qquad d(z + \zeta) \leqslant 2d(\zeta)$$

if $|Re z_j| \leqslant 1$, $|Im z_j| \leqslant 1$, $j = 1, \ldots, n$ and

$$|\varphi(z + \zeta) - \varphi(\zeta)| \leqslant C_0,$$

if $|z_j| \leqslant d(\zeta)$, $j = 1, \ldots, n$, and all $u \in (A(\mathbb{C}^n))^q$ there exists $v \in (A(\mathbb{C}^n))^q$ with

$$Pu = Pv, \qquad \int \|v\|^2 e^{-\varphi_N}\, d\lambda \leqslant K \int \|Pu\|^2 e^{-\varphi}\, d\lambda,$$

where

$$\varphi_N(z) = \varphi(z) - N \ln d(z) + N \ln(1 + \|z\|^2),$$

K is independent of u, φ, d.

The proof of Theorem 4.1 follows a standard scheme, see, for instance, [24]. We just note that as the function d we can take the function $d(z) := (R_n + \|z\|)^{-n}$, where R_n is a sufficiently large positive number. Then we can employ both Theorem 4.2 and Corollary 2.4.

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