

LOWER BOUNDS FOR ENTIRE FUNCTIONS

O.A. KRIVOSHEEVA, A.S. KRIVOSHEEV, A.I. RAFIKOV

Abstract. We study lower bounds for entire functions of proximate order and of completely regular growth. We introduce the notion of the index condensation for sequences of complex numbers of proximate order. This notion generalizes that of the index of condensation for sequences of order one. We also introduce a properly balanced set, which is a properly distributed set with a zero condensation index. We show that a regular set is properly balanced and we prove that the properly balanced property of the zero set of an entire function is a necessary and sufficient condition for the existence of family of pairwise disjoint circles with the centers at its zeros and with relatively small radii. Outside these circles, the absolute value of the function admits lower bounds asymptotically coinciding with its upper bounds in the entire plane. Thus, we show that the notion of a properly balanced set naturally generalizes the notion of a regular set in the case of arbitrary sequences including multiples sequences. A method for constructing of an exceptional set consisting of circles with centers at zeroes of entire function is also provided. In some cases, we can make the sum of the radii of these circles arbitrarily small.

Keywords: Entire function, proximate order, completely regular growth, properly balanced set, regular set.

Mathematics Subject Classification: 30D10

1. INTRODUCTION

Let f be an entire function of proximate order $\rho(r)$ and of regular growth [1, Ch. III], $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be its multiple zero set, where n_k stands for the multiplicity of the zero λ_k . In the work, we study lower bounds for the function f , which holds in the entire plane except for a special set of circles centered at the points λ_k .

A classical result by B.Ya. Levin [1, Ch. II, Thm. 2, Ch. III, Thm. 4] established a closed connection between the behavior of entire functions of regular growth f and their zero sets Λ . At that, there were obtained lower bounds for the function $|f|$ outside an exceptional C_0 -set [1, Ch. II, Sect. 1]. By an opinion of B.Ya. Levin [1, Ch. II, Sects. 1, 6], a disadvantage of the exceptional set of the circles is that it is not constructed effectively, namely, it is not said how to construct C_0 -set. Because of this, B.Ya. Levin introduced the notion of a regular set consisting of simple points λ_k (as $n_k = 1$). In this case, it is possible to provide a full description of exceptional circles being disjoint circles centered at λ_k of special radii. However, the set obtained in such way is not a C_0 -set although its linear density can be made arbitrarily small.

The regularity of the set means that its points are separated in a special way. This is why a function f with regular zero set $\Lambda = \{\lambda_k, 1\}_{k=1}^{\infty}$ possesses nice lower bounds outside an exceptional set formed by disjoint circles $B(\lambda_k, \gamma_k)$, whose radii are relatively small, namely, are infinitesimal in comparison with the centers as $k \rightarrow \infty$ [1, Ch. II, Sect. 1, Thm. 5].

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In the present work we employ another tool allowing us to separate points λ_k , a condensation index S_Λ of a sequence Λ introduced in [2]. This index is appropriate for working with arbitrary sequences including multiple ones. It is considered in the second section.

The third section is devoted to obtaining lower bounds for special polynomials defined by the sequence $\Lambda = \{\lambda_k, 1\}_{k=1}^\infty$ of normalized in a special way pieces of an infinite product representing an entire function with a multiple zero set Λ . We study the relations between these estimates and condensation indices. We construct exceptional circles centered at the points λ_k , outside which the mentioned polynomials admit appropriate lower bounds.

On the base of the results of the third section, in the fourth section we study lower bounds for entire functions of a proximate order $\rho(r)$ and of completely regular growth. We introduce the notion of a properly balanced set, which is a properly distributed set with the zero condensation index S_Λ . We show that a regular set is properly balanced, see Theorem 4.2. We also prove that a properly balanced property of the zero set $\Lambda = \{\lambda_k, 1\}_{k=1}^\infty$ of a function f is a necessary and sufficient condition ensuring the existence of a family of mutually disjoint circles with centers at the points λ_k and relatively small radii, outside which the function $|f|$ admit lower bounds coinciding asymptotically with its upper bounds in the entire plane, see Theorem 4.5. Thus, we show that the notion of a properly balanced set is a natural generalization of the notion of a regular set for the case of arbitrary sequences including the multiple ones. We provide an effective algorithm for constructing an exceptional set formed by circles centered at the points λ_k , see Theorem 4.3.

2. CONDENSATION INDEX

Let $\Lambda = \{\lambda_k, 1\}_{k=1}^\infty$ be a sequence of different complex numbers λ_k and their multiplicities n_k . We assume that $|\lambda_k|$ does not decrease and $|\lambda_k| \rightarrow \infty, k \rightarrow \infty$. We consider sequences of a proximate order $\rho(r)$ [1, Ch. I, Sect. 12]. We recall main properties of $\rho(r)$. A function $\rho(r)$, $r > 0$, satisfying the conditions

$$\lim_{r \rightarrow \infty} \rho(r) = \rho > 0, \quad \lim_{r \rightarrow \infty} r\rho'(r) \ln r = 0, \quad (2.1)$$

is called a proximate order. We have

$$(r^{\rho(r)})' = r^{\rho(r)} \left(\rho'(r) \ln r + \frac{\rho(r)}{r} \right) = r^{\rho(r)-1} (r\rho'(r) \ln r + \rho(r)).$$

Thus, by (2.1), the function $r^{\rho(r)}$ increases for sufficiently large r . We let $L(r) = r^{\rho(r)-\rho}$. The function $L(r)$ is slowly growing [1, Ch. I, Sect. 12, Lm. 5], that is,

$$\lim_{r \rightarrow \infty} \frac{L(cr)}{L(r)} = 1 \quad (2.2)$$

uniformly on each segment $0 < a \leq c \leq b < \infty$. It follows from (2.2) that for each $\varepsilon > 0$ and each $c \in [a, b]$, the inequality holds:

$$(1 - \varepsilon)c^\rho r^{\rho(r)} \leq (cr)^{\rho(cr)} \leq (1 + \varepsilon)c^\rho r^{\rho(r)}, \quad r \geq R(\varepsilon). \quad (2.3)$$

A sequence Λ is said to have a finite upper density $\bar{n}(\Lambda)$ (at order $\rho(r)$) if

$$\bar{n}(\Lambda) = \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r^{\rho(r)}} < \infty,$$

where $n(r, \Lambda)$ is the amount of the points λ_k counting their multiplicities n_k in the circle $B(0, r)$ of radius r centered at the origin. We introduce a local characteristics of the sequence Λ . Following work [2], we let

$$q_\Lambda(z, w, \delta) = \prod_{\lambda_k \in B(w, \delta|w|)} \left(\frac{z - \lambda_k}{3\delta|\lambda_k|} \right)^{n_k}.$$

As the ball $B(w, \delta|w|)$ contains none of λ_k , we let $q_\Lambda(\lambda, w, \delta) \equiv 1$. The absolute value of $q_\Lambda(\lambda, w, \delta)$ can be interpreted as a measure of accumulation of the points $\lambda_k \in B(w, \delta|w|)$ at λ . By its meaning, the quantity $\frac{\ln |q_\Lambda(\lambda, w, \delta)|}{|w|}$ is similar to the logarithm of the geometric mean (arithmetic mean of the logarithms) of the normed distances from the points $\lambda_k \in B(w, \delta|w|)$ to the point λ . If $\delta \in (0, 1)$, the absolute value of each factor q_Λ in the circle $B(w, \delta|w|)$ is bounded from above by the quantity $2(3(1 - \delta))^{-1}$. This is why

$$\frac{|\lambda - \lambda_k|}{3\delta|\lambda_k|} \leq 1, \quad \lambda, \lambda_k \in B(w, \delta|w|), \quad \delta \in \left(0, \frac{1}{3}\right). \quad (2.4)$$

Moreover, the inequality holds:

$$\frac{|\lambda - \lambda_k|}{3\delta|\lambda_k|} \geq 1, \quad \lambda \in B(w, 5\delta|w|), \quad \lambda_k \in B(w, \delta|w|), \quad \delta \in \left(0, \frac{1}{3}\right). \quad (2.5)$$

We introduce extra functions, see [2]. We let

$$q_\Lambda^m(z, \delta) = \prod_{\lambda_k \in B(\lambda_m, \delta|\lambda_m|), k \neq m} \left(\frac{z - \lambda_k}{3\delta|\lambda_k|}\right)^{n_k} = q_\Lambda(z, w, \delta) \left(\frac{z - \lambda_m}{3\delta|\lambda_m|}\right)^{n_m}, \quad m \geq 1.$$

Following [2], we define the condensation index:

$$S_\Lambda = \lim_{\delta \rightarrow 0} S_\Lambda(\delta), \quad S_\Lambda(\delta) = \lim_{m \rightarrow \infty} \frac{\ln |q_\Lambda^m(\lambda_m, \delta)|}{|\lambda_m|^{\rho(|\lambda_m|)}}.$$

It follows from (2.4) that the limit is well-defined and $S_\Lambda \leq 0$. Various examples on calculating the index S_Λ in the case $\rho(r) \equiv 1$ were given in works [3]–[5]. We consider extra examples related with the notion of a regular set. Let $\Lambda = \{\lambda_k, 1\}$. Assume that one of the following conditions holds [1, Ch. II, Sect. 1]:

(C) There exists a number $d > 0$ such that the circles of the radii

$$r_k = d|\lambda_k|^{1 - \frac{\rho(|\lambda_k|)}{2}}$$

centered at the points λ_k are mutually disjoint.

(C') The points λ_k are located in finitely many closed angles with a common vertex at the origin having no other common points. At that, for all points $\lambda_{k(p)}$ located in a same angle, the inequality

$$|\lambda_{k(p+1)}| - |\lambda_{k(p)}| \geq d|\lambda_{k(p)}|^{1 - \rho(|\lambda_{k(p)}|)}, \quad d > 0,$$

holds true.

Let us show that in both cases $S_\Lambda = 0$. First we assume that Condition (C') is satisfied. We begin with observing that for sufficiently small $\delta > 0$ ($\delta < \delta_0 < \frac{1}{3}$) and $k \geq k_0$ the circle $B(\lambda_k, \delta|\lambda_k|)$ intersects just one of the angles mentioned in the formulation of Condition (C'); we denote them by Γ_j . This is why it follows from the definition of the quantity S_Λ that it is sufficient to prove the identity $S_{\Lambda_j} = 0$, where Λ_j is a part of the sequence Λ located in the angle Γ_j . We denote $\Lambda_j = \{\lambda_{k(p)}, 1\}_{p=1}^\infty$.

Let $\delta \in (0, \delta_0)$ and $k(p) \geq k_0$. By (2.3) and Condition (C'), there exists $\alpha > 0$ such that

$$|\lambda_{k(s+1)}| - |\lambda_{k(s)}| \geq d|\lambda_{k(s)}|^{1 - \rho(|\lambda_{k(s)}|)} \geq \alpha|\lambda_{k(p)}|^{1 - \rho(|\lambda_{k(p)}|)} = h_p, \quad \lambda_{k(s)} \in B(\lambda_{k(p)}, \delta|\lambda_{k(p)}|).$$

Then by (2.4) we get

$$\begin{aligned} \sum_{\lambda_{k(s)} \in B(\lambda_{k(p)}, \delta|\lambda_{k(p)}|), s \neq p} \ln \frac{|\lambda_{k(s)} - \lambda_{k(p)}|}{3\delta|\lambda_{k(s)}|} &= \sum_{\lambda_{k(s)} \in B(\lambda_{k(p)}, \delta|\lambda_{k(p)}|), |\lambda_{k(s)}| < |\lambda_{k(p)}|} \ln \frac{|\lambda_{k(s)} - \lambda_{k(p)}|}{3\delta|\lambda_{k(s)}|} \\ &+ \sum_{\lambda_{k(s)} \in B(\lambda_{k(p)}, \delta|\lambda_{k(p)}|), |\lambda_{k(s)}| > |\lambda_{k(p)}|} \ln \frac{|\lambda_{k(s)} - \lambda_{k(p)}|}{3\delta|\lambda_{k(s)}|} \\ &\geq 2 \sum_{l=1}^{l_p} \ln \frac{lh_p}{3\delta(1+\delta)|\lambda_{k(p)}|}, \end{aligned}$$

where l_p is an integer part of the number $\frac{\delta|\lambda_{k(p)}|}{h_p}$. This yields

$$\begin{aligned} \frac{1}{|\lambda_{k(p)}|^{\rho(|\lambda_{k(p)}|)}} \sum_{\lambda_{k(s)} \in B(\lambda_{k(p)}, \delta|\lambda_{k(p)}|), s \neq p} \ln \frac{|\lambda_{k(s)} - \lambda_{k(p)}|}{3\delta|\lambda_{k(s)}|} &\geq \frac{2}{|\lambda_{k(p)}|^{\rho(|\lambda_{k(p)}|)}} \ln \left(\frac{(l_p)!(h_p)^{l_p}}{3\delta(1+\delta)|\lambda_{k(p)}|^{l_p}} \right) \\ &\geq \frac{2}{|\lambda_{k(p)}|^{\rho(|\lambda_{k(p)}|)}} \ln \left(\frac{(h_p l_p)^{l_p}}{(12\delta|\lambda_{k(p)}|)^{l_p}} \right) \\ &= \frac{2l_p}{|\lambda_{k(p)}|^{\rho(|\lambda_{k(p)}|)}} \ln \frac{h_p l_p}{12\delta|\lambda_{k(p)}|}. \end{aligned}$$

Hence,

$$\lim_{p \rightarrow \infty} \frac{1}{|\lambda_{k(p)}|^{\rho(|\lambda_{k(p)}|)}} q_{\Lambda}^{k(p)}(\lambda_{k(p)}, \delta) \geq -2 \frac{\delta}{\alpha} \ln 12.$$

This implies the identity $S_{\Lambda_j} = 0$.

Assume now that Condition (C) holds. We fix $\delta \in (0, \frac{1}{3})$. By (2.3), there exists numbers $\alpha, \beta > 0$ such that

$$\alpha d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}} \leq d|\lambda_s|^{1-\frac{\rho(|\lambda_k|)}{2}} \leq \beta d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}, \quad \lambda_s \in B(\lambda_k, \delta|\lambda_k|).$$

This is why, it follows from Condition (C) that the circle $B(\lambda_k, d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}})$ contains no points $\lambda_s, s \neq k$, and the amount $m_{n,k}$ of the points $\lambda_s \in B(\lambda_k, \delta|\lambda_k|)$ located in the annulus

$$K_{n,k} = B\left(\lambda_k, (n+1)d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}\right) \setminus B\left(\lambda_k, nd|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}\right), \quad n \geq 1,$$

does not exceeds the quotient of the annulus' area

$$B\left(\lambda_k, ((n+1)+\beta)d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}\right) \setminus B\left(\lambda_k, (n-\beta)d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}\right)$$

and the area of the circle $B(\lambda_s, d|\lambda_s|^{1-\frac{\rho(|\lambda_k|)}{2}})$ of a minimal radius for the same λ_s . The area of this circle is estimated from below by $\pi \left(\alpha d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}\right)^2$. We have

$$m_{n,k} \leq \frac{\pi(((n+1)+\beta)^2 - (n-\beta)^2)d^2|\lambda_k|^{2-\rho(|\lambda_k|)}}{\pi\alpha^2 d^2|\lambda_k|^{2-\rho(|\lambda_k|)}} = \frac{(2n+1)(2\beta+1)}{\alpha^2}.$$

Let j_k be an integer part of the number $\frac{\delta|\lambda_k|}{d|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}}$. Then

$$m_{n,k} \leq \frac{(2j_k+1)(2\beta+1)}{\alpha^2} = p_k, \quad 1 \leq n \leq j_k.$$

Taking into consideration the inequality

$$|\lambda_s - \lambda_k| \geq nd|\lambda_k|^{1-\frac{\rho(|\lambda_k|)}{2}}, \quad \lambda_s \in K_{n,k}, \quad 1 \leq n \leq j_k,$$

we obtain

$$\sum_{\lambda_s \in B(\lambda_k, \delta|\lambda_k|), s \neq k} \ln \frac{|\lambda_s - \lambda_k|}{3\delta|\lambda_s|} \geq \sum_{n=1}^{j_k} \ln \frac{np_k d |\lambda_k|^{1 - \frac{\rho(\lambda_k)}{2}}}{4\delta|\lambda_k|} \geq j_k \ln \left(\frac{p_k j_k d |\lambda_k|^{1 - \frac{\rho(\lambda_k)}{2}}}{12\delta|\lambda_k|} \right).$$

Thus,

$$\begin{aligned} S_\Lambda(\delta) &= \lim_{k \rightarrow \infty} \frac{1}{|\lambda_k|^{\rho(\lambda_k)}} q_\Lambda^k(\lambda_k, \delta) \geq \lim_{k \rightarrow \infty} \frac{j_k}{|\lambda_k|^{\rho(\lambda_k)}} \ln \left(\frac{p_k j_k d |\lambda_k|^{1 - \frac{\rho(\lambda_k)}{2}}}{12\delta|\lambda_k|} \right) \\ &= \lim_{k \rightarrow \infty} \frac{\delta}{d|\lambda_k|^{\frac{\rho(\lambda_k)}{2}}} \ln j_k = \lim_{k \rightarrow \infty} \frac{\delta}{d|\lambda_k|^{\frac{\rho(\lambda_k)}{2}}} \ln \frac{\delta}{d|\lambda_k|^{-\frac{\rho(\lambda_k)}{2}}} = 0. \end{aligned}$$

Therefore, $S_\Lambda = 0$.

3. LOWER BOUNDS FOR SPECIAL POLYNOMIALS

We provide some results concerning the index S_Λ ; we shall need them in what follows.

Lemma 3.1. *Let $\Lambda = \{\lambda_k, n_k\}$, $S_\Lambda = 0$, and B be an open set containing all points λ_k . The following statements are equivalent:*

1. *For each $\varepsilon > 0$, there exist number $R > 0$ and $\delta \in (0, \frac{1}{3})$ such that*

$$\ln |q_\Lambda(z, w, \delta)| \geq -\varepsilon |z|^{\rho(|z|)}, \quad z \in \mathbb{C} \setminus B, \quad |w| \geq R. \quad (3.1)$$

2. *For each $\varepsilon > 0$, there exist numbers $R > 0$ and $\delta \in (0, \frac{1}{3})$ such that*

$$\ln |q_\Lambda(z, w, \delta)| \geq -\varepsilon |z|^{\rho(|z|)}, \quad z \in B(w, \delta|w|) \cap \partial B, \quad |w| \geq R. \quad (3.2)$$

3. *For each $\varepsilon > 0$ there exists a number k_0 and a number $\delta \in (0, \frac{1}{3})$ such that for each $k \geq k_0$ the inequality holds:*

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq -\varepsilon |\lambda_k|^{\rho(\lambda_k)}, \quad z \in B(\lambda_k, \delta|\lambda_k|) \cap \partial B. \quad (3.3)$$

4. *For each $\varepsilon > 0$ there exists a number k_0 and a number $\delta \in (0, \frac{1}{3})$ such that for each $k \geq k_0$ the inequality holds:*

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq -\varepsilon |z|^{\rho(|z|)}, \quad z \in \mathbb{C} \setminus B. \quad (3.4)$$

Proof. Assume that (3.1) holds. Then (3.2) is obvious. Since $|z| \leq (1+\delta)|\lambda_k|$ as $z \in B(\lambda_k, \delta|\lambda_k|)$, in view of the definition of the function $q_\Lambda(z, w, \delta)$ and (2.4), (3.2), (2.3), we obtain:

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq \ln |q_\Lambda(z, \lambda_k, \delta)| \geq -\varepsilon |z|^{\rho(|z|)} \geq -\varepsilon (2|\lambda_k|)^{\rho(2|\lambda_k|)} \geq -2^{\rho+1} \varepsilon |\lambda_k|^{\rho(\lambda_k)},$$

where $k \geq k_1$ and $z \in B(\lambda_k, \delta|\lambda_k|) \cap \partial B$. This gives (3.3).

Assume (3.3) and let us prove (3.4). We take $\varepsilon > 0$ and we find k_0 and $\delta_0 \in (0, \frac{1}{3})$ such that

$$\ln \left| \frac{z - \lambda_k}{3\delta_0|\lambda_k|} \right|^{n_k} \geq -\varepsilon |\lambda_k|^{\rho(\lambda_k)}, \quad z \in B(\lambda_k, \delta_0|\lambda_k|) \cap \partial B, \quad k \geq k_0.$$

We let $\delta = \frac{\delta_0}{3}$ and we have

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq \ln \left| \frac{z - \lambda_k}{3\delta_0|\lambda_k|} \right|^{n_k} \geq -\varepsilon |\lambda_k|^{\rho(\lambda_k)}, \quad z \in B(\lambda_k, 3\delta|\lambda_k|) \cap \partial B, \quad k \geq k_0.$$

If $z \in \mathbb{C} \setminus B(\lambda_k, 3\delta|\lambda_k|)$, then

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq 0 > -\varepsilon |z|^{\rho(|z|)}.$$

In view of the above arguing and the minimum principle for the harmonic functions we obtain:

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq -\varepsilon |\lambda_k|^{\rho(|\lambda_k|)}, \quad z \in B(\lambda_k, 3\delta|\lambda_k|) \setminus B, \quad k \geq k_0.$$

Let $z \in B(\lambda_k, 3\delta|\lambda_k|)$. Then $|\lambda_k| < |z|(1 - 3\delta)^{-1} < 2|z|$, $k > 1$. In view of (2.3) we hence get:

$$\ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq -\varepsilon (2|z|)^{\rho(2|z|)} \geq -2^{\rho+1} \varepsilon |z|^{\rho(|z|)}, \quad z \in B(\lambda_k, 3\delta|\lambda_k|) \setminus B, \quad k > k_1.$$

Thus, we arrive at inequality (3.4).

Finally, assume that (3.4) holds. Suppose that (3.1) fails. Then for some $\varepsilon > 0$ there exist sequences $\{w_m\}$, $\{z_m\}$ such that $z_m \in \mathbb{C} \setminus B$, $|w_m| \rightarrow \infty$ and

$$\ln |q_\Lambda(z_m, w_m, m^{-1})| < -\varepsilon |z_m|^{\rho(|z_m|)}, \quad m \geq 1. \quad (3.5)$$

By the definition of the function q_Λ this implies that for each $m \geq 1$, the circle $B\left(w_m, \frac{|w_m|}{m}\right)$ contains at least one of the points λ_k . Let $\lambda_{k(m)}$ be a point in the circle $B\left(w_m, \frac{|w_m|}{m}\right)$ located at the minimal distance from the point z_m among all $\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right)$, $m \geq 1$; if there are several such points, we choose arbitrarily one of them. Let us show that (3.5) contradicts the assumptions of the lemma. In order to do this, we estimate from below two groups of the factors forming the quantity $|q_\Lambda(z_m, w_m, m^{-1})|$. The first one is formed by the points $\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right)$, $k \neq k(m)$, while the second corresponds to the point $\lambda_{k(m)}$. Let us estimate the first group. According the assumptions, $S_\Lambda = 0$. Then due to the definition of S_Λ , there exist $\delta \in (0, \frac{1}{3})$ and k_0 such that

$$\frac{\ln |q_\Lambda^k(\lambda_k, \delta)|}{|\lambda_k|^{\rho(|\lambda_k|)}} \geq -\frac{\varepsilon}{3}, \quad k \geq k_0. \quad (3.6)$$

We choose m_0 so that $k(m) \geq k_0$, $m \geq 2\delta^{-1}$ and $B\left(w_m, \frac{|w_m|}{m}\right) \subset B(\lambda_{k(m)}, \delta|\lambda_{k(m)}|)$, $m \geq m_0$; the latter is possible since $\lambda_{k(m)} \in B\left(w_m, \frac{|w_m|}{m}\right)$. Let $\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right)$. Assume that $|z_m - \lambda_k| < 2^{-1}|\lambda_{k(m)} - \lambda_k|$. Then according the choice of the point $\lambda_{k(m)}$ we have:

$$|\lambda_{k(m)} - \lambda_k| \leq |z_m - \lambda_k| + |z_m - \lambda_{k(m)}| \leq 2|z_m - \lambda_k| < |\lambda_{k(m)} - \lambda_k|.$$

We have obtained a contradiction. Therefore,

$$|z_m - \lambda_k| \geq 2^{-1}|\lambda_{k(m)} - \lambda_k|, \quad \lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right).$$

Taking into consideration (2.4), (3.6) and the choice of the number m_0 , we obtain:

$$\begin{aligned} \prod_{\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right), k \neq k(m)} \left(\frac{|z_m - \lambda_k|}{3m^{-1}|\lambda_k|} \right)^{n_k} &\geq \prod_{\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right), k \neq k(m)} \left(\frac{|\lambda_{k(m)} - \lambda_k|}{6m^{-1}|\lambda_k|} \right)^{n_k} \\ &\geq \prod_{\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right), k \neq k(m)} \left(\frac{|\lambda_{k(m)} - \lambda_k|}{3\delta|\lambda_k|} \right)^{n_k} \\ &\geq \prod_{\lambda_k \in B\left(w_m, \frac{|w_m|}{m}\right), k \neq k(m)} \left(\frac{|\lambda_{k(m)} - \lambda_k|}{3\delta|\lambda_k|} \right)^{n_k} \\ &= |q_\Lambda^{k(m)}(\lambda_{k(m)}, \delta)| \geq \exp \left(-\frac{\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}}{3} \right) \end{aligned} \quad (3.7)$$

as $m \geq m_0$. Since $\lambda_{k(m)} \in B\left(w_m, \frac{|w_m|}{m}\right)$ and by (2.5) and (3.5) $z_m \in B\left(w_m, \frac{5|w_m|}{m}\right)$, $m \geq 1$, in view of (2.3) there exists a number $m_1 \geq m_0$ such that

$$\exp\left(-\frac{\varepsilon|\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}}{3}\right) \geq \exp\left(-\frac{\varepsilon|z_m|^{\rho(|z_m|)}}{2}\right), \quad m \geq m_1. \quad (3.8)$$

We proceed to estimating the second group of the factors forming $|q_\Lambda(z_m, w_m, m^{-1})|$ constructed by the point $\lambda_{k(m)}$. By (3.4), there exist $m_2 \geq m_1$ and $\delta_1 \in (0, \frac{1}{3})$ such that

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3\delta_1|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq -\frac{\varepsilon|z_m|^{\rho(|z|)}}{2}, \quad m \geq m_2.$$

We choose $m_3 \geq m_2$ by the condition $m^{-1} \leq \delta_1$, $m \geq m_3$. Then

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq \ln \left| \frac{z_m - \lambda_{k(m)}}{3\delta_1|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq -\frac{\varepsilon|z_m|^{\rho(|z|)}}{2}, \quad m \geq m_3.$$

In view of (3.7) and (3.8) this implies

$$\ln |q_\Lambda(z_m, w_m, m^{-1})| \geq -\varepsilon|z_m|, \quad m \geq m_3.$$

The latter contradicts (3.5). Thus, (3.1) is true and this completes the proof. \square

We are going to describe a wide class of sets B and to provide conditions ensuring the validity of Statement 3 in Lemma 3.1. Let $\Gamma = \{\gamma_k\}_{k=1}^\infty$ be a sequence of positive numbers. We let $B(\Lambda, \Gamma) = \cup_{k=1}^\infty B(\lambda_k, \gamma_k)$.

Lemma 3.2. *Let $\Lambda = \{\lambda_k, n_k\}$ and $\Gamma = \{\gamma_k\}$ satisfy a condition:*

$$\frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{\gamma_k}{|\lambda_k|} \rightarrow 0, \quad k \rightarrow \infty. \quad (3.9)$$

Then Statement 3 in Lemma 3.1 holds with $B = B(\Lambda, \Gamma)$.

Proof. Let $\delta \in (0, \frac{1}{3})$, $\lambda_k \neq 0$ and $z \in B(\lambda_k, \delta|\lambda_k|) \cap \partial B$. By (3.9) we have:

$$\frac{1}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \left| \frac{z - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{\gamma_k}{3\delta|\lambda_k|} \geq \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{\gamma_k}{|\lambda_k|} \rightarrow 0.$$

This leads us to Statement 3 in Lemma 3.1. \square

Remark 1. a) *Suppose that $\Gamma = \{\gamma_k\}$ satisfies (3.9) and for $r_k \in (0, 1)$, $k \geq 1$, the relation holds:*

$$\frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln r_k \rightarrow 0, \quad k \rightarrow \infty. \quad (3.10)$$

Then, as one can see easily, $\{r_k \gamma_k\}$ also satisfies (3.9). In particular, if

$$m(\Lambda) = \lim_{k \rightarrow \infty} \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} = 0,$$

then for each $r_k \in (r_0, 1)$, $k \geq 1$, $r_0 > 0$, condition (3.10) holds true.

b) *Let $m(\Lambda) = 0$ and $\gamma_k = n_k |\lambda_k|^{1-\rho(|\lambda_k|)}$, $k \geq 1$. Then $\Gamma = \{\gamma_k\}$ satisfies (3.9). We have*

$$\frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{\gamma_k}{|\lambda_k|} = \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \rightarrow 0, \quad k \rightarrow \infty.$$

c) *Assume that $n_k \leq N$, $k \geq 1$. Then (3.9) holds if*

$$\frac{\ln \gamma_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \rightarrow 0, \quad k \rightarrow \infty.$$

Indeed, by (2.1), for $k \geq k_0$ we have:

$$\frac{\ln |\lambda_k|}{|\lambda_k|^{\rho(|\lambda_k|)}} \leq \frac{\ln |\lambda_k|}{|\lambda_k|^{\rho/2}} \rightarrow 0, \quad k \rightarrow \infty.$$

This is why (3.9) is true. For instance, we can choose

$$\gamma_k = \exp(-\varepsilon_k |\lambda_k|^{\rho(|\lambda_k|)}), \quad k \geq 1, \quad 0 < \varepsilon_k \rightarrow 0, \quad k \rightarrow \infty. \quad (3.11)$$

We let

$$\varepsilon_k = \frac{1}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{|\lambda_k|^{2\rho(|\lambda_k|)}}{\tau}. \quad (3.12)$$

If in addition $\bar{n}(\Lambda) < +\infty$, then the series

$$\sum_{k=1}^{\infty} \gamma_k = \sum_{k=1}^{\infty} \frac{\tau}{|\lambda_k|^{2\rho(|\lambda_k|)}}$$

converges. Thus, in the case $n_k \leq N$, $k \geq 1$, and $\bar{n}(\Lambda) < +\infty$, the exceptional set $B(\Lambda, \Gamma)$ in Lemma 3.2 consists of the circles centered at the points λ_k and the sum of their radii can be made arbitrarily small.

Let $\Gamma = \{\gamma_k\}$, $\Lambda = \{\lambda_k, n_k\}$. For each $k \geq 1$, by the symbol β_k we denote the minimal distance from the point λ_k to the points λ_s , $s \neq k$. We let

$$\Gamma_0(\tau) = \{\gamma_k^0(\tau)\}_{k=1}^{\infty}, \quad \gamma_k^0(\tau) = \tau \min\{\gamma_k, \beta_k\}, \quad \tau \in (0, 2^{-1}], \quad k \geq 1, \quad (3.13)$$

$$B(\Lambda, \Gamma_0(\tau)) = \bigcup_{k=1}^{\infty} B(\lambda_k, \gamma_k^0(\tau)).$$

By (3.13) and the definition of the numbers β_k , the circles $B(\lambda_k, \gamma_k^0(\tau))$ are mutually disjoint.

Lemma 3.3. *Let $\Lambda = \{\lambda_k, n_k\}$, $m(\Lambda) = 0$, $\tau \in (0, 2^{-1}]$, and $S_\Lambda = 0$. Assume that $\Gamma = \{\gamma_k\}$ satisfies (3.9) and $B = B(\Lambda, \Gamma_0(\tau))$. Then Statement 3 in Lemma 3.1 holds true.*

Proof. We suppose the contrary, then for some number $\varepsilon > 0$ there exists sequences $\{k(m)\}$ and $\{z_m\}$ such that $k(m) \rightarrow \infty$, $z_m \in B\left(\lambda_{k(m)}, \frac{|\lambda_{k(m)}|}{m}\right) \cap \partial B$ and

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} < -\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}, \quad m \geq 1. \quad (3.14)$$

Let Λ_1 be a sequence consisting of all pairs λ_k, n_k such that $\gamma_k^0(\tau) = \tau \gamma_k$. We let $\Gamma_1 = \{\tau \gamma_k\}_{\lambda_k, n_k \in \Lambda_1}$. By assumption, $m(\Lambda) = 0$ and hence, (3.10) holds with $r_k = \tau$. This is why according Remark 1.a), the sequence Γ_1 satisfies (3.9). Then by Lemma 3.2, the sequence Λ_1 satisfies Statement 3 in Lemma 2.1, where we let $B = B(\Lambda_1, \Gamma_1)$, and according this lemma, its Statement 4 holds as well. Therefore, there exists a number m_0 and a number $\delta \in (0, \frac{1}{3})$ such that

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq \ln \left| \frac{z_m - \lambda_{k(m)}}{3\delta |\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq -\frac{\varepsilon |z_m|^{\rho(|z_m|)}}{2}, \quad \lambda_{k(m)}, n_{k(m)} \in \Gamma_1, \quad m \geq m_0.$$

Since $z_m \in B\left(\lambda_{k(m)}, \frac{|\lambda_{k(m)}|}{m}\right)$, then $\frac{|z_m|}{|\lambda_{k(m)}|} \rightarrow 1$, $m \rightarrow \infty$. This is why due to (2.3), for some number $m_1 \geq m_0$, the inequality holds true:

$$-\frac{\varepsilon |z_m|^{\rho(|z_m|)}}{2} \geq -\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}, \quad m \geq m_1.$$

Thus,

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq -\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}, \quad \lambda_{k(m)}, n_{k(m)} \in \Gamma_1, \quad m \geq m_1. \quad (3.15)$$

According the definition of numbers β_k , we find indices p_m such that $|\lambda_{k(m)} - \lambda_{p_m}| = \beta_{k(m)}$, $m \geq 1$. By assumption, $S_\Lambda = 0$. Therefore, there exist a number $m_2 \geq m_1$ and a number $\delta_1 \in (0, \frac{1}{3})$, for which

$$\frac{\ln |q_\Lambda^{p_m}(\lambda_{p_m}, \delta_1)|}{|\lambda_{p_m}|^{\rho(|\lambda_{p_m}|)}} \geq -\frac{\varepsilon}{2}, \quad m \geq m_2. \quad (3.16)$$

Let $\lambda_{k(m)}, n_{k(m)} \in \Gamma_1$. Since $z_m \in B(\lambda_{k(m)}, \tau\beta_{k(m)})$, then $|\lambda_{k(m)} - z_m| \geq \tau\beta_{k(m)}$. Hence,

$$\frac{|\lambda_{k(m)} - \lambda_{p_m}|}{|\lambda_{k(m)} - z_m|} = \frac{\beta_{k(m)}}{|\lambda_{k(m)} - z_m|} \leq \frac{1}{\tau}.$$

This is why, in view of the belonging $z_m \in B\left(\lambda_{k(m)}, \frac{|\lambda_{k(m)}|}{m}\right)$, we obtain:

$$|\lambda_{k(m)} - \lambda_{p_m}| \leq \frac{1}{\tau} |\lambda_{k(m)} - z_m| \leq \frac{1}{\tau m} |\lambda_{k(m)}|. \quad (3.17)$$

Due to (3.17), we can assume that $m \geq m_2$, the belonging holds $\lambda_{k(m)} \in B(\lambda_{p_m}, \delta_1|\lambda_{p_m}|)$ and $(m\tau)^{-1} < \delta_1$. Therefore, according (2.4) and the first inequality in (3.17), we get:

$$\begin{aligned} \ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} &\geq \ln \left| \frac{\lambda_{k(m)} - \lambda_{p_m}}{3(\tau m)^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} \\ &\geq \ln \left| \frac{\lambda_{k(m)} - \lambda_{p_m}}{3\delta_1|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq |q_\Lambda^{p_m}(\lambda_{p_m}, \delta_1)|, \quad m \geq m_2. \end{aligned} \quad (3.18)$$

According (3.17), $\lambda_{p_m} \in B\left(\lambda_{k(m)}, \frac{|\lambda_{k(m)}|}{\tau m}\right)$. Therefore, by (2.3), for some index $m_3 \geq m_2$, the estimate holds:

$$-\frac{\varepsilon}{2} |\lambda_{p_m}|^{\rho(|\lambda_{p_m}|)} \geq -\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}, \quad m \geq m_3.$$

In view of (3.18) and (3.16) this implies:

$$\ln \left| \frac{z_m - \lambda_{k(m)}}{3m^{-1}|\lambda_{k(m)}|} \right|^{n_{k(m)}} \geq -\varepsilon |\lambda_{k(m)}|^{\rho(|\lambda_{k(m)}|)}, \quad m \geq m_3.$$

Together with (3.15), this contradicts (3.14). \square

Remark 2. Under the assumptions of Lemma 3.3, the sequence $\Gamma = \{\gamma_k\}$ can be chosen so that

$$\lim_{k \rightarrow \infty} \frac{\gamma_k}{|\lambda_k|} = 0. \quad (3.19)$$

Indeed, we let $\gamma_k = \min\{|\lambda_k|, n_k |\lambda_k|^{1-\rho(|\lambda_k|)}\}$, $k \geq 1$. According Remark 1.b), the sequence $\Gamma = \{\gamma_k\}$ satisfies (3.9). Since $m(\Lambda) = 0$, identity (3.19) follows.

In the end of the section, we provide a simple statement which is in some sense (in view of Lemma 3.1) inverse to Lemma 3.3.

Lemma 3.4. Let $\Lambda = \{\lambda_k, n_k\}$ and B_k , $k \geq 1$, be open mutually disjoint sets. Assume that $\lambda_k \in B_k$, $k \geq 1$,

$$\lim_{k \rightarrow \infty} \frac{d_k}{|\lambda_k|} = 0, \quad (3.20)$$

where d_k are diameters B_k and for each $\varepsilon > 0$ there exist k_0 and $\delta \in (0, \frac{1}{3})$ such that

$$\ln |q_\Lambda(z, \lambda_k, \delta)| \geq -\varepsilon |z|^{\rho(|z|)}, \quad z \in B(\lambda_k, \delta|\lambda_k|) \cap \partial B, \quad B = \bigcup_{k=1}^{\infty} B_k, \quad k \geq k_0. \quad (3.21)$$

Then $S_\Lambda = 0$ and $m(\Lambda) = 0$.

Proof. Let $\varepsilon > 0$. By (3.21), in view (2.4) and the definition of the function q_Λ^k we have:

$$\ln |q_\Lambda^k(z, \delta)| \geq -\varepsilon |z|^{\rho(|z|)}, \quad z \in B(\lambda_k, \delta|\lambda_k|) \cap \partial B, \quad k \geq k_0.$$

According (2.3), there exist a number $k_1 \geq k_0$ and a number $\delta_1 \in (0, \delta)$ such that

$$-\varepsilon |z|^{\rho(|z|)} \geq -2\varepsilon |\lambda_k|^{\rho(|\lambda_k|)}, \quad z \in B(\lambda_k, \delta_1|\lambda_k|), \quad k \geq k_1. \quad (3.22)$$

Therefore, in view of (2.4) we obtain:

$$\ln |q_\Lambda^k(z, \delta_1)| \geq \ln |q_\Lambda^k(z, \delta)| \geq -2\varepsilon |\lambda_k|^{\rho(|\lambda_k|)}, \quad z \in B(\lambda_k, \delta_1|\lambda_k|) \cap \partial B, \quad k \geq k_1.$$

By (3.20) we can suppose that B_k is compactly embedded into $B(\lambda_k, \delta_1|\lambda_k|)$, $k \geq k_1$. The polynomial $q_\Lambda^k(z, \delta_1)$ has no zeroes in the set B_k . Hence, by the minimum principle for harmonic functions the latter inequality is extended on B_k . In particular,

$$\ln |q_\Lambda^k(\lambda_k, \delta_1)| \geq -2\varepsilon |\lambda_k|^{\rho(|\lambda_k|)}, \quad k \geq k_1.$$

By (2.4),

$$|q_\Lambda^k(\lambda_k, \alpha)| \geq |q_\Lambda^k(\lambda_k, \delta_1)|, \quad \alpha \in (0, \delta_1).$$

Thus,

$$S_\Lambda = \lim_{\alpha \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\ln |q_\Lambda^k(\lambda_k, \alpha)|}{|\lambda_k|} \geq -2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $S_\Lambda = 0$.

Now we are going to show that $m(\Lambda) = 0$. Let $z_k \in \partial B_k \subset B(\lambda_k, \delta_1|\lambda_k|)$, $k \geq k_1$. According (3.21), (3.22) and in view of (2.4) we have:

$$\ln \left| \frac{z_k - \lambda_k}{3\delta|\lambda_k|} \right|^{n_k} \geq \ln |q_\Lambda(z_k, \lambda_k, \delta)| \geq -2\varepsilon |\lambda_k|^{\rho(|\lambda_k|)}, \quad k \geq k_1.$$

Therefore, by (2.33) and (2.32),

$$\frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{d_k}{3\delta|\lambda_k|} \geq -2\varepsilon, \quad k \geq k_1.$$

Now it follows from (3.20) that $m(\Lambda) = 0$. □

4. LOWER BOUNDS FOR ENTIRE FUNCTIONS OF COMPLETELY REGULAR GROWTH

Let $\rho(r)$ be a proximate order and f be an entire function of order at most $\rho(r)$, that is,

$$\ln |f(z)| \leq A + B|z|^{\rho(|z|)}, \quad z \in \mathbb{C}.$$

The function

$$h_f(\varphi) = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln |f(re^{i\varphi})|}{r^{\rho(r)}}, \quad \varphi \in [0, 2\pi],$$

is called an indicator of f . It is continuous [1, Ch. I., Sect. 16]. We mention one more property of the indicator [1, Ch. I, Sect. 18, Thm. 28]: for each $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that

$$\ln |f(re^{i\varphi})| \leq (h_f(\varphi) + \varepsilon)r^{\rho(r)}, \quad \varphi \in [0, 2\pi], \quad r \geq R(\varepsilon). \quad (4.1)$$

A function f is said to have a completely regular growth [1, Ch. III] if

$$h_f(\varphi) = \lim_{r \notin E, r \rightarrow +\infty} \frac{\ln |f(re^{i\varphi})|}{r^{\rho(r)}}, \quad \varphi \in [0, 2\pi],$$

where $E \subset (0, +\infty)$ is a set of relative zero measure (E_0 -set), that is,

$$\lim_{r \rightarrow +\infty} \frac{\text{mes}(E \cap (0, r))}{r} = 0;$$

here the symbol mes stands for the Lebesgue measure. A function f is said to have a regular growth on the ray $L_\varphi = \{re^{i\varphi}, r > 0\}$ if

$$h_f(\varphi) = \lim_{r \notin E_\varphi, r \rightarrow +\infty} \frac{\ln |f(re^{i\varphi})|}{r}, \quad (4.2)$$

where E_φ is E_0 -set. If f has a regular growth on each ray, then the set E_φ , generally speaking, depends on $\varphi \in [0, 2\pi]$. However, it turns out that we can find an exceptional E_0 -set suitable for all $\varphi \in [0, 2\pi]$ [1, Ch. III, Sect. 1, Thm. 1]. In other words, the function f has a completely regular growth if and only if it has a regular growth on each ray. One more equivalent definition of a function of regular growth is known. It is implied by the following result.

Lemma 4.1. *Let f be an entire function of order at most $\rho(r)$. The following statements are equivalent:*

1. *The function f has a regular growth on the ray L_φ .*
2. *There exists a sequence $\{z_m\}_{m=1}^\infty$ such that*

$$\begin{aligned} \lim_{m \rightarrow \infty} |z_m| &= \infty, & \lim_{m \rightarrow \infty} \frac{z_m}{|z_m|} &= e^{i\varphi}, \\ \lim_{m \rightarrow \infty} \frac{|z_{m+1}|}{|z_m|} &= 1, & \lim_{m \rightarrow \infty} \frac{\ln |f(z_m)|}{|z_m|^{\rho(|z_m|)}} &= h_f(\varphi). \end{aligned} \quad (4.3)$$

Proof. Let f has a regular growth on the ray L_φ . Then (4.2) holds true. We form a sequence $\{p_m\}_{m=1}^\infty$ of all natural numbers p such that the semi-interval $[p-1, p)$ is not entirely contained in E_φ . For each $m \geq 1$ we arbitrarily choose a number $r_m \in [p_m-1, p_m) \setminus E_\varphi$ and we obtain a sequence $\{z_m = r_m e^{i\varphi}\}_{m=1}^\infty$. By construction, the first and second identities in (4.3) hold, and in view of (4.2), the forth identity in (4.3) is true as well. Let us show that the third identity holds. Assume a contrary, then there exists a sequence $\{z_{m_l}\}_{l=1}^\infty$ such that for some $\alpha > 1$ the inequalities hold

$$|z_{m_{l+1}}| > \alpha |z_{m_l}|, \quad l \geq 1.$$

Then by construction, for each $l \geq 1$, the semi-interval $([|z_{m_l}|] + 1, [\alpha |z_{m_l}|])$ lies in the set E_φ , where $[x]$ denotes an integer part of a number x . Therefore,

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\text{mes}(E_\varphi \cap (0, r))}{r} \geq \overline{\lim}_{l \rightarrow \infty} \frac{\text{mes}(E_\varphi \cap (0, [\alpha |z_{m_l}|]))}{[\alpha |z_{m_l}|]} \geq \lim_{l \rightarrow \infty} \frac{[\alpha |z_{m_l}|] - [|z_{m_l}|] - 1}{[\alpha |z_{m_l}|]} = \frac{\alpha - 1}{\alpha}.$$

We have obtained a contradiction with the fact that E_φ is E_0 -set. Thus, (4.3) is true.

Assume now that Statement 2 holds. We can assume that $|z_{m+1}| \geq |z_m|$, $m \geq 1$.

Since h_f is a continuous function, by (2.3), for each $\varepsilon > 0$ there exist numbers $\delta_0 \in (0, 1)$ and $R_0(\varepsilon)$ such that

$$|t^{\rho(t)} h_f(\psi) - r^{\rho(r)} h_f(\varphi)| \leq \varepsilon r^{\rho(r)}, \quad \varphi \in [0, 2\pi], \quad te^{i\psi} \in B(re^{i\varphi}, r\delta_0), \quad r \geq R_0(\varepsilon), \quad (4.4)$$

$$|t^{\rho(t)} - r^{\rho(r)}| \leq \varepsilon r^{\rho(r)}, \quad te^{i\psi} \in B(re^{i\varphi}, r\delta_0), \quad r \geq R_0(\varepsilon). \quad (4.5)$$

Let $k \geq 1$. We choose $\delta_k \in (0, 1)$ such that for $\varepsilon = k^{-2}$ and $\delta_0 = \delta_k$ inequalities (4.4) and (4.5) hold. According (4.3), we find an index $m(k)$, for which

$$\left| \frac{z_m}{|z_m|} - e^{i\varphi} \right| < \frac{\delta_k}{12}, \quad \left| \frac{|z_{m+1}|}{|z_m|} - 1 \right| < \frac{\delta_k}{12}, \quad m \geq m(k), \quad (4.6)$$

$$\frac{\ln |f(z_m)|}{|z_m|^{\rho(|z_m|)}} \geq h_f(\varphi) - \frac{1}{k^2}, \quad m \geq m(k). \quad (4.7)$$

Let $R(k^{-2}) \geq R_0(k^{-2})$ be defined by inequality (4.1). We can assume that

$$(1 - \delta_k) |z_{m(k)}| \geq R(k^{-2}), \quad m \geq m(k). \quad (4.8)$$

We consider the functions $f_m(z) = f(z)(f(z_m))^{-1}$, $m \geq m(k)$. By (4.8), (4.1), (4.4), (4.5) and (4.7) we have:

$$\begin{aligned} \ln |f_m(re^{i\psi})| &= \ln |f(re^{i\psi})| - \ln |f(z_m)| \leq \left(h_f(\psi) + \frac{1}{k^2} \right) r^{\rho(r)} - \left(h_f(\varphi) - \frac{1}{k^2} \right) |z_m|^{\rho(|z_m|)} \\ &\leq 4k^{-2}|z_m|^{\rho(|z_m|)}, \quad z \in B(z_m, \delta_k|z_m|), \quad m \geq m(k). \end{aligned}$$

Then according theorem on lower bound for the absolute value of an analytic function in a circle [6, Ch. I, Sect. 4, Thm. 4.2] with $2\eta = k^{-1}$, the inequality holds:

$$\ln |f_m(z)| \geq -4k^{-2}(2 + \ln(3ek))|z_m|^{\rho(|z_m|)}, \quad z \in B(z_m, 6^{-1}\delta_k|z_m|) \setminus B_{k,m},$$

where $B_{k,m}$ is a finite set of circles with total sum of radii equalling to $(3k)^{-1}\delta_k|z_m|$. Hence, in view of (4.4), (4.5), and (4.7), we obtain:

$$\begin{aligned} \ln |f(re^{i\psi})| &\geq (h_f(\varphi) - k^{-2})|z_m|^{\rho(|z_m|)} - 4k^{-2}(2 + \ln(3ek))|z_m|^{\rho(|z_m|)} \\ &\geq (h_f(\psi)r - 3k^{-2})r^{\rho(r)} - 4k^{-2}(2 + \ln(6k))r^{\rho(r)} - k^{-2}r^{\rho(r)} \\ &= (h_f(\psi) - \varepsilon_k)r^{\rho(r)}, \\ \varepsilon_k &= 4k^{-2}(3 + \ln(6k)), \quad re^{i\psi} \in B(z_m, 6^{-1}\delta_k|z_m|) \setminus B_{k,m}. \end{aligned} \tag{4.9}$$

By the second inequality in (4.6), there exists a sequence $\{z_{m_{k,j}}\}_{j=1}^{\infty}$ such that

$$z_{m_{k,1}} = z_{m(k)}, \quad (\beta_k)^{j-1}|z_{m(k)}| \leq |z_{m_{k,j}}| < (\beta_k)^j|z_{m(k)}|, \quad \beta_k = 1 + \frac{\delta_k}{12}, \quad j > 1. \tag{4.10}$$

By the first inequality in (4.6) this implies the inclusions:

$$[(\beta_k)^{j-1}|z_{m(k)}|e^{i\varphi}, (\beta_k)^j|z_{m(k)}|e^{i\varphi}] \subset B(z_{z_{m_{k,j}}}, 6^{-1}\delta_k|z_{z_{m_{k,j}}}|), \quad j \geq 1.$$

We define sets $E_{k,j} \subset (0, +\infty)$ by the identities

$$\{re^{i\varphi}, r \in E_{k,j}\} = [(\beta_k)^{j-1}|z_{m(k)}|e^{i\varphi}, (\beta_k)^j|z_{m(k)}|e^{i\varphi}] \cap B_{k,m}, \quad j \geq 1.$$

We let $E_k = \cup_{j=1}^{\infty} E_{k,j}$. By (4.9) we have:

$$\ln |f(re^{i\varphi})| \geq (h_f(\varphi) - \varepsilon_k)r^{\rho(r)}, \quad r \in (|z_{m(k)}|, +\infty) \setminus E_k. \tag{4.11}$$

Let $(\beta_k)^j|z_{m(k)}| \leq r < (\beta_k)^{j+1}|z_{m(k)}|$. Then in view of (4.10) we have:

$$\begin{aligned} \frac{\text{mes}(E_k \cap (0, r))}{r} &\leq \frac{\text{mes}(E_k \cap (0, (\beta_k)^{j+1}|z_{m(k)}|))}{(\beta_k)^j|z_{m(k)}|} \leq \frac{1}{(\beta_k)^j} \sum_{s=1}^{j+1} \frac{4\delta_k|z_{m_{k,s}}|}{3k|z_{m(k)}|} \\ &\leq \frac{4\delta_k}{3k} \sum_{s=1}^{j+1} \frac{(\beta_k)^s|z_{m(k)}|}{(\beta_k)^j|z_{m(k)}|} \leq \frac{4\delta_k(\beta_k)^2}{3k(\beta_k - 1)} = \frac{16(\beta_k)^2}{3k} \leq \frac{16}{k}. \end{aligned} \tag{4.12}$$

We choose a sequence $\{R_k\}_{k=1}^{\infty}$ such that

$$R_k > |z_{m(k)}|, \quad 16 \sum_{s=1}^{k-1} \frac{R_s}{s} \leq \frac{R_k}{k}, \quad k \geq 1. \tag{4.13}$$

We let

$$\tilde{E}_1 = E_1 \cap [|z_{m(1)}|, R_1), \quad \tilde{E}_k = E_k \cap [R_{k-1}, R_k), \quad k > 1, \quad E_\varphi = \bigcup_{k=1}^{\infty} \tilde{E}_k.$$

Let $R_k < r \leq R_{k+1}$. Then by (4.12) and (4.13) we get

$$\begin{aligned} \frac{\text{mes}(E_\varphi \cap (0, r))}{r} &\leq \sum_{s=1}^{k-1} \frac{\text{mes}(E_s \cap (0, R_s))}{R_k} + \frac{\text{mes}(E_k \cap (0, R_k))}{R_k} + \frac{\text{mes}(E_{k+1} \cap (0, r))}{r} \\ &\leq \frac{16}{R_k} \sum_{s=1}^{k-1} \frac{R_s}{s} + \frac{16}{k} + \frac{16}{k} \leq \frac{33}{k}. \end{aligned}$$

Hence, E_φ is E_0 -set. Now (4.11) implies (4.2). The proof is complete. \square

Let $\Lambda = \{\lambda_k, n_k\}$. The sequence Λ is said to have an angular density [1, Ch. II] (at order $\rho(r)$) if for all $\varphi_1 < \varphi_2$ except possibly a countable set Φ_Λ there exists a limit

$$n_\Lambda(\varphi_1, \varphi_2) = \lim_{r \rightarrow +\infty} \frac{n(r, \Lambda(\varphi_1, \varphi_2))}{r^{\rho(r)}},$$

where $\Lambda(\varphi_1, \varphi_2)$ the set of all pairs λ_k, n_k such that λ_k lies in the angle $\Gamma(\varphi_1, \varphi_2) = \{z = te^{i\varphi} : \varphi \in (\varphi_1, \varphi_2), t > 0\}$, and Φ_Λ consists only of φ , for which

$$\inf_{\alpha > 0} \lim_{r \rightarrow +\infty} \frac{n(r, \Lambda(\varphi - \alpha, \varphi + \alpha))}{r^{\rho(r)}} > 0.$$

A sequence Λ is called a properly distributed set [1, Ch. II] at order $\rho(r)$ if it has an angular density for non-integer ρ , while for integer ρ it additionally satisfies the Lindelöf condition: for some $b \in \mathbb{C}$, there exists the limit

$$\Delta(\Lambda) = \lim_{r \rightarrow +\infty} r^{(\rho - \rho(r))} \left(b + \frac{1}{\rho} \sum_{|\lambda_k| < r} \frac{n_k}{(\lambda_k)^\rho} \right).$$

A classical result by B.Ya. Levin [1, Ch. II, Thm. 2, Ch. III, Thm. 4] states that f has a regular growth if and only if its zero set $\Lambda = \{\lambda_k, n_k\}$ is properly distributed. At that, the inequality holds:

$$\ln |f(re^{i\varphi})| \geq r^{\rho(r)} h_f(\varphi) + \alpha(r), \quad re^{i\varphi} \in \mathbb{C} \setminus B_f, \quad \frac{\alpha(r)}{r^{\rho(r)}} \rightarrow 0, \quad r \rightarrow \infty, \quad (4.14)$$

where B_f is C_0 -set, that is, it can be covered by circles $B(z_j, r_j)$, $j \geq 1$, such that

$$\sigma(B_f) = \overline{\lim}_{r \rightarrow \infty} \frac{1}{r} \sum_{|z_j| \leq r} r_j = 0. \quad (4.15)$$

The aim of the present section is an effective constructing of an exceptional set B_f in a form convenient for applying. In order to make such effective constructing, B.Ya. Levin introduced the notion of a regular set [1, Ch. II, Sect. 1]. This is a properly distributed set of simple points $\Lambda = \{\lambda_k, 1\}$ satisfying Condition (C) or (C') in Section 2. The set B_Λ , which consists of the circles $B\left(\lambda_k, d|\lambda_k|^{1 - \frac{\rho(|\lambda_k|)}{2}}\right)$ in the first case, $k \geq 1$, of the circles $B\left(\lambda_k, d'|\lambda_{k(p)}|^{1 - \rho(|\lambda_{k(p)}|)}\right)$, $d' \leq \frac{d}{2}$, $k \geq 1$, in the second case, is called a R -set. We note that a R -set is not a C_0 -set. Its upper linear density $\sigma(B_\Lambda)$ is non-zero. However, under Condition (C'), by choosing $d > 0$, the quantity $\sigma(B_\Lambda)$ can be made arbitrarily small. Under Condition (C), the sum of areas of exceptional circles can be made arbitrarily small. B.Ya. Levin proved [1, Ch. II, Sect. 1, Thm. 5] that in the case as the zero set of the function f is a R -set, as an exceptional set B_f , the set B_Λ can serve in inequality (4.14).

A simple exceptional set in estimate (4.14) consisting of mutually disjoint circles arises owing to Conditions (C) and (C'), which separate simple points λ_k . In the present work we employ an essentially weaker condition allowing us to separate appropriately the points λ_k even in the case of multiple points. This condition is just vanishing of the condensation index S_Λ .

A properly distributed set Λ is called properly balanced if the condition $S_\Lambda = 0$ holds.

Let us establish a connection between the notions of regular and properly balanced set. We mention first that in distinction to a properly balanced set, the notion of a regular set is defined only for simple points λ_k .

We consider an example from work [3]. We denote $\Lambda_1 = \{\lambda_k, 1\}_{k=1}^\infty$ and we let $\lambda_{2k} = k$ and $\lambda_{2k-1} = k - e^{-\varepsilon(k)k}$, $k \geq 1$, where $0 < \varepsilon(k) \rightarrow 0$, $e^{-\varepsilon(k)k} \rightarrow 0$, $k \rightarrow \infty$ (for instance, $\varepsilon(k) = \frac{1}{\sqrt{k}}$). We also let $\Lambda_2 = \{-\lambda_k, 1\}_{k=1}^\infty$. It is easy to see that the sequence $\Lambda = \Lambda_1 \cup \Lambda_2$ has an angular density at order $\rho(r) \equiv 1$ and satisfies the Lindelöf condition, that is, this is a properly distributed set at order $\rho(r) \equiv 1$. It was proved in work [3] that the identity $S_{\Lambda_1} = 0$ holds true. This implies easily that $S_\Lambda = 0$, i.e., Λ is a properly balanced set. At that, Λ is not a regular set since $\lambda_{2k} - \lambda_{2k-1} = e^{-\varepsilon(k)k} \rightarrow 0$, $k \rightarrow \infty$ and Conditions (C) and (C') fail.

Thus, a properly balanced set (in the case of simple points) is not necessarily a regular set. The following inverse statements holds; it is implied by the examples considered in the second section.

Theorem 4.2. *Let $\Lambda = \{\lambda_k, 1\}$ be a regular set. Then Λ is a properly balanced set.*

Theorem 4.2 means that the notion of a properly balanced set is more general than the notion of a regular set. In what follows we shall show that both notions are introduced due to the same way.

Theorem 4.3. *Let f be an entire function of order at most $\rho(r)$ and of a completely regular growth, $\Lambda = \{\lambda_k, n_k\}$ be its multiple zero set. Assume that $\Gamma = \{\gamma_k\}$ satisfies (3.9), $\tau \in (0, \frac{1}{2}]$ and $S_\Lambda = 0$. Then (4.14) holds, where $B_f = B(\Lambda, \Gamma_0(\tau))$.*

Proof. By the assumption, the function f has a completely regular growth. Therefore, according the aforementioned result by B.Ya. Levin, Λ is a properly distributed set. In particular, Λ has an angular density. This implies that Λ has a density, that is, there exists a limit

$$\lim_{r \rightarrow \infty} \frac{n(r, \Lambda)}{r^{\rho(r)}} = n(\Lambda) < +\infty.$$

Hence,

$$\lim_{k \rightarrow \infty} \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \leq \lim_{k \rightarrow \infty} \frac{n(|\lambda_k| + 1, \Lambda) - n(|\lambda_k|, \Lambda)}{|\lambda_k|^{\rho(|\lambda_k|)}} = \lim_{k \rightarrow \infty} \frac{n(|\lambda_k| + 1, \Lambda)}{(|\lambda_k| + 1)^{\rho(|\lambda_k|)}} - \lim_{k \rightarrow \infty} \frac{n(|\lambda_k|, \Lambda)}{|\lambda_k|^{\rho(|\lambda_k|)}} = 0.$$

We have obtained the identity $m(\Lambda) = 0$. Thus, all assumptions of Lemma 3.3 are satisfied. According this lemma and Lemma 3.1, for each $\varepsilon > 0$ there exist numbers $R_0 > 0$ and $\delta \in (0, \frac{1}{3})$ such that

$$\ln |q_\Lambda(z, w, \delta)| \geq -\frac{\varepsilon |z|^{\rho(|z|)}}{4}, \quad z \in B(w, \delta |w|) \setminus B(\Lambda, \Gamma_0(\tau)), \quad |w| \geq R_0; \quad (4.16)$$

Now we find $\delta_0 \in (0, \delta)$ and $R_0(\varepsilon) \geq R_0$ such that inequalities (4.4) and (4.5) hold true.

By the assumption, the function f has a completely regular growth. In particular, (4.14) holds, where B_f is covered by circles $B(z_j, r_j)$ obeying (4.15). Hence, there exists $R_1 \geq R_0(\varepsilon)$ such that

$$\frac{1}{r} \sum_{|z_j| \leq 2r} r_j \leq \frac{\delta_0}{3}, \quad r \geq R_1. \quad (4.17)$$

According (4.14), we can assume that

$$\alpha(r) \geq -\frac{\varepsilon r^{\rho(r)}}{4}, \quad r \geq (1 - \delta_0)R_1. \quad (4.18)$$

Let $|z| \geq R_1$ and a circle $B(z_j, r_j)$ intersects the circle $4B(z, \delta_0|z|)$ at a point w . Assume that $|z_j| > |z|$. Then by (4.17)

$$\frac{r_j}{|z_j|} \leq \frac{1}{|z_j|} \sum_{|z_s| \leq |z_j|} r_s \leq \frac{\delta_0}{3}.$$

Therefore,

$$\left(1 - \frac{\delta_0}{3}\right) |z_j| < |w| < (1 + \delta_0)|z|$$

In view of the inequality $\delta_0 < \frac{1}{3}$ this implies that $|z_j| < 2|z|$. Thus, according (4.17), the sum d of the diameters of all circles $B(z_j, r_j)$ intersecting $B(z, \delta_0|z|)$ satisfies the estimate

$$\frac{d}{|z|} \leq \frac{1}{|z|} \sum_{|z_j| \leq 2|z|} 2r_j \leq \frac{2\delta_0}{3}.$$

This is why there exists $t \in (0, 1)$ such that the circumference $S(z, t\delta_0|z|)$ does not intersect the set B_f . We let $z = r_0 e^{i\varphi_0}$. Then by (4.14), (4.4), (4.5) and (4.18) we have:

$$\begin{aligned} \ln |f(re^{i\varphi})| &\geq r^{\rho(r)} h_f(\varphi) + \alpha(r) \geq r_0^{\rho(r_0)} h_f(\varphi_0) - \frac{\varepsilon r_0^{\rho(r_0)}}{4} - \frac{\varepsilon r^{\rho(r)}}{4} \\ &\geq r_0^{\rho(r_0)} h_f(\varphi_0) - \frac{3\varepsilon}{4} r_0^{\rho(r_0)}, \quad re^{i\varphi} \in S(z, t\delta_0|z|). \end{aligned} \quad (4.19)$$

We consider the function $g(\lambda) = f(\lambda)(q_\Lambda(\lambda, z, \delta_0))^{-1}$. It is analytic in the circle $B(z, \delta_0|z|)$ and by (2.4), (4.19) it satisfies the estimate

$$\ln |g(re^{i\varphi})| \geq r_0^{\rho(r_0)} h_f(\varphi_0) - \frac{3\varepsilon}{4} r_0^{\rho(r_0)}, \quad re^{i\varphi} \in S(z, t\delta_0|z|).$$

Since g has no zeroes in the ball $B(z, \delta_0|z|)$, by the minimum principle for harmonic functions we get:

$$\ln |g(z)| \geq r_0^{\rho(r_0)} h_f(\varphi_0) - \frac{3\varepsilon}{4} r_0^{\rho(r_0)}. \quad (4.20)$$

Let $z \in B(\Lambda, \Gamma_0(\tau))$ and $|z| \geq R_2$. Then by (4.16) and (4.20) and in view of (2.4) we have:

$$\begin{aligned} \ln |f(z)| &= \ln |g(z)| + \ln |q_\Lambda(z, z, \delta_0)| \geq \ln |g(z)| + \ln |q_\Lambda(z, z, \delta)| \\ &\geq |z|^{\rho(|z|)} h_f(\varphi_0) - \frac{3\varepsilon}{4} |z|^{\rho(|z|)} - \frac{\varepsilon |z|^{\rho(|z|)}}{4} = |z|^{\rho(|z|)} h_f(\varphi_0) - \varepsilon |z|^{\rho(|z|)}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives a required statement. \square

Remark 3. a) As it has been mentioned, under the assumptions of Theorem 4.3, the identity $m(\Lambda) = 0$ holds. Therefore, according Remark 2, the sequence $\Gamma = \{\gamma_k\}$ in Theorem 4.3 can be chosen so that (3.19) holds.

b) The set $B(\Lambda, \Gamma_0(\tau))$ is the union of mutually disjoint circles $B(\lambda_k, \gamma_k^0(\tau))$, where

$$\gamma_k^0(\tau) = \tau \min\{\gamma_k, \beta_k/2\}, \quad \beta_k = \min_{s \neq k} |\lambda_k - \lambda_s|, \quad k \geq 1, \quad \lim_{k \rightarrow \infty} \frac{n_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \ln \frac{\gamma_k}{|\lambda_k|} = 0.$$

According Remark 1.c), for simple points with $n_k = 1$, the latter identity is equivalent to the relation

$$\frac{\ln \gamma_k}{|\lambda_k|^{\rho(|\lambda_k|)}} \rightarrow 0, \quad k \rightarrow \infty.$$

In this case, by this remark, the sum of radii of exceptional circles $B(\lambda_k, \gamma_k^0(\tau))$ can be made arbitrarily small. In particular, in view of Theorem 4.2, this can be made as Λ is a regular set. We mention that an exceptional set in Theorem 5 in [1, Ch. II, Sect. 1] is essentially larger and it possesses only a finite upper linear density $\sigma(B_f)$.

Theorem 4.4. *Let f be an entire function of order at most $\rho(r)$, $\Lambda = \{\lambda_k, n_k\}$ be its multiple zero set and B_k , $k \geq 1$, be open mutually disjoint sets, $B = \cup_{k=1}^{\infty} B_k$. Assume that $\lambda_k \in B_k$, $k \geq 1$, the diameters d_k of the sets B_k satisfies (3.20) and that for each $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that*

$$\ln |f(re^{i\varphi})| \geq (h_f(\varphi) - \varepsilon)r^{\rho(r)}, \quad re^{i\varphi} \in \partial B \setminus B(0, r(\varepsilon)). \quad (4.21)$$

Then $S_{\Lambda} = 0$ and $m(\Lambda) = 0$.

Proof. Let us prove inequality (3.21). We fix $\varepsilon > 0$ and choose numbers $\delta_0 \in (0, 1)$ and $R_0(\varepsilon) \geq r(\varepsilon)$ such that (4.4) and (4.5) hold. We choose $R(\varepsilon) \geq R_0(\varepsilon)$ such that (4.1) is true. Finally, we choose a number k_0 such that

$$|\lambda_k| > R(\varepsilon)(1 - \delta_0)^{-1}, \quad k \geq k_0. \quad (4.22)$$

Let $k \geq k_0$ and $\lambda_k = r_k e^{i\varphi_k}$. We consider a function $g(z) = f(z)(q_{\Lambda}(z, \lambda_k, \frac{\delta_0}{5}))^{-1}$. By (2.5), (4.1), (4.22), (4.4) and (4.5) we have

$$\ln |g(re^{i\varphi})| \leq \ln |f(re^{i\varphi})| \leq (h_f(\varphi) + \varepsilon)r^{\rho(r)} \leq (h_f(\varphi_k) + 2\varepsilon)r_k^{\rho(r_k)}, \quad re^{i\varphi} \in S(\lambda_k, \delta_0|\lambda_k|).$$

By the minimum principle for subharmonic functions this yields:

$$\ln |g(re^{i\varphi})| \leq (h_f(\varphi_k) + 2\varepsilon)r_k^{\rho(r_k)}, \quad re^{i\varphi} \in B(\lambda_k, \delta_0|\lambda_k|).$$

Let $re^{i\varphi} \in B(\lambda_k, \frac{\delta_0|\lambda_k|}{5})$. Then $r > R(\varepsilon)$ and $\lambda_k \in B(re^{i\varphi}, \delta_0 r)$. Therefore, in view of (4.4) and (4.5), we have

$$\ln |g(re^{i\varphi})| \leq (h_f(\varphi) + 4\varepsilon)r^{\rho(r)}.$$

By (4.21) this implies that for all $k \geq k_0$ the inequality holds:

$$\ln |q_{\Lambda}(re^{i\varphi}, \lambda_k, \frac{\delta_0}{5})| = \ln |f(re^{i\varphi})| - \ln |g(re^{i\varphi})| \geq -5\varepsilon r^{\rho(r)}, \quad re^{i\varphi} \in B(\lambda_k, \frac{\delta_0|\lambda_k|}{5}) \cap \partial B.$$

This gives (3.21). Thus, in view of the assumptions of the theorem, all assumptions of Lemma 3.4 are satisfied. Hence, its statement is true and $S_{\Lambda} = m(\Lambda) = 0$. \square

Theorem 4.5. *Let f be an entire function of order at most $\rho(r)$ and $\Lambda = \{\lambda_k, n_k\}$ is its multiple zero set. The following statements are equivalent:*

1. Λ is a properly balanced set.

2. There exist numbers $\gamma_k^0 > 0$, $k \geq 1$, such that $\frac{\gamma_k^0}{|\lambda_k|} \rightarrow 0$, $k \rightarrow \infty$, the circles $B(\lambda_k, \gamma_k^0)$ are mutually disjoint and (4.14) holds, where $B_f = \cup_{k=1}^{\infty} B(\lambda_k, \gamma_k^0)$.

Proof. Assume that Statement 1 is true. Then $S_{\Lambda} = 0$ and Λ is a properly distributed set. According the above result by B.Ya. Levin, the function f has a completely regular growth. Thus, by Theorem 4.3 and Remark 3.a), Statement 2 holds.

Assume that Statement 2 holds. Then by Theorem 4.4 we have $S_{\Lambda} = 0$. Let us show that f is a function of a completely regular growth. As it has been observed in the beginning of the section, it is sufficient to show that f has a regular growth on each ray L_{φ} .

We form a sequence $\{p_m\}_{m=1}^{\infty}$ of all natural numbers p such that the semi-interval $[(p-1)e^{i\varphi}, pe^{i\varphi})$ does not lie entirely in B_f . For each $m \geq 1$, we arbitrarily choose a point $z_m \in [(p_m-1)e^{i\varphi}, p_m e^{i\varphi}) \setminus B_f$ and we obtain a sequence $\{z_m\}_{m=1}^{\infty}$. By construction, the first and second identity in (4.3) and in view of (4.14), the fourth identity in (4.3) holds as well. Let us show that also the third identity is true. Assume a contrary, then there exists a subsequence $\{z_{m_j}\}_{j=1}^{\infty}$ such that for some $\alpha > 1$ the inequalities

$$|z_{m_{j+1}}| > \alpha |z_{m_j}|, \quad j \geq 1,$$

hold. Then by construction, for each $j \geq 1$, the semi-interval $[(|z_{m_j}| + 1)e^{i\varphi}, [\alpha |z_{m_j}|]e^{i\varphi})$ is contained in B_f ; we recall that $[x]$ is an integer part of a number x . Since $B(\lambda_k, \gamma_k^0)$ are mutually

disjoint, this semi-interval is contained in some circle $B(\lambda_{k_j}, \gamma_{k_j}^0)$. Since $\frac{\gamma_{k_j}^0}{|\lambda_{k_j}|} \rightarrow 0$, then $\frac{|w_j|}{|\lambda_{k_j}|} \rightarrow 1$, $j \rightarrow \infty$, where $w_j = (|z_{m_j}| + 1)e^{i\varphi}$. This is why

$$0 = \lim_{j \rightarrow \infty} \frac{2\gamma_{k_j}^0}{|\lambda_{k_j}|} \geq \lim_{j \rightarrow \infty} \frac{[\alpha|z_{m_j}|] - [|z_{m_j}|] - 1}{|\lambda_{k_j}|} = \lim_{j \rightarrow \infty} \frac{[\alpha|z_{m_j}|] - [|z_{m_j}|]}{|w_j|} - 1 = \alpha - 1 > 0.$$

We have obtained a contradiction. Thus, identities (4.3) hold and by Lemma 4.1, the function f has a completely regular growth on the ray L_φ . \square

Remark 4. *By Theorem 4.2, a regular set is a particular case of a properly balanced set. The first appeared with the only aim to make the exceptional set in (4.14) consisting of mutually intersecting circles centered at zero with relatively small radii. According Theorem 4.5, a necessary and sufficient condition of this is a proper balancing of the zero set of the function f . Thus, the notion of a properly balanced set is a natural generalization of the notion of a regular set. We also note that then notion of a properly balanced set is effective for a multiple zero set.*

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