

DECAY OF SOLUTION OF ANISOTROPIC DOUBLY NONLINEAR PARABOLIC EQUATION IN UNBOUNDED DOMAINS

L.M. KOZHEVNIKOVA, A.A. LEONTIEV

Abstract. This work is devoted to a class of parabolic equations with a double nonlinearity whose representative is a model equation

$$(|u|^{k-2}u)_t = \sum_{\alpha=1}^n (|u_{x_\alpha}|^{p_\alpha-2}u_{x_\alpha})_{x_\alpha}, \quad p_n \geq \dots \geq p_1 > k, \quad k \in (1, 2).$$

For the solution of Dirichlet initial boundary value problem in a cylindrical domain $D = (0, \infty) \times \Omega$, $\Omega \subset \mathbb{R}_n$, $n \geq 2$, with homogeneous Dirichlet boundary condition and compactly supported initial function, precise estimates the decay rate as $t \rightarrow \infty$ are established. Earlier these results were obtained by the authors for $k \geq 2$. The case $k \in (1, 2)$ differs by the method of constructing Galerkin approximations that for an isotropic model equation was proposed by E.R. Andriyanova and F.Kh. Mukminov.

Keywords: anisotropic equation, doubly nonlinear parabolic equations, existence of strong solution, decay rate of solution.

1. INTRODUCTION

Let Ω be an unbounded domain in the space $\mathbb{R}_n = \{\mathbf{x} = (x_1, x_2, \dots, x_n)\}$, $n \geq 2$. In the cylindrical domain $D = \{t > 0\} \times \Omega$ we consider a Dirichlet initial boundary value problem for a second order anisotropic quasilinear parabolic equation

$$(|u|^{k-2}u)_t = \sum_{\alpha=1}^n (a_\alpha(u_{x_\alpha}^2)u_{x_\alpha})_{x_\alpha}, \quad k \in (1, 2), \quad (t, \mathbf{x}) \in D; \quad (1)$$

$$u(t, \mathbf{x}) \Big|_S = 0, \quad S = \{t > 0\} \times \partial\Omega; \quad (2)$$

$$u(0, \mathbf{x}) = \varphi(\mathbf{x}), \quad \varphi(\mathbf{x}) \in L_k(\Omega), \quad \varphi_{x_\alpha}(\mathbf{x}) \in L_{p_\alpha}(\Omega), \quad \alpha = \overline{1, n}. \quad (3)$$

Nonnegative functions $a_\alpha(s)$, $s \geq 0$, $\alpha = \overline{1, n}$ are assumed to obey the conditions $a_\alpha(0) = 0$, $a_\alpha(s) \in C^1(0, \infty)$,

$$\bar{a}s^{(p_\alpha-2)/2} \leq a_\alpha(s) \leq \hat{a}s^{(p_\alpha-2)/2}, \quad (4)$$

$$\frac{p_1}{2}a_\alpha(s) \leq a_\alpha(s) + a'_\alpha(s)s \leq \hat{b}a_\alpha(s) \quad (5)$$

with positive constants $\hat{a} \geq \bar{a}$, $2\hat{b} \geq p_1 > k$ ($p_1 \leq p_2 \leq \dots \leq p_n$). For example, $a_\alpha(s) = s^{(p_\alpha-2)/2}$, $\alpha = \overline{1, n}$, $\hat{b} = p_n/2$.

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The work is devoted to studying the stabilization rate as $t \rightarrow \infty$ of the solution to problem (1)–(3) with a compactly supported initial function $\varphi(\mathbf{x})$.

The study of behavior of solutions to initial boundary value problems for linear and quasi-linear parabolic equations of second and higher orders as $t \rightarrow \infty$ was done in the works of A.K. Gushchin, V.I. Ushakov, F.Kh. Mukminov, A.F. Tedeev, L.M. Kozhevnikova, R.Kh. Karimov, and others. The surveys of appropriate results can be found in [1], [2], [3].

In the isotropic case, i.e., as all p_α are same and equal to p , $p \geq 2$, for $k = 2$ problem (1)–(3) was studied in work [3]. Estimates for the decay rate of the solution to a Cauchy problem for a parabolic degenerate equation with the anisotropic p -Laplacian and a double nonlinearity as $k \in (1, 2)$ were established in the work of S.P. Degtyarev, A.F. Tedeev [4].

The questions on existence and uniqueness of solutions to an isotropic parabolic equation with a double nonlinearity were considered in the works by P.A. Raviart, J.L. Lions, A. Bamberger, O. Grange, F. Mignot, H.W. Alt, S. Luckhaus, F. Bernis, and others. However, to obtain a lower bound for the decay of the solution as $t \rightarrow \infty$ one needs additional smoothness.

F.Kh. Mukminov and E.R. Andriyanova [5] suggested a usual approach of construction a strong solution to a model isotropic parabolic equation with a double non-linearity in an unbounded domain on the basis of Galerkin approximations which in the cases $k \in (1, 2)$ and $k \geq 2$ are constructed in different ways. In work [6] this method was adapted to a certain class of anisotropic parabolic equation like (1) as $k \geq 2$ and on the basis of Galerkin approximation they obtained an estimate for the admissible decay rate of the solution in an unbounded domain. The present work is the continuation of work [6] for the case $k \in (1, 2)$.

We consider the domains located along a selected axis Ox_s , $s \in \overline{1, n}$ (the domain Ω lies in the half-space $\mathbb{R}_n^+[s] = \{\mathbf{x} \in \mathbb{R}_n \mid x_s > 0\}$, the cross-section $\gamma_r = \{\mathbf{x} \in \Omega \mid x_s = r\}$ is non-empty and bounded for each $r > 0$). In what follows we shall employ the notation $\Omega_a^b = \{\mathbf{x} \in \Omega \mid a < x_s < b\}$, at that the values $a = 0$, $b = \infty$ are omitted.

The initial function is assumed to be bounded and compactly supported so that

$$\text{supp } \varphi \subset \Omega^{R_0}, \quad R_0 > 0. \quad (6)$$

Theorem 1. *Let the domain is located along the axis Ox_s , $s \in \overline{1, n}$ and condition (6) is satisfied. Then there exist positive numbers $\kappa(p_s, k)$, $\mathcal{M}(p_s, k)$ and a bounded solution $u(t, \mathbf{x})$ to problem (1)–(3) such that for all $t > 0$, $r \geq 2R_0$ the estimate*

$$\|u(t)\|_{L_k(\Omega_r)} \leq \mathcal{M} \exp\left(-\kappa \left[\frac{r^{p_s}}{t}\right]^{1/(p_s-1)}\right) \|\varphi\|_{L_k(\Omega)} \quad (7)$$

holds true.

On the basis of inequality (7) we establish a lower estimate for decay of a solution to problem (1)–(3) as $t \rightarrow \infty$.

The admissible stabilization rate for a solution to an high order isotropic quasilinear parabolic equation as $k = 2$ was studied by A.F. Tedeev [7] for a Dirichlet initial boundary value problem and by N. Alikakos, R. Rostmanian [8] for a Cauchy problem.

Theorem 2. *Suppose the domain is located along the axis Ox_s , $s \in \overline{1, n}$ and condition (6) is satisfied. Then there exists a positive number $C(\varphi, k, p_1, \hat{a}, \hat{b})$ and a bounded solution $u(t, \mathbf{x})$ to problem (1)–(3) such that for all $t \geq 0$ the inequality*

$$\|u(t)\|_{L_k(\Omega)} \geq \|\varphi\|_{L_k(\Omega)} (C(\varphi)t + 1)^{-1/(p_1-k)} \quad (8)$$

holds true.

We define a function

$$\mu_1(r) = \inf \left\{ \|g_{x_1}\|_{L_{p_1}(\Omega^r)} \mid g(\mathbf{x}) \in C_0^\infty(\Omega), \|g\|_{L_k(\Omega^r)} = 1 \right\}, \quad r > 0. \quad (9)$$

we shall study the decay in the domain obeying the condition

$$\lim_{r \rightarrow \infty} \mu_1(r) = 0. \quad (10)$$

It is shown that if this condition is not satisfied, the maximal decay rate of a solution is attained, i.e., the estimate

$$\|u(t)\|_{L_k(\Omega)} \leq Mt^{-1/(p_1-k)}, \quad t > 0, \quad (11)$$

is valid (see [6, Cor. 2]).

We let

$$\nu(r) = \inf \left\{ \|g_{x_1}\|_{L_{p_1}(\gamma_r)} \mid g(\mathbf{x}) \in C_0^\infty(\Omega), \|g\|_{L_{p_1}(\gamma_r)} = 1 \right\}, \quad r > 0. \quad (12)$$

We assume that the domain Ω satisfies the condition

$$\int_1^\infty \nu^{p_1/p_s}(r) dr = \infty. \quad (13)$$

Let $r(t)$ be an arbitrary positive function obeying the inequality

$$(\mu_1^{p_1}(r(t))t)^{-1/(p_1-k)} \exp \left(\kappa \int_1^{r(t)} \nu^{p_1/p_s}(\rho) d\rho \right) \geq 1, \quad t > 0. \quad (14)$$

The existence of such function follows from (10). Moreover, it follows from (14), (10) that

$$\lim_{t \rightarrow \infty} r(t) = \infty.$$

Theorem 3. *Suppose the domain is located along the axis Ox_s , $s \in \overline{2, n}$ and the conditions (6), (10), (13) are satisfied. Then there exist a positive number $M(p_s, p_1, \|\varphi\|_{L_k(\Omega)})$ and a bounded solution $u(t, \mathbf{x})$ to problem (1)–(3) such that the estimate*

$$\|u(t)\|_{L_k(\Omega)} \leq M (t\mu_1^{p_1}(r(t)))^{-1/(p_1-k)}, \quad t > 0, \quad (15)$$

holds true.

If the conditions

$$\mu_1(r) \geq Cr^{-a}, \quad r > 1, \quad a, C > 0,$$

$$\lim_{r \rightarrow \infty} \frac{1}{\ln r} \int_1^r \nu^{p_1/p_s}(\rho) d\rho = \infty,$$

are satisfied, one can let

$$r(t) = t^{\varepsilon/(ap_1)}, \quad t > 0, \quad \varepsilon \in (0, 1), \quad (16)$$

and estimate (15) casts into the form

$$\|u(t)\|_{L_k(\Omega)} \leq Mt^{-(1-\varepsilon)/(p_1-k)}, \quad t > 0. \quad (17)$$

The choice of the function $r(t)$ by formula (16) is satisfactory since estimate (17) has an exponent close to the exponent $1/(p_1 - k)$ of lower bound (8). Other examples of the domains of revolution are provided in work [6].

2. AUXILIARY STATEMENTS

Let $\|\cdot\|_{p,Q}$ be the norm in $L_p(Q)$, $p \geq 1$, $(\cdot, \cdot)_Q$ be the scalar product in $L_2(Q)$ and the values $p = 2$, $Q = \Omega$ are omitted. By $D_a^b = (a, b) \times \Omega$ we denote the cylinder, the values $a = 0$ and $b = \infty$ can be absent.

A Banach space $\overset{\circ}{W}_{k,p}^1(\Omega)$ is introduced as the completion of the space $C_0^\infty(\Omega)$ w.r.t. the norm

$$\|u\|_{\overset{\circ}{W}_{k,p}^1(\Omega)} = \sum_{\alpha=1}^n \|u_{x_\alpha}\|_{p_\alpha} + \|u\|_k.$$

We introduce Banach spaces $\overset{\circ}{W}_{k,p}^{0,1}(D^T)$, $\overset{\circ}{W}_{k,p}^{1,1}(D^T)$ as the completions of the space $C_0^\infty(D_{-1}^{T+1})$ w.r.t. the norms

$$\begin{aligned} \|u\|_{\overset{\circ}{W}_{k,p}^{0,1}(D^T)} &= \|u\|_{k,D^T} + \sum_{\alpha=1}^n \|u_{x_\alpha}\|_{p_\alpha,D^T}, \\ \|u\|_{\overset{\circ}{W}_{k,p}^{1,1}(D^T)} &= \|u\|_{k,D^T} + \|u_t\|_{k,D^T} + \sum_{\alpha=1}^n \|u_{x_\alpha}\|_{p_\alpha,D^T}, \end{aligned}$$

respectively.

Definition 1. By a generalized solution to problem (1)–(3), we call a function $u(t, \mathbf{x})$ such that for all $T > 0$ $u(t, \mathbf{x}) \in \overset{\circ}{W}_{k,p}^{0,1}(D^T)$ and the integral identity

$$\int_{D^T} \left(-|u|^{k-2} u v_t + \sum_{\alpha=1}^n a_\alpha(u_{x_\alpha}^2) u_{x_\alpha} v_{x_\alpha} \right) d\mathbf{x} dt = \int_{\Omega} |\varphi(\mathbf{x})|^{k-2} \varphi(\mathbf{x}) v(0, \mathbf{x}) d\mathbf{x} \quad (18)$$

is satisfied for each function $v(t, \mathbf{x}) \in \overset{\circ}{W}_{k,p}^{1,1}(D^T)$, $v(T, \mathbf{x}) = 0$.

Conditions (5) imply inequalities

$$(p_1 - 1)a_\alpha(s) \leq a_\alpha(s) + 2a'_\alpha(s)s \leq \widehat{c}a_\alpha(s), \quad \widehat{c} = 2\widehat{b} - 1, \quad s \geq 0, \quad \alpha = \overline{1, n}, \quad (19)$$

which can be rewritten as

$$0 \leq (a_\alpha(z^2)z)' \leq \widehat{c}a_\alpha(z^2), \quad z \in \mathbb{R}, \quad \alpha = \overline{1, n}. \quad (20)$$

We let $A_\alpha(s) = \int_0^s a_\alpha(\tau) d\tau$, then, employing conditions (5), we deduce the inequalities

$$\frac{p_1}{2} A_\alpha(s) \leq a_\alpha(s)s \leq \widehat{b} A_\alpha(s), \quad s \geq 0, \quad \alpha = \overline{1, n}. \quad (21)$$

Lemma 1. Each bounded set of reflexive Banach space is weakly compact (see [9, Ch. V, Sec. 19.7, Th. 1]).

Remark 1. The spaces $\overset{\circ}{W}_{k,p}^1(\Omega)$, $\overset{\circ}{W}_{k,p}^{0,1}(D^T)$ are reflexive separable Banach spaces (cf. [6, Rem. 1]).

Remark 2. In what follows, in order to avoid bulking while arguing, instead of the statements like “in a sequence u^M one can select a subsequence u^{M_i} converging in $L_2(\Omega)$ as $i \rightarrow \infty$ ” we shall say shortly “a sequence u^M selectively converges in $L_2(\Omega)$ as $M \rightarrow \infty$ ”. In a similar way we shall employ the notion “selectively weakly converges”, etc.

Lemma 2. Let $g^M(t, \mathbf{x})$, $M = \overline{1, \infty}$, $g(t, \mathbf{x})$ be functions in $L_p(Q)$, $1 < p < \infty$ such that

$$\|g^M\|_{p,Q} \leq C, \quad g^M \rightarrow g \text{ as } M \rightarrow \infty \text{ a.e. in } Q,$$

then $g^M \rightharpoonup g$ as $M \rightarrow \infty$ weakly in $L_p(Q)$ (see [10, Ch. I, Sec. 1.4, Lem. 1.3])

Remark 3. Lemma 2 is formulated in [10] for a bounded domain Q , but it is true also for an arbitrary unbounded domain. We shall apply Lemma 2 for $Q = \Omega$ and for $Q = (0, T) \times \Omega$.

Lemma 3. Let the system of functions $\psi_i(\mathbf{x}) \in C_0^\infty(\Omega)$, $i = \overline{1, \infty}$, is linearly independent and its linear span is a dense set in the space $\mathring{W}_{k, \mathbf{p}}^1(\Omega)$. By P_L we denote the set of the functions $\sum_{i=1}^L d_i(t)\psi_i(\mathbf{x})$, where $d_i(t) \in C^\infty[0, T]$. Then the set $P = \bigcup_{L=1}^\infty P_L$ is dense in the space $\mathring{W}_{k, \mathbf{p}}^{1,1}(D^T)$.

Proof. Let us prove the density of the set P in the space $C_0^\infty(D_{-1}^{T+1})$. Let $v(t, \mathbf{x}) \in C_0^\infty(D_{-1}^{T+1})$, obviously, $v(t, \mathbf{x}) \in C([-1, T+1] \rightarrow \mathring{W}_{k, \mathbf{p}}^1(\Omega))$. We choose an arbitrary ε and fix δ such that for all $t, t^* \in [-1, T+1]$ obeying $|t - t^*| < 2\delta$ the inequality

$$\|v(t) - v(t^*)\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} < \varepsilon \quad (22)$$

holds true.

We choose a finite sequence of points t_j , $j = \overline{1, N}$ such that $(-1, T+1) = \bigcup_{j=1}^N (t_j - \delta, t_j + \delta)$ and a partition of the unity

$$\sum_{j=1}^N w_j(t) = 1, \quad w_j(t) \in C_0^\infty((t_j - \delta, t_j + \delta)) \quad 0 \leq w_j(t) \leq 1.$$

The definition of the system of the functions $\psi_k(\mathbf{x})$ implies that for each j , $j = \overline{1, N}$, there exists an index $L_j(\varepsilon)$ and numbers f_{jk} such that

$$\|v(t_j, \mathbf{x}) - \sum_{k=1}^{L_j} f_{jk}\psi_k(\mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} < \varepsilon, \quad j = \overline{1, N}. \quad (23)$$

Let us show that the functions $\sum_{j=1}^N \sum_{k=1}^{L_j} w_j(t)f_{jk}\psi_k(\mathbf{x}) = \sum_{k=1}^L \left(\sum_{j=1}^N w_j(t)f_{jk} \right) \psi_k(\mathbf{x})$, $L = \max_{j=1, N} L_j$,

$f_{jk} = 0$, $k > L_j$, approximate the function $v(t, \mathbf{x})$ in the norm of the space $\mathring{W}_{k, \mathbf{p}}^1(\Omega)$ uniformly in $t \in [-1, T+1]$. Indeed, employing (22), (23), we deduce

$$\begin{aligned} & \max_{t \in [-1, T+1]} \|v(t, \mathbf{x}) - \sum_{j=1}^N \sum_{k=1}^{L_j} w_j(t)f_{jk}\psi_k(\mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} = \\ & = \max_{t \in [-1, T+1]} \left\| \sum_{j=1}^N w_j(t)v(t, \mathbf{x}) - \sum_{j=1}^N \sum_{k=1}^{L_j} w_j(t)f_{jk}\psi_k(\mathbf{x}) \right\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} \leq \\ & \leq \sum_{j=1}^N \max_{[-\delta+t_j, \delta+t_j]} \|w_j(t)v(t, \mathbf{x}) - w_j(t) \sum_{k=1}^L f_{jk}\psi_k(\mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} \leq \\ & \leq \sum_{j=1}^N \max_{[-\delta+t_j, \delta+t_j]} \|v(t, \mathbf{x}) - \sum_{k=1}^L f_{jk}\psi_k(\mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} \leq \\ & \leq \sum_{j=1}^N \max_{[-\delta+t_j, \delta+t_j]} \|v(t, \mathbf{x}) - v(t_j, \mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} + \\ & + \sum_{j=1}^N \|v(t_j, \mathbf{x}) - \sum_{k=1}^L f_{jk}\psi_k(\mathbf{x})\|_{\mathring{W}_{k, \mathbf{p}}^1(\Omega)} \leq N\varepsilon + N\varepsilon = 2N\varepsilon = \varepsilon_1. \end{aligned}$$

We introduce the notation $f_k(t) = \sum_{k=1}^L w_j(t) f_{jk}$. We take $w \in C_0^\infty(D_{-1}^{T+1})$ and let $v(t, \mathbf{x}) = w_t(t, \mathbf{x}) \in C_0^\infty(D_{-1}^{T+1})$. According to proven above, for each $\varepsilon_1 > 0$ there exists $L(\varepsilon_1)$ such that

$$\max_{t \in [-1, T+1]} \left\| w_t - \sum_{k=1}^{L(\varepsilon_1)} f_k(t) \psi_k(\mathbf{x}) \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} < \varepsilon_1. \quad (24)$$

Consider the function $w(t, \mathbf{x}) = \int_{-1}^t w_\tau(\tau, \mathbf{x}) d\tau$ and let us show that the functions $\sum_{k=1}^L \left(\int_{-1}^t f_k(\tau) d\tau \right) \psi_k(\mathbf{x})$ approximate the function $w(t, \mathbf{x})$ in the space $\dot{W}_{k, \mathbf{p}}^1(\Omega)$ uniformly in $t \in [-1, T+1]$. Indeed, as $L \rightarrow \infty$

$$\begin{aligned} & \max_{t \in [-1, T+1]} \left\| w(t, \mathbf{x}) - \sum_{k=1}^L \int_{-1}^t f_k(\tau) \psi_k(\mathbf{x}) d\tau \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} = \\ &= \max_{t \in [-1, T+1]} \left\| \int_{-1}^t w_\tau(\tau, \mathbf{x}) d\tau - \sum_{k=1}^L \int_{-1}^t f_k(\tau) \psi_k(\mathbf{x}) d\tau \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} = \\ &= \max_{t \in [-1, T+1]} \left\| \int_{-1}^t \left(w_\tau(\tau, \mathbf{x}) - \sum_{k=1}^L f_k(\tau) \psi_k(\mathbf{x}) \right) d\tau \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} \leq \\ &\leq (T+2) \max_{\tau \in [-1, T+1]} \left\| w_\tau(\tau, \mathbf{x}) - \sum_{k=1}^L f_k(\tau) \psi_k(\mathbf{x}) \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} \rightarrow 0. \end{aligned}$$

Denote $d_k(\tau) = \int_{-1}^{\tau} f_k(\rho) d\rho$, then inequality (24) and the latter relations as $L \rightarrow \infty$ imply

$$\begin{aligned} & \max_{\tau \in [-1, T+1]} \left\| w_\tau(\tau, \mathbf{x}) - \sum_{k=1}^L d'_k(\tau) \psi_k(\mathbf{x}) \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} \rightarrow 0, \\ & \max_{[-1, T+1]} \left\| w(\tau, \mathbf{x}) - \sum_{k=1}^L d_k(\tau) \psi_k(\mathbf{x}) \right\|_{\dot{W}_{k, \mathbf{p}}^1(\Omega)} \rightarrow 0, \end{aligned}$$

that yields $\left\| w(\tau, \mathbf{x}) - \sum_{k=1}^L d_k(\tau) \psi_k(\mathbf{x}) \right\|_{\dot{W}_{k, \mathbf{p}}^1(D_{-1}^{T+1})} \rightarrow 0$. \square

Theorem 4. Let $\varphi(\mathbf{x}) \in \dot{W}_{k, \mathbf{p}}^1(\Omega)$, $p_1 \geq k$, $k \in (1, 2)$, then there exists a generalized solution $u(t, \mathbf{x})$ to problem (1)–(3) which for each $T > 0$ satisfies the conditions

$$u \in L_\infty((0, \infty), \dot{W}_{k, \mathbf{p}}^1(\Omega)); \quad (25)$$

$$|u|^{(k-2)/2} u_t \in L_2(D^T), \quad u \in C([0, \infty), L_k(\Omega)); \quad (26)$$

$$u_t \in L_k(D^T). \quad (27)$$

At that, the inequalities

$$(k-1) \|u(t)\|_k^k + k\bar{a} \sum_{\alpha=1}^n \int_0^t \|u_{x_\alpha}(\tau)\|_{p_\alpha}^{p_\alpha} d\tau \leq (k-1) \|\varphi\|_k^k, \quad t \geq 0, \quad (28)$$

$$(k-1) \frac{d}{dt} \|u(t)\|_k^k + k\bar{a} \sum_{\alpha=1}^n \|u_{x_\alpha}(t)\|_{p_\alpha}^{p_\alpha} \leq 0, \quad t > 0, \quad (29)$$

hold true.

Proof. We choose a linearly independent system of functions $\psi_i(\mathbf{x}) \in C_0^\infty(\Omega)$, $i = \overline{1, \infty}$, such that its linear span is a dense set in the space $\mathring{W}_{k, \mathbf{p}}^1(\Omega)$. This system is assumed to be orthonormalized in $L_2(\Omega)$. We let $I^M = \bigcup_{i=1}^M \text{supp } \psi_i(\mathbf{x})$, $m_i = \max_{\mathbf{x} \in I^M} |\psi_i(\mathbf{x})|$.

We seek approximate solutions $u^M(t, \mathbf{x})$ as $u^M(t, \mathbf{x}) = \sum_{i=1}^M c_i^M(t) \psi_i(\mathbf{x})$, $M = \overline{1, \infty}$. At that, the functions $c_i^M(t)$, $t \in [0, \infty)$, are determined by the system of ordinary differential equations

$$\left((\omega^M)^{k/2-1} u^M \right)_t, \psi_j + \sum_{\alpha=1}^n \left(a_\alpha (u_{x_\alpha}^M)^2 \right) u_{x_\alpha}^M, (\psi_j)_{x_\alpha} = 0, \quad (30)$$

$$\omega^M = (u^M)^2 + \frac{k}{2} \varepsilon^M, \quad j = \overline{1, M},$$

(we shall choose the numbers $\varepsilon^M > 0$ later) and by initial conditions

$$c_i^M(0) = c_i^M, \quad i = \overline{1, M}, \quad (31)$$

chosen so that

$$u^M(0, \mathbf{x}) = \sum_{i=1}^M c_i^M \psi_i(\mathbf{x}) \rightarrow \varphi(\mathbf{x}) \text{ in } \mathring{W}_{k, \mathbf{p}}^1(\Omega) \text{ as } M \rightarrow \infty. \quad (32)$$

It implies immediately that

$$\|u^M(0)\|_{W_{k, \mathbf{p}}^1(\Omega)} \leq E_1(\|\varphi\|_{W_{k, \mathbf{p}}^1(\Omega)}), \quad M = \overline{1, \infty}. \quad (33)$$

Let us make sure that equations (30) are solvable w.r.t. the derivatives $\frac{d}{dt} c_i^M(t)$. It is obvious that equations (30) read as

$$\sum_{i=1}^M A_{ji}(c_1^M(t), \dots, c_M^M(t)) \frac{d}{dt} c_i^M(t) = F_j(c_1^M(t), \dots, c_M^M(t)), \quad j = \overline{1, M}, \quad (34)$$

$$\begin{aligned} A_{ji}(c_1, \dots, c_M) &= \left(\left(\varepsilon^M \frac{k}{2} + (k-1) \left(\sum_{l=1}^M c_l \psi_l \right)^2 \right) (\omega^M)^{k/2-2} \psi_i, \psi_j \right) = \\ &= (\psi_i, \psi_j)_M, \quad i, j = \overline{1, M}, \quad F_j(c_1, \dots, c_M) = \\ &= - \sum_{\alpha=1}^n \sum_{i=1}^M c_i \left(\left(\sum_{l=1}^M c_l (\psi_l)_{x_\alpha} \right)^2 \right) (\psi_i)_{x_\alpha}, \quad j = \overline{1, M}. \end{aligned}$$

It is easy to check that $(g, h)_M$, $g, h \in C_0^\infty(\Omega)$, is a scalar product. Therefore, for each t the matrix of the coefficients $A_{ji}(c_1^M(t), \dots, c_M^M(t))$ is the Gram matrix of the system of linearly independent vectors ψ_i , $i = \overline{1, M}$, and is invertible. This is why system (34) can be rewritten as

$$\frac{d}{dt} c_i^M(t) = \sum_{j=1}^M A_{ij}^{-1}(c_1^M(t), \dots, c_M^M(t)) F_j(c_1^M(t), \dots, c_M^M(t)), \quad i = \overline{1, M}. \quad (35)$$

Let us establish the estimate for Galerkin approximations. We multiply j th equation in (30) by $c_j^M(t)$ and sum up then all the equations in j from 1 to M that results in the identities

$$\left((\omega^M)^{k/2-1} u^M \right)_t, u^M + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, u_{x_\alpha}^M) = 0, \quad M = \overline{1, \infty},$$

which can be rewritten as

$$\frac{d}{dt} \left(\frac{k-1}{k} \int_{I^M} (\omega^M)^{k/2} d\mathbf{x} - \varepsilon^M \frac{k}{2} \int_{I^M} (\omega^M)^{k/2-1} \right) + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, u_{x_\alpha}^M) = 0, \quad M = \overline{1, \infty}. \quad (36)$$

After integration from 0 to t we get

$$\begin{aligned} & \frac{k-1}{k} \|(\omega^M)^{1/2}(t)\|_{k, I^M}^k - \varepsilon^M \frac{k}{2} \int_{I^M} (\omega^M(t, \mathbf{x}))^{k/2-1} d\mathbf{x} + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, u_{x_\alpha}^M)_{Dt} = \\ & = \frac{k-1}{k} \|(\omega^M)^{1/2}(0)\|_{k, I^M}^k - \varepsilon^M \frac{k}{2} \int_{I^M} (\omega^M(0, \mathbf{x}))^{k/2-1} d\mathbf{x}, \quad M = \overline{1, \infty}. \end{aligned} \quad (37)$$

We choose the numbers $\varepsilon^M \leq 1/M$ to satisfy inequalities

$$\text{mes } I^M \leq (\varepsilon^M)^{-k/4}, \quad M = \overline{1, \infty}. \quad (38)$$

Employing (38), (33), we deduce the inequalities

$$\begin{aligned} & \|(\omega^M)^{1/2}(0)\|_{k, I^M} \leq \| |u^M(0)| + \left(\frac{k}{2} \varepsilon^M \right)^{1/2} \|_{k, I^M} \leq \|u^M(0)\|_k + \\ & + \left(\frac{k}{2} \varepsilon^M \right)^{1/2} (\text{mes } I^M)^{1/k} \leq E_1 + \left(\frac{k}{2} \right)^{1/2} (\varepsilon^M)^{1/4} \leq E_1 + \left(\frac{k}{2} \right)^{1/2}, \end{aligned} \quad (39)$$

$$\frac{k}{2} \varepsilon^M \int_{I^M} (\omega^M(t, \mathbf{x}))^{k/2-1} d\mathbf{x} \leq \left(\frac{k}{2} \varepsilon^M \right)^{k/2} \text{mes } I^M \leq \left(\frac{k}{2} \right)^{k/2} (\varepsilon^M)^{k/4} \leq \left(\frac{k}{2} \right)^{k/2}. \quad (40)$$

Taking into consideration (4) and mimicking (39), (40), (37), for $t \geq 0$ we obtain

$$\|(\omega^M)^{1/2}(t)\|_k + \sum_{\alpha=1}^n \|u_{x_\alpha}^M\|_{p_\alpha, Dt}^{p_\alpha} \leq E_2, \quad M = \overline{1, \infty}. \quad (41)$$

Moreover, inequalities (4), (41) allows one to establish the estimates

$$\sum_{\alpha=1}^n \|a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M\|_{p_\alpha/(p_\alpha-1), Dt} \leq \widehat{a} \sum_{\alpha=1}^n \|u_{x_\alpha}^M\|_{p_\alpha, Dt}^{p_\alpha-1} \leq E_3, \quad M = \overline{1, \infty}. \quad (42)$$

Hereinafter the constants E_i depend only on \widehat{a} , \widehat{a} , \widehat{b} , \mathbf{p} , $\|\varphi\|_{W_{k, \mathbf{p}}^1(\Omega)}$.

Let us show that all possible solutions to problem (31), (35) are uniformly bounded as $t \geq 0$. Indeed, employing (41), for $t \geq 0$ we deduce

$$\begin{aligned} & \sum_{j=1}^M |c_j^M(t)|^2 = \|u^M(t)\|^2 = \\ & = \int_{\Omega} |u^M(t)|^k |u^M(t)|^{2-k} d\mathbf{x} \leq \|u^M(t)\|_k^k \max_{\mathbf{x} \in I^M} \left| \sum_{j=1}^M c_j^M(t) \psi_j(\mathbf{x}) \right|^{2-k} \leq \end{aligned}$$

$$\leq E_2 \left(\sum_{j=1}^M |c_j^M(t)|^2 \right)^{(2-k)/2} \left(\sum_{j=1}^M m_j^2 \right)^{(2-k)/2}.$$

It yields

$$|c_i^M(t)|^k \leq \left(\sum_{j=1}^M |c_j^M(t)|^2 \right)^{k/2} \leq E_2 \left(\sum_{j=1}^M m_j^2 \right)^{(2-k)/2}, \quad i = \overline{1, M}.$$

In view of the continuity of the right hand side of equations (35), there exist absolute continuous functions $c_i^M(t)$, $t \in [0, \infty)$, $i = \overline{1, M}$, which almost everywhere satisfy system (35) and initial condition (31) (see [11, Ch. VIII, Sec. 8]).

We multiply j th equation in (30) by $\frac{d}{dt}c_j^M(t)$ and sum up then all the equations in j from 1 to M that results in the identities

$$\left((\omega^M)^{k/2-1} u^M \right)_t, u_t^M + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, u_{t x_\alpha}^M) = 0, \quad M = \overline{1, \infty},$$

which can be rewritten as

$$\begin{aligned} (k-1) \|(\omega^M)^{k/4-1} u_t^M u^M\|^2 + \varepsilon^M \frac{k}{2} \|(\omega^M)^{k/4-1} u_t^M\|^2 + \\ + \frac{1}{2} \frac{d}{dt} \sum_{\alpha=1}^n \int_{\Omega} A_\alpha ((u_{x_\alpha}^M(t))^2) d\mathbf{x} = 0, \quad M = \overline{1, \infty}. \end{aligned} \quad (43)$$

After integration from 0 to t , employing (21), we get

$$\begin{aligned} (k-1) \|(\omega^M)^{k/4-1} u_t^M u^M\|_{D^t}^2 + \varepsilon^M \frac{k}{2} \|(\omega^M)^{k/4-1} u_t^M\|_{D^t}^2 + \\ + \frac{1}{2b} \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M(t))^2) u_{x_\alpha}^M(t), u_{x_\alpha}^M(t)) \leq \\ \leq \frac{1}{p_1} \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M(0))^2) u_{x_\alpha}^M(0), u_{x_\alpha}^M(0)), \quad M = \overline{1, \infty}. \end{aligned}$$

Further, due to the inequalities $(k-1)(u^M)^2 + \varepsilon^M \frac{k}{2} \geq (k-1)\omega^M$, applying (4) and using (33), we deduce

$$\|(\omega^M)^{(k-2)/4} u_t^M\|_{D^t}^2 + \sum_{\alpha=1}^n \|u_{x_\alpha}^M(t)\|_{p_\alpha}^{p_\alpha} \leq E_5, \quad M = \overline{1, \infty}. \quad (44)$$

Let T be an arbitrary positive number. Inequalities (41), (44) imply the boundedness of the sequence $\{(\omega^M)^{1/2}\}_{M=1}^\infty$ in the spaces $C([0, \infty), L_k(\Omega))$, $L_k(D^T)$ and the same for the sequence $\{u^M\}_{M=1}^\infty$ in the space $C([0, \infty), \dot{W}_{k, \mathbf{p}}^1(\Omega))$, $\dot{W}_{k, \mathbf{p}}^{0,1}(D^T)$ and for the sequence $\{(\omega^M)^{(k-2)/4} u_t^M\}_{M=1}^\infty$ in $L_2(D^T)$. Moreover, inequalities (42) yield the boundedness of the sequences $a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M$ in the spaces $L_{p_\alpha/(p_\alpha-1)}(D^T)$, $\alpha = \overline{1, n}$. These facts ensure the selective weak convergence of the mentioned sequences as $M \rightarrow \infty$ in the following spaces,

$$\begin{aligned} u^M \rightharpoonup u \quad \text{in} \quad \dot{W}_{k, \mathbf{p}}^{0,1}(D^T), \\ a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M \rightharpoonup b_\alpha \quad \text{in} \quad L_{p_\alpha/(p_\alpha-1)}(D^T), \quad \alpha = \overline{1, n}. \end{aligned}$$

Moreover, consider the sequence $v^M = (\omega^M)^{(k-2)/4}u^M$, $M = \overline{1, \infty}$, and the sequence of its derivatives $v_t^M = (\omega^M)^{(k-6)/4}u_t^M(\frac{k-2}{2}u^2 + \omega^M)$, $M = \overline{1, \infty}$. It is obvious that (44) implies inequalities

$$\|v_t^M\|_{D^t} \leq \frac{k}{2} \|(\omega^M)^{(k-2)/4}u_t^M\|_{D^t} \leq E_6, \quad M = \overline{1, \infty}, \quad (45)$$

which yield the selective weak convergence

$$v_t^M \rightharpoonup g \quad \text{in } L_2(D^T).$$

In what follows we shall prove that u^M selectively a.e. in D converges to u and it will allow us to establish that $g = (|u|^{(k-2)/2}u)_t$.

The sequence $u^M \in C([0, \infty), \overset{\circ}{W}_{k,p}^1(\Omega))$, $M = \overline{1, \infty}$, is bounded in this space. For each bounded domain $Q \subset \Omega$ with a smooth boundary we have a compact embedding $L_1(Q) \subset W_1^1(Q)$. This is why by a diagonal process one can establish a strong selective convergence $u^M(t_j, \mathbf{x}) \rightarrow h(t_j, \mathbf{x})$ in $L_1(Q)$ on a countable dense set $\{t_j\}_{j=1}^\infty \subset [0, T]$. We shall assume that $0, T \in \{t_j\}_{j=1}^\infty$. One can also assume that $u^M(t_j, \mathbf{x}) \rightarrow h(t_j, \mathbf{x})$ selectively a.e. in Q for each t_j , $j = \overline{1, \infty}$. Completely in the same way as $k \leq p_1$ we can assume that the sequence $u^M(t_j, \mathbf{x}) \rightarrow h(t_j, \mathbf{x})$ strongly in $L_k(Q)$ for each t_j , $j = \overline{1, \infty}$.

Following J.L. Lions [10, Ch. I, Sec. 12.2], let us prove the selective strong convergence of the sequence $v^M = (\omega^M)^{(k-2)/4}u^M$ in the space $C([0, T], L_1(Q))$. First, applying (45), let us prove the equicontinuity in t of the sequence v^M in $L_2(\Omega)$,

$$\begin{aligned} \|v^M(t_2) - v^M(t_1)\| &= \left\| \int_{t_1}^{t_2} v_t^M(t) dt \right\| \leq \int_{t_1}^{t_2} \|v_t^M(t)\| dt \leq \\ &\leq |t_2 - t_1|^{1/2} \left(\int_{t_1}^{t_2} \|v_t^M(t)\|^2 dt \right)^{1/2} \leq E_6 |t_2 - t_1|^{1/2}, \quad t_1, t_2 \in [0, T], \quad M = \overline{1, \infty}. \end{aligned} \quad (46)$$

Inequalities (41) imply the uniform in $t \in [0, T]$ boundedness of the sequence $v^M(t, \mathbf{x})$ in $L_2(\Omega)$,

$$\|v^M(t)\| = \|(\omega^M)^{(k-2)/4}u^M(t)\| \leq \|(\omega^M)^{k/4}\| = \|(\omega^M)^{1/2}\|_k^{k/2} \leq E_7, \quad M = \overline{1, \infty}.$$

Due to the boundedness of the sequence $v^M(t, \mathbf{x})$, $M = \overline{1, \infty}$, in the space $C([0, T], L_2(\Omega))$, it selectively weakly converges in $L_2(\Omega)$ for the same t_j as above. The established selective convergence $u^M(t_j, \mathbf{x}) \rightarrow h(t_j, \mathbf{x})$ a.e. in Q for each t_j implies selective convergence $v^M(t_j, \mathbf{x}) \rightarrow v(t_j, \mathbf{x}) = |h(t_j, \mathbf{x})|^{(k-2)/2}h(t_j, \mathbf{x})$ a.e. in Q . Then, by Egorov theorem, for each $\delta > 0$ we obtain the uniform convergence $v^M(t_j, \mathbf{x}) \rightrightarrows v(t_j, \mathbf{x})$ on Q_δ , $\text{mes}(Q \setminus Q_\delta) < \delta$. By inequalities

$$\begin{aligned} \|v^M(t_j) - v(t_j)\|_{1,Q} &\leq \text{mes } Q \max_{\mathbf{x} \in Q_\delta} |v^M(t_j, \mathbf{x}) - v(t_j, \mathbf{x})| + \|v^M(t_j) - v(t_j)\|_{1,Q \setminus Q_\delta} \leq \\ &\leq \text{mes } Q \max_{\mathbf{x} \in Q_\delta} |v^M(t_j, \mathbf{x}) - v(t_j, \mathbf{x})| + \delta^{1/2} \|v^M(t_j) - v(t_j)\|_{2,Q \setminus Q_\delta} \end{aligned}$$

it implies the strong convergence $v^M(t_j, \mathbf{x}) \rightarrow v(t_j, \mathbf{x})$ in $L_1(Q)$ for each t_j .

For a bounded domain Q by (46) one can easily establish the uniform fundamentality of the sequence $v^M(t, \mathbf{x})$ in the norm of $L_1(Q)$,

$$\begin{aligned} \|v^N(t) - v^M(t)\|_{1,Q} &= \|v^N(t) - v^N(t_{j_i}) + v^N(t_{j_i}) - v^M(t_{j_i}) + v^M(t_{j_i}) - v^M(t)\|_{1,Q} \leq \\ &\leq 2(\text{mes } Q)^{1/2} E_6 |t - t_{j_i}|^{1/2} + \|v^N(t_{j_i}) - v^M(t_{j_i})\|_{1,Q}. \end{aligned}$$

Choosing a finite set of the numbers t_{j_i} with a small step and increasing then N, M , we achieve the uniform in t smallness of the right hand side.

Thus, we have established the selective strong convergence of $v^M \rightarrow v$ in $C([0, T], L_1(Q))$. The convergence holds also in $L_1((0, T) \times Q)$, and hence $v^M \rightarrow v$ selectively converges a.e. in $(0, T) \times Q$. Due to the arbitrariness of Q , the sequence v^M selectively converges to v a.e. in D^T . Moreover, due to the the arbitrariness of T , choosing $T = 1, 2, \dots$, by a diagonal process one can select a subsequence $v^M \rightarrow v$ a.e. in D as $M \rightarrow \infty$. Then the subsequence $u^M(t, \mathbf{x})$ selectively converges to $h(t, \mathbf{x})$ a.e. in D . According to Lemma 2, $u^M(t, \mathbf{x}) \rightarrow h(t, \mathbf{x})$ in $L_k(D^T)$ for any $T > 0$, and by the uniqueness of the limit $h(t, \mathbf{x}) = u(t, \mathbf{x})$ a.e. in D . Thus, v^M selectively converges to $v = |u|^{(k-2)/2}u$ a.e. in D .

According to Lemma 2, $v^M \rightharpoonup v$ weakly in $L_2(D^T)$. Further, $(v_t^M, w)_{D^T} = -(v^M, w_t)_{D^T}$ for each function $w \in C_0^\infty(D^T)$, and passing to the limit as $M \rightarrow \infty$, we obtain

$$(g, w)_{D^T} = -(v, w_t)_{D^T}.$$

It follows that $g = v_t = (|u|^{(k-2)/2}u)_t$. We observe that the belonging $v, v_t \in L_2(D^T)$ implies $v \in C([0, \infty), L_2(\Omega))$.

Let us show that the sequence u_t^M , $M = \overline{1, \infty}$, is bounded in $L_k(D^T)$. Indeed, it follows from (41), (44) that

$$\begin{aligned} \|u_t^M\|_{k, D^T} &= \left(\int_{D^T} (\omega^M)^{k(k-2)/4} |u_t^M|^k (\omega^M)^{(2-k)k/4} d\mathbf{x} dt \right)^{1/k} \leq \\ &\leq \|(\omega^M)^{(k-2)/4} u_t^M\|_{2, D^T} \|(\omega^M)^{1/2}\|_{k, D^T}^{(2-k)/2} \leq E_8. \end{aligned}$$

The boundedness of $\|u_t^M\|_{k, D^T}$ implies $u_t^M \rightharpoonup b$ in $L_k(D^T)$. Then $(u_t^M, w)_{D^T} = -(u^M, w_t)_{D^T}$ for each function $w \in C_0^\infty(D^T)$, and passing to the limit as $M \rightarrow \infty$, we get

$$(b, w)_{D^T} = -(u, w_t)_{D^T},$$

hence, $b = u_t$. Then one can assume that $u_t^M \rightharpoonup u_t$ weakly in $L_k(D^T)$. We note that the belonging $u, u_t \in L_k(D^T)$ yields $u \in C([0, \infty), L_k(\Omega))$.

On one hand, estimate (33) and the convergence $u^M(0, \mathbf{x}) \rightarrow u(0, \mathbf{x})$ a.e. as $M \rightarrow \infty$ by Lemma 2 imply the weak convergence $u^M(0, \mathbf{x}) \rightharpoonup u(0, \mathbf{x})$ in $L_k(\Omega)$ as $M \rightarrow \infty$. On the other hand, by the choice (32), $u^M(0, \mathbf{x})$ strongly converges to $\varphi(\mathbf{x})$ in $L_k(\Omega)$. Due to the uniqueness of the weak limit, $u(0, \mathbf{x}) = \varphi(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$.

Let us prove that the function $u(t, \mathbf{x})$ satisfies integral identity. (18). Equations (30) imply the identities

$$\left(((\omega^M)^{(k-2)/2} u^M)_t, w \right)_{D^T} + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, w_{x_\alpha})_{D^T} = 0, \quad M = \overline{1, \infty}, \quad (47)$$

satisfied for each function $w(\tau, \mathbf{x}) \in P = \bigcup_{L=1}^\infty P_L$. We integrate the first term by parts,

$$\begin{aligned} \left((\omega^M)^{(k-2)/2} u^M, w \right) \Big|_{t=0}^{t=T} - \left((\omega^M)^{(k-2)/2} u^M, w_t \right)_{D^T} + \\ + \sum_{\alpha=1}^n (a_\alpha ((u_{x_\alpha}^M)^2) u_{x_\alpha}^M, w_{x_\alpha})_{D^T} = 0, \quad M = \overline{1, \infty}. \end{aligned} \quad (48)$$

We observe that the inequality $(\omega^M)^{(k-2)/2} |u^M| \leq (\omega^M)^{(k-1)/2} \in C([0, \infty), L_{k'}(\Omega))$ holds true, since $\|(\omega^M)^{(k-1)/2}\|_{k'} = \|(\omega^M)^{1/2}\|_k^{k-1}$ is a bounded sequence in $C[0, \infty)$. Hence, by Lemma 2, the convergence $(\omega^M)^{(k-2)/2} u^M$ selectively weakly converges to $|u|^{k-2}u$ in $L_{k'}(D^T)$ and $(\omega^M(T))^{(k-2)/2} u^M(T) \rightharpoonup |u(T)|^{k-2}u(T)$, $(\omega^M(0))^{(k-2)/2} u^M(0) \rightharpoonup |u(0)|^{k-2}u(0)$ in $L_{k'}(\Omega)$. One can also state that $(\omega^M(T))^{1/2} \rightharpoonup |u(T)|$, $(\omega^M(0))^{1/2} \rightharpoonup |u(0)|$ in $L_k(\Omega)$. The fact that the

limiting functions look exactly like this is justified by the above proven convergence of the subsequence u^M a.e. in D^T , and also a.e. in Ω for $t = 0, T$.

In (48) we can pass to the limit as $M \rightarrow \infty$ that results in the identity

$$\left(|u|^{k-2}u, w \right) \Big|_{t=0}^{t=T} - \left(|u|^{k-2}u, w_t \right)_{D^T} + \sum_{\alpha=1}^n (b_\alpha, w_{x_\alpha})_{D^T} = 0, \quad (49)$$

valid for each function $w \in P$. Since P is dense in the space $\mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$ (Lemma 3), identity (49) holds true for arbitrary $w \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$. Here we employ that $|u|^{k-2}u \in L_{k'}(D^T)$, $b_\alpha \in L_{p_\alpha/(p_\alpha-1)}(D^T)$, $\alpha = \overline{1, n}$. In particular, for $w = u$, we employ the identity

$$\int_0^t (|u|^{k-2}u, u_\tau) d\tau = \frac{1}{k} \|u(t)\|_k^k \Big|_{t=0}^{t=T} \quad (50)$$

to get

$$\begin{aligned} \sum_{\alpha=1}^n (b_\alpha, u_{x_\alpha})_{D^T} + \|u(t)\|_k^k \Big|_{t=0}^{t=T} - \left(|u|^{k-2}u, u_t \right)_{D^T} = \\ = \frac{k-1}{k} \|u(t)\|_k^k \Big|_{t=0}^{t=T} + \sum_{\alpha=1}^n (b_\alpha, u_{x_\alpha})_{D^T} = 0. \end{aligned} \quad (51)$$

Let us show that for each function $v \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$ the identity

$$\sum_{\alpha=1}^n (b_\alpha, v_{x_\alpha})_{D^T} = \sum_{\alpha=1}^n (a_\alpha((v_{x_\alpha})^2)u_{x_\alpha}, v_{x_\alpha})_{D^T} \quad (52)$$

holds true. We deduct identity (48) from (37) with $t = T$ and for $w \in P$ we obtain the relations

$$\begin{aligned} - \left((\omega^M)^{(k-2)/2}u^M, w \right) \Big|_{t=0}^{t=T} + \left((\omega^M)^{(k-2)/2}u^M, w_t \right)_{D^T} + \\ + \sum_{\alpha=1}^n (a_\alpha((u_{x_\alpha}^M)^2)u_{x_\alpha}^M, (u^M - w)_{x_\alpha})_{D^T} + \\ + \frac{k-1}{k} \|(\omega^M)^{1/2}(t)\|_k^k \Big|_{t=0}^{t=T} - \varepsilon^M \frac{k}{2} \int_{I^M} (\omega^M(t, \mathbf{x}))^{k/2-1} d\mathbf{x} \Big|_{t=0}^{t=T} = 0, \quad M = \overline{1, \infty}, \end{aligned}$$

that by the condition of monotonic non-decay of the functions $a_\alpha(z^2)z$, $z \in \mathbb{R}$, $\alpha = \overline{1, n}$, (see (20)) and inequality (40) imply the estimates

$$\begin{aligned} - \left((\omega^M)^{(k-2)/2}u^M, w \right) \Big|_{t=0}^{t=T} + \left((\omega^M)^{(k-2)/2}u^M, w_t \right)_{D^T} + \\ + \sum_{\alpha=1}^n (a_\alpha((w_{x_\alpha})^2)w_{x_\alpha}, (u^M - w)_{x_\alpha})_{D^T} + \\ + \frac{k-1}{k} \|(\omega^M)^{1/2}(t)\|_k^k \Big|_{t=0}^{t=T} - (\varepsilon^M)^{k/4} \left(\frac{k}{2} \right)^{k/2} \leq 0, \quad M = \overline{1, \infty}. \end{aligned}$$

Further, we pass to the limit as $M \rightarrow \infty$ for a fixed $w \in P$ employing at that the above proven convergence.

Thus, for arbitrary $w \in P$ the inequality

$$\begin{aligned}
 - (|u|^{k-2}u, w) \Big|_{t=0}^{t=T} + (|u|^{k-2}u, w_t)_{D^T} + \sum_{\alpha=1}^n (a_\alpha(w_{x_\alpha}^2)w_{x_\alpha}, (u-w)_{x_\alpha})_{D^T} + \\
 + \frac{k-1}{k} \|u(t)\|_k^k \Big|_{t=0}^{t=T} \leq 0
 \end{aligned} \tag{53}$$

holds true. According to Lemma 3 the set P is dense in the space $\mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$. Then for an arbitrary function $w \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$ there exists a sequence $w^l \in P$ such that $\|w^l - w\|_{W_{k,\mathbf{p}}^{1,1}(D^T)} \rightarrow 0$ as $l \rightarrow \infty$. We write (53) for $w = w^l$ and pass then to the limit as $l \rightarrow \infty$. Justification of passage to the limits as $l \rightarrow \infty$

$$(a_\alpha((w_{x_\alpha}^l)^2)w_{x_\alpha}^l, (u-w^l)_{x_\alpha})_{D^T} \rightarrow (a_\alpha(w_{x_\alpha}^2)w_{x_\alpha}, (u-w)_{x_\alpha})_{D^T}, \quad \alpha = \overline{1, n},$$

was provided in [6]. Thus, identity (53) is established for an arbitrary $w \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$.

We deduct (51) from (53) and add (49) that yields the inequality

$$\sum_{\alpha=1}^n (a_\alpha(w_{x_\alpha}^2)w_{x_\alpha} - b_\alpha, (u-w)_{x_\alpha})_{D^T} \leq 0, \tag{54}$$

being true for $w \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$. In (54) we let $w = u + \varepsilon v$, $\varepsilon > 0$, where $v \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$ that gives us

$$\sum_{\alpha=1}^n (a_\alpha((u_{x_\alpha} + \varepsilon v_{x_\alpha})^2)(u_{x_\alpha} + \varepsilon v_{x_\alpha}) - b_\alpha, v_{x_\alpha})_{D^T} \geq 0.$$

The latter inequality as $\varepsilon \rightarrow 0$ yields the relation

$$\sum_{\alpha=1}^n (a_\alpha(u_{x_\alpha}^2)u_{x_\alpha} - b_\alpha, v_{x_\alpha})_{D^T} \geq 0,$$

which by the arbitrariness of v lead us to identity (52). By (49) and (52) for $v \in \mathring{W}_{k,\mathbf{p}}^{1,1}(D^T)$ we conclude that

$$- (|u|^{k-2}u, v_t)_{D^T} + \sum_{\alpha=1}^n (a_\alpha(u_{x_\alpha}^2)u_{x_\alpha}, v_{x_\alpha})_{D^T} + (|u|^{k-2}u, v) \Big|_{t=0}^{t=T} = 0. \tag{55}$$

Thus, (18) is proven.

By (51), (52) we get the identity

$$\frac{k-1}{k} \|u(t)\|_k^k + \sum_{\alpha=1}^n \int_0^t (a_\alpha(u_{x_\alpha}^2)u_{x_\alpha}, u_{x_\alpha}) d\tau = \frac{k-1}{k} \|\varphi\|_k^k, \quad t \geq 0, \tag{56}$$

differentiating which w.r.t. t , we obtain

$$\frac{k-1}{k} \frac{d}{dt} \|u(t)\|_k^k + \sum_{\alpha=1}^n (a_\alpha(u_{x_\alpha}^2)u_{x_\alpha}, u_{x_\alpha}) = 0, \quad t > 0. \tag{57}$$

Applying then (4), by (56), (57) we get (28), (29). \square

Proposition 1. *A generalized solution $u(t, \mathbf{x})$ of problem (1)–(3) with a bounded initial function $\varphi(\mathbf{x}) \in L_\infty(\Omega) \cap \mathring{W}_{k,\mathbf{p}}^1(\Omega)$ is bounded, i.e.,*

$$\text{vrai sup}_D |u(t, \mathbf{x})| \leq B < \infty. \tag{58}$$

We omit the proof of this proposition.

3. ADMISSIBLE DECAY RATE OF SOLUTION

Since the unique solvability of the problem (1)–(3) is not established, we in fact obtain the lower estimate for the constructed solution.

Proof of Theorem 2. We first assume that the domain Ω is bounded and let us prove estimate (8) for Galerkin approximations.

We introduce the notations

$$G^M(t) = \sum_{\alpha=1}^n \int_{\Omega} a_{\alpha}((u_{x_{\alpha}}^M)^2)(u_{x_{\alpha}}^M)^2 d\mathbf{x}, \quad H^M(t) = \sum_{\alpha=1}^n \int_{\Omega} A_{\alpha}((u_{x_{\alpha}}^M)^2) d\mathbf{x},$$

$$E^M(t) = \int_{I^M} \left(\frac{k-1}{k} (\omega^M(t))^{k/2} - \varepsilon^M \frac{k}{2} (\omega^M(t))^{(k-2)/2} \right) d\mathbf{x} + \left(\frac{2}{k} \right)^{1/2} (\varepsilon^M)^{k/4},$$

employing (21), we obtain inequalities

$$\frac{p_1}{2} H^M(t) \leq G^M(t) \leq \widehat{b} H^M(t), \quad t \geq 0. \quad (59)$$

We rewrite identities (36), (43) as

$$\frac{dE^M(t)}{dt} + G^M(t) = 0, \quad t > 0, \quad (60)$$

$$\int_{I^M} \left((k-1)(u^M)^2 + \frac{k}{2} \varepsilon^M \right) (\omega^M)^{(k-4)/2} (u_t^M)^2 + \frac{1}{2} \frac{dH^M(t)}{dt} = 0, \quad t > 0. \quad (61)$$

Applying Cauchy-Schwarz inequality for integrals, we obtain the relations

$$\begin{aligned} \left(\frac{dE^M(t)}{dt} \right)^2 &= \left(\int_{\mathcal{V}^M} \left((k-1)(\omega^M)^{(k-2)/2} + \varepsilon^M \frac{k}{2} (2-k)(\omega^M)^{(k-4)/2} \right) u^M u_t^M d\mathbf{x} \right)^2 \leq \\ &\leq \left((k-1) \left(\int_{\mathcal{V}^M} (\omega^M)^{(k-4)/2} (u^M)^2 (u_t^M)^2 \right)^{1/2} \left(\int_{\mathcal{V}^M} (\omega^M)^{k/2} \right)^{1/2} + \right. \\ &\quad \left. + \varepsilon^M \frac{k}{2} \left(\int_{\mathcal{V}^M} (\omega^M)^{(k-4)/2} (u_t^M)^2 \right)^{1/2} \left(\int_{\mathcal{V}^M} (2-k)(\omega^M)^{(k-2)/2} \right)^{1/2} \right)^2. \end{aligned}$$

Employing Cauchy-Schwarz inequality for sums, by (61), (40) we deduce

$$\begin{aligned} \left(\frac{dE^M(t)}{dt} \right)^2 &\leq \int_{I^M} \left((k-1)(u^M)^2 + \varepsilon^M \frac{k}{2} \right) (\omega^M)^{(k-4)/2} (u_t^M)^2 d\mathbf{x} \times \\ &\quad \times \int_{I^M} \left((k-1)(\omega^M)^{k/2} + (2-k)\varepsilon^M \frac{k}{2} (\omega^M)^{(k-2)/2} \right) d\mathbf{x} \leq \\ &\leq -\frac{1}{2} \frac{dH^M(t)}{dt} \left\{ \int_{I^M} \left((k-1)(\omega^M)^{k/2} - \varepsilon^M \frac{k^2}{2} (\omega^M)^{(k-2)/2} \right) d\mathbf{x} + k \left(\frac{2}{k} \right)^{1/2} (\varepsilon^M)^{k/4} \right\} = \\ &= -\frac{k}{2} \frac{dH^M(t)}{dt} E(t). \end{aligned} \quad (62)$$

Relations (62), (60), (59) imply inequalities

$$\frac{k}{2}E^M(t)\frac{dH^M(t)}{dt} \leq \frac{dE^M(t)}{dt}G^M(t) \leq \frac{p_1}{2}\frac{dE^M(t)}{dt}H^M(t),$$

which we rewrite as

$$\frac{dH^M(t)}{dt}/H^M(t) \leq \frac{p_1}{k}\frac{dE^M(t)}{dt}/E^M(t).$$

Solving differential inequality and applying (59), we obtain the estimates

$$\frac{1}{\widehat{b}}G^M(t) \leq H^M(t) \leq H^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k}, \quad t > 0. \quad (63)$$

Mimicking (60), (63), (59), we arrive at the relations

$$\begin{aligned} \frac{dE^M(t)}{dt} &\geq -\widehat{b}H^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k} \geq \\ &\geq -\frac{2\widehat{b}}{p_1}G^M(0)(E^M(t))^{p_1/k}/(E^M(0))^{p_1/k}, \end{aligned}$$

which we rewrite as

$$\frac{dE^M(t)}{dt}/(E^M(t))^{p_1/k} \geq -\frac{2\widehat{b}}{p_1}G^M(0)/(E^M(0))^{p_1/k}.$$

Solving differential inequality, we obtain the estimate

$$E^M(t) \geq E^M(0) \left(t \frac{2(p_1 - k)\widehat{b}}{kp_1} G^M(0)/E^M(0) + 1 \right)^{-k/(p_1 - k)}, \quad t > 0. \quad (64)$$

For a fixed $t > 0$ as $k \leq p_1$ in the case of the bounded domain Ω the sequence $u^M(t, \mathbf{x})$ is selectively strongly converges as $M \rightarrow \infty$ to $u(t, \mathbf{x})$ in the space $L_k(\Omega)$. It is obvious that

$$E^M(t) \leq \frac{k-1}{k} \|(\omega^M)^{1/2}(t)\|_k^k + \left(\frac{2}{k}\right)^{1/2} (\varepsilon^M)^{k/4}, \quad M = \overline{1, \infty},$$

and according to (38), the inequality

$$\lim_{M \rightarrow \infty} E^M(t) \leq \frac{k-1}{k} \|u(t)\|_k^k = E(t)$$

holds true.

Moreover, due to (40) the inequalities

$$\lim_{M \rightarrow \infty} E^M(0) \geq \lim_{M \rightarrow \infty} \left(\frac{k-1}{k} \|u^M(0)\|_k^k + \left(\frac{2}{k}\right)^{1/2} \frac{2-k}{2} (\varepsilon^M)^{k/4} \right) = \frac{k-1}{k} \|\varphi\|_k^k,$$

$$\lim_{M \rightarrow \infty} G^M(0) \leq \lim_{M \rightarrow \infty} \widehat{a} \sum_{\alpha=1}^n \|u_{x_\alpha}^M\|_{p_\alpha}^{p_\alpha} = \widehat{a} \sum_{\alpha=1}^n \|\varphi_{x_\alpha}\|_{p_\alpha}^{p_\alpha}$$

hold true. After the passage to the limit in (64) as $M \rightarrow \infty$, we obtain

$$\|u(t)\|_k^k \geq \|\varphi\|_k^k (1 + C(\|\varphi\|_{W_{k,p}^1(\Omega)})t)^{-k/(p_1 - k)}. \quad (65)$$

Let us establish estimate (65) for a solution to problem (1)–(3) in an unbounded domain Ω . Let $\Omega^{(l)} \subset \Omega$ be bounded subdomains such that $\Omega^{(l)} \subset \Omega^{(l+1)}$, $l = \overline{1, \infty}$, $\bigcup_{l=1}^{\infty} \Omega^{(l)} = \Omega$. By $u^{(l)}$ we denote the solutions in $\Omega^{(l)}$ with a compactly supported initial function ($\text{supp } \varphi \subset \Omega^{(1)}$); one can assume that these solutions are extended by zero outside $\Omega^{(l)}$. The convergence of the

sequence $u^{(l)}(t, \mathbf{x})$ to the solution $u(t, \mathbf{x})$ of problem (1)–(3) as $l \rightarrow \infty$ can be shown in the same way as in Theorem 4.

Property (25) ensures the estimate

$$\|u^{(l)}\|_{W_{k,p}^1(\Omega)} \leq C, \quad t > 0, \quad l = \overline{1, \infty}.$$

Then for a fixed $t > 0$ one can assume that $u^{(l)}(t, \mathbf{x}) \rightarrow u(t, \mathbf{x})$ in $\overset{\circ}{W}_k^1(\Omega^r)$ as $l \rightarrow \infty$. Employing the compactness of the embedding $W_k^1(\Omega^r) \subset L_k(\Omega^r)$, we establish the strong convergence $u^{(l)}(t, \mathbf{x}) \rightarrow u(t, \mathbf{x})$ in $L_k(\Omega^r)$ as $l \rightarrow \infty$ for each $r > 0$. Owing to estimate (7), for any ε there exists r such that the inequality

$$\|u^{(l)}(t)\|_{k, \Omega_r}^k \leq \varepsilon$$

holds true. The function $u^{(l)}$ satisfies estimate(65) that yields

$$\|u^{(l)}(t)\|_{k, \Omega_r}^k \geq \|\varphi\|_{k, \Omega_r}^k (1 + C(\|\varphi\|_{W_{k,p}^1(\Omega)})t)^{-k/(p_1-k)} - \varepsilon.$$

Employing the strong convergence in $L_k(\Omega^r)$, we pass to the limit as $l \rightarrow \infty$ and then as $r \rightarrow \infty$ ($\varepsilon \rightarrow 0$). Thus, estimate (8) is proven for the unbounded domain Ω for arbitrary $t \geq 0$. \square

4. UPPER ESTIMATES

In this section we prove Theorem 1,3.

Proof of Theorem 1. Let $\xi(x_s)$ be a Lipschitz nonnegative cut-off function. In (55) we let $v = u\xi$ and employ (50) to obtain the relation

$$\frac{k-1}{k} \int_{\Omega} |u|^k \xi \Big|_{\tau=0}^{\tau=t} d\mathbf{x} + \sum_{\alpha=1}^n \int_{D^t} a_{\alpha}(u_{x_{\alpha}}^2) u_{x_{\alpha}} (u\xi)_{x_{\alpha}} d\mathbf{x} d\tau = 0.$$

Using (4), we obtain (taking into consideration that $\xi\varphi = 0$)

$$\begin{aligned} \frac{k-1}{k} \int_{\Omega} |u(t, \mathbf{x})|^k \xi(x_s) d\mathbf{x} + \bar{a} \sum_{\alpha=1}^n \int_{D^t} \xi |u_{x_{\alpha}}|^{p_{\alpha}} d\mathbf{x} d\tau &\leq \\ &\leq \hat{a} \int_{D^t} |u| |u_{x_s}|^{p_s-1} \xi'(x_s) d\mathbf{x} d\tau \equiv I^t. \end{aligned} \quad (66)$$

Let $\theta(x)$, $x > 0$, be an absolute continuous function being one as $x \geq 1$, vanishing as $x \leq 0$, and being linear as $x \in [0, 1]$. In (66) we let $\xi(x_s) = \theta((x_s - r)/\rho)$. It is clear that

$$\xi'(x_s) = \frac{1}{\rho}, \quad x \in (r, r + \rho), \quad \xi'(x_s) = 0, \quad x \notin (r, r + \rho). \quad (67)$$

Let us estimate the integral

$$I^t = \frac{\hat{a}}{\rho} \int_0^t \int_{\Omega_r^{r+\rho}} |u| |u_{x_s}|^{p_s-1} d\mathbf{x} d\tau.$$

Employing Young inequality and (58), for each $\varepsilon > 0$ we deduce

$$I^t \leq \frac{\hat{a}}{\varepsilon\rho} \left(\frac{p_s-1}{p_s} \int_0^t \int_{\Omega_r^{r+\rho}} |u_{x_s}|^{p_s} d\mathbf{x} d\tau + \frac{\varepsilon^{p_s}}{p_s} B^{p_s-k} \int_0^t \int_{\Omega_r^{r+\rho}} |u|^k d\mathbf{x} d\tau \right). \quad (68)$$

Mimicking (66), (68), we obtain the inequality

$$\begin{aligned} & \frac{k-1}{k} \int_{\Omega_{r+\rho}} |u(t, \mathbf{x})|^k d\mathbf{x} + \bar{a} \sum_{\alpha=1}^n \int_0^t \int_{\Omega_{r+\rho}} |u_{x_\alpha}|^{p_\alpha} d\mathbf{x} d\tau \leq \\ & \leq \frac{C_1}{\varepsilon \rho} \left(\int_0^t \int_{\Omega_{r+\rho}} |u_{x_s}|^{p_s} d\mathbf{x} d\tau + \varepsilon^{p_s} \int_0^t \int_{\Omega_{r+\rho}} |u|^k d\mathbf{x} d\tau \right). \end{aligned} \quad (69)$$

We introduce the notation

$$F_r(t) = \int_{\Omega_r} |u(t, \mathbf{x})|^k d\mathbf{x} + \sum_{\alpha=1}^n \int_0^t \int_{\Omega_r} |u_{x_\alpha}|^{p_\alpha} d\mathbf{x} d\tau,$$

then (69) can be rewritten as

$$F_{r+\rho}(t) \leq \frac{C_2}{\varepsilon \rho} \left(F_r(t) + \varepsilon^{p_s} \int_0^t F_r(\tau) d\tau \right). \quad (70)$$

Let us establish the inequality

$$F_{R_0+l\rho}(t) \leq C \left(\frac{2C_2}{\rho} \right)^l t^{l/p_s} \left\{ \prod_{i=0}^{l-1} (1 + i/p_s) \right\}^{-1/p_s} \|\varphi\|_k^k, \quad l = \overline{0, \infty} \quad (71)$$

by induction in l . As the zeroth step of the induction, inequality (28) for each $t > 0$ implies inequality $F_{R_0}(t) \leq C \|\varphi\|_k^k$. Suppose that (71) holds true for some integer $l \geq 0$. Substituting $\varepsilon = \left[\frac{(1+l/p_s)}{t} \right]^{1/p_s}$, $r = R_0 + l\rho$ into (70) and bearing in mind (71), we obtain

$$\begin{aligned} F_{R_0+(l+1)\rho}(t) & \leq C 2^l \left(\frac{C_2}{\rho} \right)^{l+1} t^{l/p_s} \left\{ \prod_{i=0}^l (1 + i/p_s) \right\}^{-1/p_s} \|\varphi\|_k^k \times \\ & \times \left\{ t^{l/p_s} + \frac{1+l/p_s}{t} \int_0^t \tau^{l/p_s} d\tau \right\} = C \left(\frac{2C_2}{\rho} \right)^{l+1} t^{(l+1)/p_s} \left\{ \prod_{i=0}^l (1 + i/p_s) \right\}^{-1/p_s} \|\varphi\|_k^k. \end{aligned}$$

Inequality (71) is proven.

Let $\rho = (r - R_0)/l$. Employing Stirling's approximation, by (71) one can easily get

$$F_r(t) \leq C_3 \exp \left(-\frac{l}{p_s} \ln \frac{(r - R_0)^{p_s}}{C_4 t l^{p_s-1}} \right) \|\varphi\|_k^k. \quad (72)$$

Letting l being the integer part of the expression $\left[\frac{(r - R_0)^{p_s}}{e C_4 t} \right]^{1/(p_s-1)}$, by inequality (72) we obtain

$$F_r(t) \leq C_5 \exp \left(-C_6 \left[\frac{(r - R_0)^{p_s}}{t} \right]^{1/(p_s-1)} \right) \|\varphi\|_k^k. \quad (73)$$

In the case $l = 0$ inequality (73) follows from relation (28). Finally, as $r \geq 2R_0$ by (73) we get estimate (7). \square

Theorem 3 is proven on the basis of the following statement.

Lemma 4. *Suppose the domain is located along the axis Ox_s , $s \in \overline{2, n}$ and conditions (13), (6) hold true. Then there exists positive numbers $\kappa(p_s, k)$, $\mathcal{M}(p_s, k)$ such that for the constructed bounded solution $u(t, \mathbf{x})$ to problem (1)–(3) for all $t \geq 0$, $r \geq 2R_0$ the estimate*

$$\|u(t)\|_{k, \Omega_r} \leq \mathcal{M} \exp \left(-\kappa \int_1^r \nu^{p_1/p_s}(\rho) d\rho \right) \|\varphi\|_k \quad (74)$$

holds true.

Proof. Let $\theta(x)$, $x > 0$, be an absolute continuous function being one as $x \geq r$, vanishing as $x \leq R_0$, being linear as $x \in [R_0, 2R_0]$, and satisfying equation

$$\theta'(x) = \delta \nu^{p_1/p_s}(x) \theta(x), \quad x \in (2R_0, r), \quad (75)$$

(we shall determine the constant δ later). Solving this equation, we find, in particular, that

$$\theta'(x) = \frac{\theta(2R_0)}{R_0} = \frac{1}{R_0} \exp \left(-\delta \int_{2R_0}^r \nu^{p_1/p_s}(\rho) d\rho \right), \quad x \in (R_0, 2R_0). \quad (76)$$

For each function $v(\mathbf{x}) \in C_0^\infty(\Omega)$ the definition of the function $\nu(\rho)$ implies the inequality

$$\nu(\rho) \|v\|_{p_1, \gamma_\rho} \leq \|v_{x_1}\|_{p_1, \gamma_\rho}, \quad \rho > 0,$$

that yields the relations

$$\int_{2R_0}^r \theta^{p_s}(\rho) \nu^{p_1}(\rho) \|v\|_{p_1, \gamma_\rho}^{p_1} d\rho \leq \int_{2R_0}^r \theta^{p_s}(\rho) \|v_{x_1}\|_{p_1, \gamma_\rho}^{p_1} d\rho. \quad (77)$$

Applying (77) for each function $v \in C_0^\infty(\Omega)$ as $s \in \overline{2, n}$, we deduce

$$\begin{aligned} \int_{2R_0}^r \nu^{p_1}(\rho) \theta^{p_s}(\rho) \|v\|_{p_s, \gamma_\rho}^{p_s} d\rho &\leq \max_{\Omega} |v(\mathbf{x})|^{p_s - p_1} \int_{2R_0}^r \nu^{p_1}(\rho) \theta^{p_s}(\rho) \|v\|_{p_1, \gamma_\rho}^{p_1} d\rho \leq \\ &\leq \max_{\Omega} |v(\mathbf{x})|^{p_s - p_1} \int_{2R_0}^r \theta^{p_s}(\rho) \|v_{x_1}\|_{p_1, \gamma_\rho}^{p_1} d\rho. \end{aligned} \quad (78)$$

We note that inequalities (1) hold true for any bounded function $v \in \mathring{W}_{k, \mathbf{p}}^1(\Omega)$ (see [6, Corollary 1]).

In (66) we let $\xi(x_s) = \theta^{p_s}(x_s)$ and obtain

$$\begin{aligned} \frac{k-1}{k} \int_{\Omega} |u(t, \mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \bar{a} \sum_{\alpha=1}^n \int_{D^t} \theta^{p_s} |u_{x_\alpha}|^{p_\alpha} d\mathbf{x} d\tau &\leq \\ &\leq \hat{a} \int_0^t \int_{\Omega} |u| |u_{x_s}|^{p_s - 1} p_s \theta'(x_s) \theta^{p_s - 1}(x_s) d\mathbf{x} d\tau \equiv \hat{a} I^t. \end{aligned} \quad (79)$$

Employing Young inequality, we get

$$I^t \leq \varepsilon(p_s - 1) \int_0^t \int_{\Omega} |u_{x_s}|^{p_s} \theta^{p_s} d\mathbf{x} d\tau + \frac{1}{\varepsilon^{p_s - 1}} \int_0^t \int_{\Omega} |u|^{p_s} (\theta'(x_s))^{p_s} d\mathbf{x} d\tau. \quad (80)$$

We choose $\varepsilon = \frac{\bar{a}}{\bar{a}} \frac{1}{p_s - 1}$ and mimicking (79), (80), we obtain the inequality

$$\frac{k-1}{k} \int_{\Omega} |u(t, \mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \bar{a} \sum_{\alpha=1, \alpha \neq s}^n \int_{D^t} \theta^{p_s} |u_{x_\alpha}|^{p_\alpha} d\mathbf{x} d\tau \leq C_7 \int_{D^t} |u|^{p_s} (\theta'(x_s))^{p_s} d\mathbf{x} d\tau. \quad (81)$$

Employing (75), (76), one can easily reduce (81) to

$$\begin{aligned} & \frac{k-1}{k} \int_{\Omega} |u(t, \mathbf{x})|^k \theta^{p_s}(x_s) d\mathbf{x} + \bar{a} \sum_{\alpha=1, \alpha \neq s}^n \int_{D^t} \theta^{p_s} |u_{x_\alpha}|^{p_\alpha} d\mathbf{x} d\tau \leq \\ & \leq C_7 \frac{1}{R_0^{p_s}} \exp \left(-\delta p_s \int_{2R_0}^r \nu^{p_1/p_s}(\rho) d\rho \right) \int_0^t \int_{\Omega_{R_0}^{2R_0}} |u|^{p_s} d\mathbf{x} d\tau + \\ & + C_7 \delta^{p_s} \int_0^t \int_{\Omega_{2R_0}^r} |u|^{p_s} \nu^{p_1}(x_s) \theta^{p_s}(x_s) d\mathbf{x} d\tau = I_1^t + I_2^t. \end{aligned} \quad (82)$$

Employing [6, Ineq. (73)] and relation (28), we deduce

$$I_1^t \leq C_8 \exp \left(-\delta p_s \int_{2R_0}^r \nu^{p_1/p_s}(\rho) d\rho \right) \int_0^t \|u_{x_s}\|_{p_s}^{p_s} d\tau \leq C_9 \exp \left(-\delta p_s \int_{2R_0}^r \nu^{p_1/p_s}(\rho) d\rho \right) \|\varphi\|_k^k. \quad (83)$$

Applying (1), we get

$$I_2^t \leq C_{10} \delta^{p_s} \int_0^t \int_{\Omega_{2R_0}^r} |u_{x_1}|^{p_1} \theta^{p_s} d\mathbf{x} d\tau. \quad (84)$$

Choosing $\delta = \left(\frac{\bar{a}}{C_{10}} \right)^{1/p_s}$, mimicking (82) – (84), we obtain

$$\frac{k-1}{k} \|u(t)\|_{k, \Omega_r}^k + \bar{a} \sum_{\alpha=2, \alpha \neq s}^n \int_0^t \|u_{x_\alpha}(t)\|_{\Omega_r}^{p_\alpha} d\tau \leq C_9 \exp \left(-C_{11} \int_1^r \nu^{p_1/p_s}(\rho) d\rho \right) \|\varphi\|_k^k.$$

Inequality (74) is proven. \square

Theorem 3 is proven on the basis of estimate (74) by analogy with the proof of Theorem 3 in [6].

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Larisa Mikhailovna Kozhevnikova,
Sterlitamak Branch of Bashkir State University,
Lenin av., 37,
453103, Sterlitamak, Russia
E-mail: kosul@mail.ru

Alexei Alexandrovich Leontiev,
Sterlitamak Branch of Bashkir State University
Lenin av., 37,
453103, Sterlitamak, Russia