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ON THE GROWTH OF SOLUTIONS OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. In this paper, by using the value distribution theory, we study the growth and the oscillation of meromorphic solutions of the linear differential equation

$$f^{(k)}+\left(A_{k-1,1}(z)e^{P_{k-1}(z)}+A_{k-1,2}(z)e^{Q_{k-1}(z)}\right)f^{(k-1)}$$
$$+\cdots+\left(A_{0,1}(z)e^{P_0(z)}+A_{0,2}(z)e^{Q_0(z)}\right)f=F(z),$$

where $A_{j,i}(z)\,(\not\equiv 0)$ $(j=0,\ldots,k-1)$, $F(z)$ are meromorphic functions of a finite order, and $P_j(z),Q_j(z)$ $(j=0,1,\ldots,k-1;i=1,2)$ are polynomials with degree $n\geqslant 1$. Under some conditions, we prove that as $F\equiv 0$, each meromorphic solution $f\not\equiv 0$ with poles of uniformly bounded multiplicity is of infinite order and satisfies $\rho_2(f)=n$ and as $F\not\equiv 0$, there exists at most one exceptional solution f_0 of a finite order, and all other transcendental meromorphic solutions f with poles of uniformly bounded multiplicities satisfy $\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty$ and $\overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)\leqslant\max\{n,\rho(F)\}$. Our results extend the previous results due Zhan and Xiao [19].

Keywords: Order of growth, hyper-order, exponent of convergence of zero sequence, differential equation, meromorphic function.

Mathematics Subject Classification: 34M10, 30D35

1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory, see [12], [18]. Let $\rho(f)$ stands for the order of growth of a meromorphic function f and the hyper-order of f is defined by

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f , see [12], [14], [18].

Definition 1.1. *(*[15], [17]*) Let f be a meromorphic function. The convergence exponent of the zero-sequence of a meromorphic function f is defined by*

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N\left(r,\frac{1}{f}\right)}{\log r},$$

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where $N\left(r,\frac{1}{f}\right)$ *is the integrated counting function of zeros of* f *in* $\{z:|z|\leqslant r\}$, *and the exponent of convergence the sequence of distinct zeros of* f *is defined by*

$$\overline{\lambda}(f)=\limsup_{r\to+\infty}\frac{\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ *is the integrated counting function of distinct zeros of* f *in* $\{z:|z|\leqslant r\}$. *The hyper convergence exponents of the zero-sequence and the distinct zeros of* f *are defined respectively by*

$$\lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},\qquad \overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

Several authors [3], [9], [14] have study the growth of solutions of the second order linear differential equation

$$f''+A_1(z)e^{P(z)}f'+A_2(z)e^{Q(z)}f=0, \tag{1.1}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_2(z)$ ($\not\equiv 0$) are entire functions such that $\rho(A_1)<\deg P(z)$, $\rho(A_2)<\deg Q(z)$. Gundersen showed in [9] that if $\deg P(z)\neq\deg Q(z)$, then each nonconstant solution of (1.1) is of infinite order. If $\deg P(z)=\deg Q(z)$, then (1.1) may have nonconstant solutions of a finite order. For instance $f(z)=e^z+1$ satisfies $f''+e^zf'-e^zf=0$.

In [10], Habib and Belaïdi studied the order and hyper-order of solutions of some higher order linear differential equations and they proved the following result.

Theorem 1.1. *([10]) Let* $A_j(z)$ ($\not\equiv 0$), ($j=1,2$), $B_l(z)$ ($\not\equiv\ 0$) ($l=1,\dots,k-1$), D_m ($m=0,\dots,k-1$) *be entire functions with*

$$\max\{\rho(A_j),\rho(B_l),\rho(D_m)\}<1,$$

b_l ($l=1,\dots,k-1$) *be complex constants such that* (i) $\arg b_l=\arg a_1$ *and* $b_l=c_la_1$ ($0<c_l<1$) ($l\in I_1$) *and* (ii) b_l *is a real constant such that* $b_l\leqslant 0$ ($l\in I_2$), *where* $I_1\neq\varnothing$, $I_2\neq\varnothing$, $I_1\cap I_2=\varnothing$, $I_1\cup I_2=\{1,2,\dots,k-1\}$, *and* a_1, a_2 *are complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<\frac{b}{1-c}$, *where* $c=\max\{c_l:l\in I_1\}$ *and* $b=\min\{b_l:l\in I_2\}$, *then each solution* $f\not\equiv 0$ *of the equation*

$$\begin{aligned}f^{(k)}&+\left(D_{k-1}+B_{k-1}e^{b_{k-1}z}\right)f^{(k-1)}+\ldots+\left(D_1+B_1e^{b_1z}\right)f'\\&+\left(D_0+A_1e^{a_1z}+A_2e^{a_2z}\right)f=0\end{aligned} \tag{1.2}$$

satisfies $\rho(f)=+\infty$ *and* $\rho_2(f)=1$.

And in [2], they studied the order and hyper-order of solutions of some higher order linear differential equations with meromorphic coefficient and they proved the following result.

Theorem 1.2. *([2]) Let* $A_j(z)$ ($\not\equiv 0$) ($j=1,2$), $B_l(z)$ ($\not\equiv 0$) ($l=1,\dots,k-1$) *be meromorphic functions with*

$$\max\{\rho(A_j)\ \ (j=1,2),\rho(B_l)\ \ (l=1,\dots,k-1)\}<1,$$

b_l ($l=1,\dots,k-1$) *be complex constants such that* (i) $b_l=c_la_1$ ($0<c_l<1$) ($l\in I_1$) *and* (ii) b_l *is a real constant such that* $b_l<0$ ($l\in I_2$), *where* $I_1\neq\emptyset$, $I_2\neq\emptyset$, $I_1\cap I_2=\emptyset$, $I_1\cup I_2=\{1,2,\dots,k-1\}$, *and* a_1, a_2 *are complex numbers such that* $a_1a_2\neq 0$, $a_1\neq a_2$ *(suppose that* $|a_1|\leqslant|a_2|$*). If* $\arg a_1\neq\pi$ *or* a_1 *is a real number such that* $a_1<\frac{b}{1-c}$, *where*

$c = \max\{c_l, l \in I_1\}$ and $b = \min\{b_l, l \in I_2\}$, then each meromorphic solution $f\,(\not\equiv 0)$ with poles of uniformly bounded multiplicities of the equation

$$f^{(k)} + B_{k-1}e^{b_{k-1}z}f^{(k-1)} + \cdots + B_1e^{b_1z}f' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)f = 0 \tag{1.3}$$

satisfies $\rho(f) = +\infty$ and $\rho_2(f) = 1$.

In [19], Zhan and Xiao studied the homogeneous and nonhomogeneous higher order differential equations and obtained the following results.

Theorem 1.3. *([19]) Let $A_{ji}(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_{ji}) < n$, $n \geqslant 1$ is a positive integer, $j = 0, 1, \ldots, k-1$; $i = 1, 2$. Let $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ are distinct numbers. Then each solution $f\,(\not\equiv 0)$ of the equation*

$$\begin{aligned} &f^{(k)} + \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right)f^{(k-1)} \\ &\quad + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right)f = 0 \end{aligned} \tag{1.4}$$

of a finite order.

Theorem 1.4. *([19]) Let $A_{ji}(z)$ $(\not\equiv 0)$ be entire functions with $\rho(A_{ji}) < n$, where $n \geqslant 1$ is a positive integer, $j = 0, 1, \ldots, k-1; i = 1, 2$. Let $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ are distinct numbers. $F(z)(\not\equiv 0)$ is an entire function of a finite order. Then the equation*

$$\begin{aligned} &f^{(k)} + \left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right)f^{(k-1)} \\ &\quad + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right)f = F(z) \end{aligned} \tag{1.5}$$

satisfies the following statements:
(i) There exists at most one exceptional solution f_0 of a finite order, and all other solutions satisfy $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \max\{n, \rho(F)\}$.
(ii) If there exists f_0 of a finite order, then $\rho(f_0) \leqslant \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}$.
(iii) If $F(z)$ is an entire function of order less than n and $\arg a_{0,n} \neq \arg b_{0,n}$, then each solution of (1.5) is of infinite order.

In this paper, we are concerned with a more general problem. We extend and improve Theorem 1.3 and Theorem 1.4. In fact, we will prove the following theorems.

Theorem 1.5. *Let $A_{ji}(z)$ $(\not\equiv 0)$ be meromorphic functions of a finite order such that $\max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1; i = 1, 2\} < n$, where $n \geqslant 1$ is a positive integer. Let $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_ja_{0,n}, b_{j,n} = c_jb_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$ are distinct numbers. Then each meromorphic solution $f\,(\not\equiv 0)$ of equation (1.4) with poles of uniformly bounded multiplicity is of infinite order and satisfies $\rho_2(f) = n$.*

Theorem 1.6. *Let $A_{ji}(z)$ $(\not\equiv 0)$, $F(z)(\not\equiv 0)$ be meromorphic functions of a finite order with $\max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1; i = 1, 2\} < n$, where $n \geqslant 1$ is a positive integer. Let $P_j(z) = a_{j,n}z^n + \cdots + a_{j,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials, where $a_{j,q}, b_{j,q}$ $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{j,n}b_{j,n} \neq 0$, $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_ja_{0,n}$, $b_{j,n} = c_jb_{0,n}$, $c_j > 1$, $j = 1, \ldots, k-1$ are distinct numbers. Then the equation (1.5) satisfies:*

(i) There exists at most one exceptional meromorphic solution f_0 with finite order, and all other transcendental meromorphic solutions f with poles of uniformly bounded multiplicities satisfy

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leqslant \max\{n, \rho(F)\}.$$

(ii) If there exists f_0 of a finite order, then $\rho(f_0) \leqslant \max\{n, \overline{\lambda}(f_0), \rho(F)\}$.
(iii) If $F(z)$ is a meromorphic function of order less than n and $\arg a_{0,n} \neq \arg b_{0,n}$, then each meromorphic solution f of (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

2. Auxiliary lemmata

First, we recall the following definitions. The linear measure of a set $E \subset [0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt$$

and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by

$$lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt,$$

where $\chi_H(t)$ is the characteristic function of a set H.

Lemma 2.1. *([1]) Let $P_j(z)\,(j = 0, 1, \ldots, k)$ be polynomials with $\deg P_0 = n$ $(n \geqslant 1)$ and $\deg P_j \leqslant n$ $(j = 1, \ldots, k)$. Let $A_j(z)$ $(j = 0, 1, \ldots, k)$ be meromorphic functions of a finite order and* $\max\{\rho(A_j), j = 0, 1, \ldots, k\} < n$ *such that $A_0(z) \not\equiv 0$. We denote*

$$F(z) = A_k e^{P_k(z)} + A_{k-1} e^{P_{k-1}(z)} + \cdots + A_1 e^{P_1(z)} + A_0 e^{P_0(z)}.$$

If $\deg(P_0(z) - P_j(z)) = n$ for all $j = 1, \ldots, k$, then F is a nontrivial meromophic function with finite order satisfying $\rho(F) = n$.

Lemma 2.2. *([8]) Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset (1, +\infty)$ of a finite logarithmic measure and a constant $B > 0$ that depends only on α and positive integers (n, m) obeying $n > m \geqslant 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leqslant B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f)\right]^{n-m}.$$

Lemma 2.3. *([11]) Let $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) be a polynomial with degree $n \geqslant 1$ and $A(z)$ be a meromorphic function with $\rho(A) < n$. Let*

$$f(z) = A(z) e^{P(z)}, \quad z = re^{i\theta}, \quad \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta.$$

Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for each $\theta \in [0, 2\pi) \setminus H$ $(H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\})$ and for $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$, we have
(i) if $\delta(P, \theta) > 0$, then

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \leqslant \left|f\left(re^{i\theta}\right)\right| \leqslant \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\},$$

(ii) if $\delta(P, \theta) < 0$, then

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} \leqslant \left|f\left(re^{i\theta}\right)\right| \leqslant \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\}.$$

Lemma 2.4. *([5]) Let $f(z)$ be a meromorphic function of order $\rho(f) = \rho < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that as $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$, we have $|f(z)| \leqslant \exp\left(r^{\rho+\varepsilon}\right)$.*

It is well known that due to the Wiman-Valiron theory [13], [15], it is important to studyt the properties of entire solutions of differential equations. In [4], Chen extended the Wiman-Valiron theory from entire functions to meromorphic functions. Here we give a special form of the result given by Wang and Yi in [17], when meromorphic function has infinite order.

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. By $\mu(r) = \max\{|a_n| r^n;\ n = 0, 1, \ldots\}$ we denote the maximum term of g and by $\nu_g(r) = \max\{m :\ \mu(r) = |a_m| r^m\}$ we denote the central index of g.

Lemma 2.5. *([17]) Let $f(z) = g(z)/d(z)$ be a meromorphic function of infinite order obeying $\rho_2(f) = \sigma$, $g(z)$ and $d(z)$ are entire functions, where $\rho(d) < +\infty$. Then there exists a sequence of complex numbers $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfying*

$$r_m \to +\infty, \quad \theta_m \in [0, 2\pi); \quad m \in \mathbb{N}, \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0, 2\pi), \quad |g(z_m)| = M(r_m, g)$$

and for sufficiently large m we have

$$\frac{f^{(n)}(z_m)}{f(z_m)} = \left(\frac{\nu_g(r_m)}{z_m}\right)^n (1 + o(1)) \quad (n \in \mathbb{N}),$$

$$\limsup_{r_m\to+\infty} \frac{\log\log \nu_g(r_m)}{\log r_m} = \rho_2(g) = \sigma.$$

Lemma 2.6. *([9]) Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be a monotone nondecreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin (E_4 \cup [0,1])$, where E_4 is a set of a finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \leqslant \psi(\alpha r)$ for all $r > r_1$.*

Lemma 2.7. *Suppose that $k \geqslant 2$ and F, A_0, $A_1, \ldots, A_{k-1}$ are meromorphic functions such that $\rho = \max\{\rho(A_j)\ j = 0, 1, 2, \ldots, k-1,\ \rho(F)\} < +\infty$. Let $f(z)$ be a transcendental meromorphic solution with all poles of f are of uniformly bounded multiplicity, of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F. \tag{2.1}$$

Then $\rho_2(f) \leqslant \rho$.

Proof. We assume that f is a transcendental meromorphic solution of equation (2.1). If $\rho(f) < +\infty$, then $\rho_2(f) = 0 \leqslant \rho$. Assume that f is a meromorphic solution to equation (2.1) of infinite order with poles of uniformly bounded multiplicity. By (2.1) we have

$$\left|\frac{f^{(k)}}{f}\right| \leqslant |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right| + |A_0(z)|. \tag{2.2}$$

By (2.1) it follows that the poles of f can locate only at the poles of A_j $(j = 0, \ldots, k-1)$ and F. Note that the poles of f are of uniformly bounded multiplicity. Hence, $\lambda(1/f) \leqslant \rho$. By the Hadamard factorization theorem, we know that f can be expressed as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda(1/f) \leqslant \rho < \rho(f) = \rho(g) = +\infty$$

and $\rho_2(f) = \rho_2(g)$. By Lemma 2.5, there exists a sequence $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfying

$$r_m \to +\infty, \quad \theta_m \in [0, 2\pi), \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0, 2\pi), \quad |g(z_m)| = M(r_m, g)$$

such that for m sufficiently large we have

$$\frac{f^{(n)}(z_m)}{f(z_m)} = \left(\frac{\nu_g(r_m)}{z_m}\right)^n (1+o(1)) \ \ (n \in \mathbb{N}) \tag{2.3}$$

and

$$\limsup_{r_m \to +\infty} \frac{\log\log \nu_g(r_m)}{\log r_m} = \rho_2(g). \tag{2.4}$$

By Lemma 2.4, for each given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ of a finite logarithmic measure such that

$$|F(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\}, \ \ |d(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\} \tag{2.5}$$

and

$$|A_j(z)| \leqslant \exp\left\{r^{\rho+\varepsilon}\right\} \ \ (j = 0, \dots, k-1) \tag{2.6}$$

hold for $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$. Since $M(r,g) \geqslant 1$ for r sufficiently large, it follows from (2.5) that

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r,g)} \leqslant \exp\left\{2r^{\rho+\varepsilon}\right\}. \tag{2.7}$$

Substituting (2.3), (2.6) and (2.7) into (2.2), we obtain

$$\left(\frac{\nu_g(r_m)}{r_m}\right)^k |1+o(1)| \leqslant \sum_{j=1}^{k-1} e^{r_m^{\rho+\varepsilon}} \left(\frac{\nu_g(r_m)}{r_m}\right)^j |1+o(1)| + e^{r_m^{\rho+\varepsilon}} + e^{2r_m^{\rho+\varepsilon}}.$$

It follows that

$$(\nu_g(r_m))^k |1+o(1)| \leqslant (k+1)\, e^{2r_m^{\rho+\varepsilon}} r_m^k \,(\nu_g(r_m))^{k-1} |1+o(1)|.$$

Hence,

$$\nu_g(r_m) \leqslant (k+1)\, A r_m^k e^{2r_m^{\rho+\varepsilon}}, \tag{2.8}$$

where the sequence $\left\{z_m = r_m e^{i\theta_m}\right\}_{m\in\mathbb{N}}$ satisfies

$$r_m \notin [0,1] \cup E_3, \quad r_m \to +\infty, \quad \theta_m \in [0, 2\pi), \quad \lim_{m\to+\infty} \theta_m = \theta_0 \in [0,2\pi), \quad |g(z_m)| = M(r_m, g)$$

and $A > 0$ is some constant. Then by (2.8), Lemma 2.6 and $\varepsilon > 0$ being arbitrary, we obtain that $\rho_2(g) = \rho_2(f) \leqslant \rho$. □

Remark 2.1. For $F \equiv 0$, Lemma 2.7 was proved by Chen and Xu in [7].

Lemma 2.8. *([16]) Let $g(z)$ be a transcendental entire function and $\nu_g(r)$ be the central index of g. For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r,g)$. Then there exist a constant δ_r (> 0) and a set E_5 of a finite logarithmic measure such that for all z satisfying $|z| = r \notin E_5$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\frac{g^{(n)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)) \ \ (n \geqslant 1 \textit{ is an integer}).$$

Lemma 2.9. *([8]) Let $f(z)$ be a transcendental meromorphic function of a finite order ρ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ denote a set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, 2, \dots, m)$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_6 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z obeying $|z| = r \notin [0,1] \cup E_6$ and $(k,j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.10. *Let $f(z) = g(z)/d(z)$ be a meromorphic function with $\rho(f) = \rho \leqslant +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions:*
(i) g is transcendental and d is polynomial,
(ii) g, d are transcendental and $\lambda(d) = \rho(d) = \beta < \rho(g) = \rho$.
For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$ and let $\nu_g(r)$ be the central index of g. Then there exist a constant $\delta_r\ (> 0)$, a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_7 of finite logarithmic measure such that the estimation

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z}\right)^n (1 + o(1)) \quad (n \geqslant 1 \text{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_7$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. By mathematical induction, we obtain

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.9}$$

where $C_{jj_1\cdots j_n}$ are constants and $j + j_1 + 2j_2 + \cdots + nj_n = n$. Hence,

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.10}$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.8, there exist a constant $\delta_r\ (> 0)$ and a set E_5 of a finite logarithmic measure such that for all z obeying $|z| = r \notin E_5$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, 2, \ldots, n), \tag{2.11}$$

where $\nu_g(r)$ is the central index of g. Substituting (2.11) into (2.10) yields

$$\begin{aligned} \frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n \Bigg[&(1 + o(1)) \\ &+ \sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z}\right)^{j-n} (1 + o(1)) \sum_{(j_1\cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \Bigg]. \end{aligned} \tag{2.12}$$

We can choose a constant σ such that $\beta < \sigma < \rho$. By Lemma 2.9, for any given ε $(0 < 2\varepsilon < \sigma - \beta)$, we have

$$\left|\frac{d^{(s)}(z)}{d(z)}\right| \leqslant r^{s(\beta - 1 + \varepsilon)} \quad (s = 1, 2, \ldots, n), \tag{2.13}$$

where $|z| = r \notin [0,1] \cup E_6$, $E_6 \subset (1, +\infty)$ with $lm(E_6) < +\infty$. From this and $j_1 + 2j_2 + \cdots + nj_n = n - j$, we have

$$|z|^{n-j} \left| \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leqslant |z|^{(n-j)(\beta+\varepsilon)} \tag{2.14}$$

for $|z| = r \notin [0,1] \cup E_6$. By $\rho(g) = \rho$, there exists a sequence $\{r'_m\}$ $(r'_m \to +\infty)$ satisfying

$$\lim_{r'_m \to +\infty} \frac{\log \nu_g(r'_m)}{\log r'_m} = \rho. \tag{2.15}$$

Setting the logarithmic measure of $E_7 = [0,1] \cup E_5 \cup E_6$, $lm\,(E_7) = \delta < +\infty$, there exists a point $r_m \in [r'_m, (\delta+1)\, r'_m] \setminus E_7$. Since

$$\frac{\log \nu_g(r_m)}{\log r_m} \geqslant \frac{\log \nu_g(r'_m)}{\log [(\delta+1)\, r'_m]} = \frac{\log \nu_g(r'_m)}{(\log r'_m)\left[1 + \frac{\log(\delta+1)}{\log r'_m}\right]}, \tag{2.16}$$

we get

$$\lim_{r_m \to +\infty} \frac{\log \nu_g(r_m)}{\log r_m} = \rho. \tag{2.17}$$

Hence, for sufficiently large m, we obtain

$$\nu_g\,(r_m) \geqslant r_m^{\rho-\varepsilon} \geqslant r_m^{\sigma-\varepsilon}, \tag{2.18}$$

where $\rho - \varepsilon$ can be replaced by a large enough number M if $\rho = +\infty$. This and (2.14) imply

$$\left| \left(\frac{\nu_g\,(r)}{z}\right)^{j-n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leqslant r_m^{(n-j)(\beta-\sigma+2\varepsilon)} \to 0,\ r_m \to +\infty, \tag{2.19}$$

where $|z| = r_m \notin E_7$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. From (2.12) and (2.19), we obtain our result. □

Lemma 2.11. *Let $f(z) = g(z)/d(z)$ be a meromorphic function with $\rho\,(f) = \rho \leqslant +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions*
(i) g is transcendental and d is polynomial,
(ii) g, d are transcendental and $\lambda\,(d) = \rho\,(d) = \beta < \rho\,(g) = \rho$.
For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g\,(z_r)| = M\,(r, g)$. Then there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \leqslant r_m^{2n} \quad (n \geqslant 1 \textit{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_8, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. Let $z_r = re^{i\theta_r}$ be a point satisfying $|g\,(z_r)| = M\,(r, g)$. By Lemma 2.10, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g\,(r_m)}{z}\right)^n (1 + o\,(1)) \quad (n \geqslant 1 \text{ is an integer}) \tag{2.20}$$

holds for all z satisfying $|z| = r_m \notin E_8, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. On the other hand, for any given $\varepsilon > 0$ and sufficiently large m we obtain

$$\nu_g\,(r_m) \geqslant r_m^{\rho-\varepsilon}, \tag{2.21}$$

where $\rho - \varepsilon$ can be replaced by a large enough number M if $\rho = +\infty$. Hence, we have

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \leqslant r_m^{2n}. \tag{2.22}$$

This completes the proof. □

Lemma 2.12. *([12]) Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S\,(r, f),$$

where $S(r,f)=O(\log T(r,f)+\log r)$, *possibly outside a set* $E_9\subset(0,+\infty)$ *with a finite linear measure. If* f *is of a finite order of growth, then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=O(\log r).$$

Lemma 2.13. *([6]) Let* $A_0, A_1,\dots,A_{k-1}, F\not\equiv 0$ *are meromorphic functions of a finite order. If* f *is a meromorphic solution with* $\rho(f)=+\infty$ *of the equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=F,$$

then

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

3. Proof of Theorem 1.5

First, we prove that each meromorphic solution $f\,(\not\equiv 0)$ of the equation (1.4) is transcendental of order $\rho(f)\geqslant n$. We assume that $f\,(\not\equiv 0)$ is a meromorphic solution of equation (1.4) with $\rho(f)<n$. We can rewrite equation (1.4) as

$$\begin{aligned}(A_{k-1,1}(z)e^{P_{k-1}(z)}&+A_{k-1,2}(z)e^{Q_{k-1}(z)})f^{(k-1)}\\&+\cdots+\left(A_{0,1}(z)e^{P_0(z)}+A_{0,2}(z)e^{Q_0(z)}\right)f=-f^{(k)}.\end{aligned}\tag{3.1}$$

Since

$$\max\{\rho(A_{ji}),\, j=0,1,\dots,k-1; i=1,2\}<n$$

and

$$\rho(f)<n,$$

then $A_{ji}f^{(j)}$, $j=0,1,\dots,k-1; i=1,2$ and $f^{(k)}$ are meromorphic functions of a finite order with

$$\rho\left(A_{ji}f^{(j)}\right)<n \quad\text{and}\quad \rho\left(f^{(k)}\right)<n.$$

We have also $a_{0,n}\neq b_{0,n}$ and $a_{j,n}=c_ja_{0,n}, b_{j,n}=c_jb_{0,n}, c_j>1$, $j=1,\dots,k-1$. Hence, $a_{j,n}\neq b_{j,n}$ and therefore $\deg(P_j-P_0)=\deg(Q_j-Q_0)=n$. Since $A_{0,1}(z)f\neq 0, A_{0,2}(z)f\neq 0$, by Lemma 2.1, we find that the order of growth of the left side of equation (3.1) is n, this contradicts the inequality $\rho\left(f^{(k)}\right)<n$. Thus, each meromorphic solution $f\,(\not\equiv 0)$ of equation (1.4) is transcendental with order $\rho(f)\geqslant n$.

Let $z=re^{i\theta}$, $a_{0,n}=|a_{0,n}|e^{i\theta_1}$, $b_{0,n}=|b_{0,n}|e^{i\theta_2}$, $\theta_1,\theta_2\in[0,2\pi)$. Then

$$\delta(P_0,\theta)=|a_{0,n}|\cos(n\theta+\theta_1),\delta(Q_0,\theta)=|b_{0,n}|\cos(n\theta+\theta_2).\tag{3.2}$$

Since $a_{j,n}=c_ja_{0,n}$, $b_{j,n}=c_jb_{0,n}, c_j>1$, $j=1,\dots,k-1$, and c_j are distinct numbers, we have

$$\delta(P_j,\theta)=c_j\delta(P_0,\theta),\ \delta(Q_j,\theta)=c_j\delta(Q_0,\theta),\tag{3.3}$$

and there exists exactly one c_s such that $c_s=\max\{c_j, j=0,1,\dots,k-1\}$. Let $c_0=1$.

We split our proof into two cases: $\theta_1=\theta_2$ and $\theta_1\neq\theta_2$

Case 1. As $\theta_1=\theta_2$, because of $a_{0,n}\neq b_{0,n}$, we suppose $|a_{0,n}|<|b_{0,n}|$ without loss of generality. Assume that f is a meromorphic solution to equation (1.4) with poles of uniformly bounded multiplicity. From (1.4), we have

$$\begin{aligned}&|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}|\\&\qquad\leqslant\left|\frac{f}{f^{(s)}}\right|\left(\left|\frac{f^{(k)}}{f}\right|+\sum_{j=0,j\neq s}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right).\end{aligned}\tag{3.4}$$

Since f is transcendental, then by Lemma 2.2, for $\alpha = 2$, there exist a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\left[T(2r, f)\right]^{k+1}, \ j = 1, 2, \ldots, k, \ j \neq s. \tag{3.5}$$

By (1.4), it follows that the poles of f can be located only at the poles of $A_{ji}(z)$, $j = 0, 1, \ldots, k-1$; $i = 1, 2$. We observe that the poles of f are of uniformly bounded multiplicity. Hence,

$$\lambda(1/f) \leqslant \max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1; i = 1, 2\} < n.$$

By Hadamard factorization theorem, we know that f can be expressed as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda(1/f) < n \leqslant \rho(f) = \rho(g).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.11, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of a finite logarithmic measure such that the estimate

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant r_m^{2s} \tag{3.6}$$

holds for all z satisfying $|z| = r_m \notin E_8$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

(i) If $\delta(P_0, \theta) > 0$, then by (3.3) we have

$$\delta(Q_j, \theta) > \delta(Q_0, \theta) > 0, \quad \delta(Q_j, \theta) > \delta(P_j, \theta) > \delta(P_0, \theta) > 0.$$

By Lemma 2.3, for any given ε obeying

$$0 < \varepsilon < \min\left\{\frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right), j \neq s\right\},$$

there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$, where

$$H = \{\theta \in [0; 2\pi) : \delta(P_0, \theta) = 0, \delta(Q_0, \theta) = 0\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,2}(z)e^{Q_s(z)}\right| - \left|A_{s,1}(z)e^{P_s(z)}\right| \\ &\geqslant \exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r^n\} - \exp\{(1+\varepsilon)c_s\delta(P_0, \theta)r^n\} \\ &\geqslant \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r^n\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} |A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\{(1+\varepsilon)c_j\delta(P_0, \theta)r^n\} + \exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r^n\} \\ &\leqslant 2\exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r^n\}, \ j = 0, 1, 2, \ldots, k-1, \ j \neq s. \end{aligned} \tag{3.8}$$

Substituting (3.5), (3.6), (3.7), (3.8) into (3.4), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_2 \cup E_8$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H$ we obtain

$$\begin{aligned} \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r_m^n\} \leqslant r_m^{2s}\Bigg(& B\left[T(2r_m, f)\right]^{k+1} \\ & + B\left[T(2r_m, f)\right]^{k+1}\sum_{j=0, j\neq s}^{k-1} 2\exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r_m^n\}\Bigg) \end{aligned}$$

$$\leqslant 4r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0,\theta\right)r_m^n\right\}$$

which gives

$$\exp\left\{\left(1-\varepsilon\right)c_s\delta\left(Q_0,\theta\right)r_m^n\right\}\leqslant 8r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{\left(1+\varepsilon\right)c_j\delta\left(Q_0,\theta\right)r_m^n\right\}. \quad (3.9)$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.9) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

(ii) If $\delta\left(P_0,\theta\right)<0$, then by (3.2) and (3.3) we have

$$\delta\left(Q_j,\theta\right)<\delta\left(Q_0,\theta\right)<\delta\left(P_0,\theta\right)<0,\quad \delta\left(P_j,\theta\right)<\delta\left(P_0,\theta\right)<0.$$

By Lemma 2.3, for any given $0<\varepsilon<1$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H$, where $H=\{\theta\in[0;2\pi):\delta\left(P_0,\theta\right)=0,\delta\left(Q_0,\theta\right)=0\}$ is a finite set, we get

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|&\leqslant\left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\&\leqslant\exp\left\{\left(1-\varepsilon\right)\delta\left(P_j,\theta\right)r^n\right\}+\exp\left\{\left(1-\varepsilon\right)\delta\left(Q_j,\theta\right)r^n\right\}\\&\leqslant 2\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r^n\right\},\quad j=0,1,2,\ldots,k-1.\end{aligned}\quad (3.10)$$

By (1.4) we have

$$1\leqslant\left|\frac{f}{f^{(k)}}\right|\sum_{j=0}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}. \quad (3.11)$$

Substituting (3.5), (3.6) and (3.10) into (3.11), for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H$ we obtain

$$\begin{aligned}1&\leqslant r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left(\sum_{j=0}^{k-1}2\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r_m^n\right\}\right)\\&\leqslant 2kr_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{\left(1-\varepsilon\right)\delta\left(P_0,\theta\right)r_m^n\right\}\end{aligned}\quad (3.12)$$

which gives

$$\exp\left\{\left(\varepsilon-1\right)\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 2kr_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}. \quad (3.13)$$

By Lemma 2.6 and (3.13) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log^+T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r\to+\infty}\frac{\log_2^+T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

Case 2 Assume that $\theta_1\neq\theta_2$.

(i) If $\delta(P_0,\theta)>0$, $\delta(Q_0,\theta)<0$, then by (3.3), we get

$$\delta(P_j,\theta)>\delta(P_0,\theta)>0,\quad \delta(Q_j,\theta)<\delta(Q_0,\theta)<0,$$

by Lemma 2.3, for any given $0<\varepsilon<\min\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\}$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\{\theta\in[0,2\pi):\delta(P_0,\theta)=0,\delta(Q_0,\theta)=0,\delta(P_0,\theta)=\delta(Q_0,\theta)\}$$

is a finite set, we have

$$\begin{aligned}|A_{s,1}(z)e^{P_s(z)}+A_{s,2}(z)e^{Q_s(z)}|&\geqslant\left|A_{s,1}(z)e^{P_s(z)}\right|-\left|A_{s,2}(z)e^{Q_s(z)}\right|\\&\geqslant\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\}-\exp\{(1-\varepsilon)c_s\delta(Q_0,\theta)r^n\}\\&\geqslant\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\},\end{aligned}\tag{3.14}$$

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}|&\leqslant\left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right|\\&\leqslant\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\}+\exp\{(1-\varepsilon)c_j\delta(Q_0,\theta)r^n\}\\&\leqslant 2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\},\ j=0,1,2,\dots,k-1,\quad j\neq s.\end{aligned}\tag{3.15}$$

By (3.4), (3.5), (3.6), (3.14) and (3.15), for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$ we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\}\leqslant&r_m^{2s}\Bigg(B[T(2r_m,f)]^{k+1}\\&+B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}\Bigg)\\\leqslant&4r_m^{2s}B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}\end{aligned}$$

which gives

$$\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\}\leqslant 8r_m^{2s}B[T(2r_m,f)]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}.\tag{3.16}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.16) we obtain

$$\rho(f)=\limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m}=+\infty,$$

and

$$\rho_2(f)=\limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2(f)\leqslant n$, so $\rho_2(f)=n$.

(ii) If $\delta(P_0,\theta)<0$, $\delta(Q_0,\theta)>0$, by (3.3), we have

$$\delta(P_j,\theta)<\delta(P_0,\theta)<0,\quad \delta(Q_j,\theta)>\delta(Q_0,\theta)>0.$$

By Lemma 2.3, for any given $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\{\theta\in[0,2\pi):\delta(P_0,\theta)=0,\delta(Q_0,\theta)=0,\delta(P_0,\theta)=\delta(Q_0,\theta)\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,2}(z)e^{Q_s(z)}\right| - \left|A_{s,1}(z)e^{P_s(z)}\right| \\ &\geqslant \exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\} - \exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\} \\ &\geqslant \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} |A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{(1+\varepsilon)\, c_j\delta\left({}_0,\theta\right) r^n\right\} + \exp\left\{(1-\varepsilon)\, c_j\delta\left(P_0,\theta\right) r^n\right\} \\ &\leqslant 2\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r^n\right\},\ j=0,1,2,\ldots,k-1,\ j\neq s. \end{aligned} \tag{3.18}$$

Proceeding as in the proof of (i) , for all z satisfying $|z| = r_m \notin [0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$ we obtain

$$\begin{aligned} \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r_m^n\right\} \leqslant r_m^{2s}\Bigg(& B\left[T\left(2r_m,f\right)\right]^{k+1} \\ & + B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1} 2\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}\Bigg) \\ \leqslant & 4r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}, \end{aligned}$$

which gives

$$\exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r_m^n\right\} \leqslant 8r_m^{2s}B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right) r_m^n\right\}. \tag{3.19}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.19) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2(f)\leqslant n$, so $\rho_2(f)=n$.

(iii) If $\delta(P_0,\theta)>0$, $\delta(Q_0,\theta)>0$, then by (3.3), we have

$$\delta\left(P_j,\theta\right)>\delta\left(P_0,\theta\right)>0, \delta\left(Q_j,\theta\right)>\delta\left(Q_0,\theta\right)>0.$$

We suppose $\delta(P_0,\theta)>\delta(Q_0,\theta)$ without loss of generality. By Lemma 2.3, for any given $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\left\{\theta\in[0,2\pi):\delta\left(P_0,\theta\right)=0,\delta\left(Q_0,\theta\right)=0,\delta\left(P_0,\theta\right)=\delta\left(Q_0,\theta\right)\right\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,1}(z)e^{P_s(z)}\right| - \left|A_{s,2}(z)e^{Q_s(z)}\right| \\ &\geqslant \exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\} - \exp\left\{(1-\varepsilon)\, c_s\delta\left(Q_0,\theta\right) r^n\right\} \\ &\geqslant \frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right) r^n\right\}, \end{aligned} \tag{3.20}$$

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{(1+\varepsilon)\, c_j\delta\left(P_0,\theta\right)r^n\right\}+\exp\left\{(1+\varepsilon)\, c_j\delta\left(Q_0,\theta\right)r^n\right\} \\ &\leqslant 2\exp\left\{(1+\varepsilon)\, c_j\delta\left(P_0,\theta\right)r^n\right\},\ j=0,1,2,\ldots,k-1,\ j\neq s.\end{aligned} \tag{3.21}$$

From (3.4), (3.5), (3.6), (3.20) and (3.21), we have for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$

$$\frac{1}{2}\exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 4r_m^{2s}\ B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\, c_j\delta\left(P_0,\theta\right)r_m^n\right\},$$

which gives

$$\exp\left\{(1-\varepsilon)\, c_s\delta\left(P_0,\theta\right)r_m^n\right\}\leqslant 8r_m^{2s}\ B\left[T\left(2r_m,f\right)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\left\{(1+\varepsilon)\, c_j\delta\left(P_0,\theta\right)r_m^n\right\}. \tag{3.22}$$

Since $0<\varepsilon<\min\left\{\frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right),\ j\neq s\right\}$, then by Lemma 2.6 and (3.22) we obtain

$$\rho\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log T\left(r_m,f\right)}{\log r_m}=+\infty,$$

and

$$\rho_2\left(f\right)=\limsup_{r_m\to+\infty}\frac{\log\log T\left(r_m,f\right)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2\left(f\right)\leqslant n$, so $\rho_2\left(f\right)=n$.

(iv) If $\delta\left(P_0,\theta\right)<0$, $\delta\left(Q_0,\theta\right)<0$, then by (3.3), we have

$$\delta\left(P_j,\theta\right)<\delta\left(P_0,\theta\right)<0,\delta\left(Q_j,\theta\right)<\delta\left(Q_0,\theta\right)<0.$$

Let $\delta=\max\left\{\delta\left(P_0,\theta\right),\delta\left(Q_0,\theta\right)\right\}$. Then, by Lemma 2.3, for any given $0<\varepsilon<1$, there exists a set $E_2\subset[1,+\infty)$ of a finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_2$, $r\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$, where

$$H_1=\left\{\theta\in[0,2\pi):\delta\left(P_0,\theta\right)=0,\delta\left(Q_0,\theta\right)=0,\delta\left(P_0,\theta\right)=\delta\left(Q_0,\theta\right)\right\}$$

is a finite set, we get

$$\begin{aligned}\left|A_{j,1}(z)e^{P_j(z)}+A_{j,2}(z)e^{Q_j(z)}\right| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right|+\left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\left\{(1-\varepsilon)\, c_j\delta\left(P_0,\theta\right)r^n\right\}+\exp\left\{(1-\varepsilon)\, c_j\delta\left(Q_0,\theta\right)r^n\right\} \\ &\leqslant 2\exp\left\{(1-\varepsilon)\, c_j\delta r^n\right\},\quad j=0,1,\ldots,k-1.\end{aligned} \tag{3.23}$$

By (3.5), (3.6), (3.11) and (3.23) for all z satisfying $|z|=r_m\notin[0,1]\cup E_1\cup E_2\cup E_8$, $r_m\to+\infty$ and $\arg z=\theta\in[\theta_r-\delta_r,\theta_r+\delta_r]\setminus H_1$ we have

$$\begin{aligned}1 &\leqslant r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left\{\sum_{j=0}^{k-1}2\exp\left\{(1-\varepsilon)\, c_j\delta r_m^n\right\}\right\} \\ &\leqslant 2r_m^{2k}B\left[T\left(2r_m,f\right)\right]^{k+1}\left\{\sum_{j=0}^{k-1}\exp\left\{(1-\varepsilon)\, c_j\delta r_m^n\right\}\right\}.\end{aligned} \tag{3.24}$$

Since $c_j>1$, $j=1,\ldots,k-1$ and $\delta<0$, we obtain

$$\exp\left\{(1-\varepsilon)\, c_j\delta r_m^n\right\}\leqslant\exp\left\{(1-\varepsilon)\,\delta r_m^n\right\},\quad j=1,\ldots,k-1$$

so (3.24) becomes

$$1\leqslant 2r_m^{2k}kB\left[T\left(2r_m,f\right)\right]^{k+1}\exp\left\{(1-\varepsilon)\,\delta r_m^n\right\}$$

which gives

$$\exp\{(\varepsilon-1)\,\delta r_m^n\} \leqslant 2r_m^{2k}Bk\,[T(2r_m,f)]^{k+1}. \tag{3.25}$$

By Lemma 2.6 and (3.25) we obtain

$$\rho(f)=\limsup_{r_m\to+\infty}\frac{\log T(r_m,f)}{\log r_m}=+\infty$$

and

$$\rho_2(f)=\limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m}\geqslant n.$$

In addition, by Lemma 2.7 and from equation (1.4), we have $\rho_2(f)\leqslant n$, so $\rho_2(f)=n$. This completes the proof of Theorem 1.5.

4. Proof of Theorem 1.6

(i) Suppose f_0 is a meromorphic solution of a finite order to equation (1.5) with poles of uniformly bounded multiplicities. If $f_1(\not\equiv f_0)$ is an another meromorphic solution of a finite order to equation (1.5) with poles of uniformly bounded multiplicities, the function f_1-f_0 is a nonzero meromorphic solution to equation (1.4) with $\rho(f_1-f_0)<+\infty$. This contradicts Theorem 1.5. Hence, equation (1.5) has at most one meromorphic solution of a finite order. We assume that $f(z)$ is a meromorphic solution of infinite order to (1.5) with poles of uniformly bounded multiplicity. By (1.5), it is easy to see that if f has a zero of order $\alpha\,(\alpha>k)$at z_0, and B_0, $B_1,\ldots,B_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order at least $\alpha-k$. Hence,

$$n\left(r,\frac{1}{f}\right)\leqslant k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}n(r,B_j)$$

and

$$N\left(r,\frac{1}{f}\right)\leqslant k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}N(r,B_j), \tag{4.1}$$

where $B_j(z)=A_{j1}(z)e^{P_j(z)}+A_{j2}(z)e^{Q_j(z)}$, $j=0,1,2,\ldots,k-1$. Now (1.5) can be rewritten as

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+B_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+B_1(z)\frac{f'}{f}+B_0(z)\right). \tag{4.2}$$

By Lemma 2.12 and (4.2), we get for $|z|=r$ outside a set E_9 of finite linear measure, we have

$$\begin{aligned}m\left(r,\frac{1}{f}\right)&\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+\sum_{j=0}^{k-1}m(r,B_j)+O(1)\\&\leqslant m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m(r,B_j)+O(\log rT(r,f)).\end{aligned} \tag{4.3}$$

Therefore, by (4.1), (4.3) and the first main theorem, there holds

$$T(r,f)=T(r,\frac{1}{f})+O(1)\leqslant T(r,F)+\sum_{j=0}^{k-1}T(r,B_j)+k\overline{N}\left(r,\frac{1}{f}\right)+O(\log rT(r,f)) \tag{4.4}$$

for all sufficiently large $r\notin E_9$. For sufficiently large r, we have

$$O(\log rT(r,f))\leqslant\frac{1}{2}T(r,f). \tag{4.5}$$

Let $\rho_1 = \max\{n, \rho(F)\}$. By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ of a finite logarithmic measure such that

$$T(r, F) \leqslant r^{\rho_1+\varepsilon},\ T(r, B_j) \leqslant r^{\rho_1+\varepsilon},\ j = 0, 1, \dots, k-1, \tag{4.6}$$

when $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$. By (4.4), (4.5) and (4.6), for $r \notin [0,1] \cup E_3 \cup E_9$ sufficiently large, we obtain

$$T(r, f) \leqslant r^{\rho_1+\varepsilon} + kr^{\rho_1+\varepsilon} + k\overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{2}T(r, f)$$

which gives

$$T(r, f) \leqslant 2(k+1) r^{\rho_1+\varepsilon} + 2k\overline{N}\left(r, \frac{1}{f}\right). \tag{4.7}$$

Hence,

$$\rho_2(f) \leqslant \overline{\lambda}_2(f)$$

and therefore,

$$\rho_2(f) \leqslant \overline{\lambda}_2(f) \leqslant \lambda_2(f).$$

Since by the definition we have $\overline{\lambda}_2(f) \leqslant \lambda_2(f) \leqslant \rho_2(f)$, we get

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f).$$

On the other hand, $\max\{\rho(A_{ji}), j = 0, 1, \dots, k-1; i = 1, 2\} < n$ and $\rho(B_j) < +\infty$ for all $j = 0, 1, \dots, k-1$, and $f(z)$ is a solution to (1.5) of infinite order. Hence, by Lemma 2.13 we obtain $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$. Since $\rho(B_j) \leqslant n$, by Lemma 2.7, we have $\rho_2(f) \leqslant \max\{n, \rho(F)\}$.

(ii) Suppose f_0 is a meromorphic solution of the equation (1.5) with finite order, by Lemma 2.12, we have $m\left(r, \frac{f_0^{(j)}}{f_0}\right) = O(\log r)$, $j = 1, \dots, k-1$. Using (4.2), we can get for $|z| = r$ outside a set E_9 of finite linear measure, we have

$$\begin{aligned} m\left(r, \frac{1}{f_0}\right) \leqslant & m\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} m\left(r, \frac{f_0^{(j)}}{f_0}\right) + \sum_{j=0}^{k-1} m(r, B_j) + O(1) \\ \leqslant & m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, B_j) + O(\log r) \end{aligned} \tag{4.8}$$

and

$$N\left(r, \frac{1}{f_0}\right) \leqslant k\overline{N}\left(r, \frac{1}{f_0}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, B_j). \tag{4.9}$$

By (4.8) and (4.9), we get

$$T(r, f_0) = T(r, \frac{1}{f_0}) + O(1) \leqslant T\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} T(r, B_j) + k\overline{N}\left(r, \frac{1}{f_0}\right) + O(\log r). \tag{4.10}$$

By (4.6) and (4.10), we get

$$T(r, f_0) \leqslant (k+1) r^{\rho_1+\varepsilon} + k\overline{N}\left(r, \frac{1}{f_0}\right) + O(\log r).$$

Hence, we obtain

$$\rho(f_0) \leqslant \max\left\{\overline{\lambda}(f_0), \rho_1\right\} = \max\left\{n, \overline{\lambda}(f_0), \rho(F)\right\}.$$

(iii) First we prove that each meromorphic solution f to equation (1.5) is transcendental of order $\rho(f) \geqslant n$. We assume that f is a meromorphic solution to equation (1.5) with $\rho(f) < n$. We can rewrite equation (1.5) as

$$\left(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)}\right) f = B(z), \tag{4.11}$$

where

$$B(z) = F(z) - f^{(k)}.$$

Since $\max\{\rho(A_{ji}), j = 0, 1, \ldots, k-1; i = 1, 2, \rho(F)\} < n$ and $\rho(f) < n$, then $A_{ji}f^{(j)}$, $j = 0, 1, \ldots, k-1$, $i = 1, 2$, and $B(z)$ are meromorphic functions of a finite order with $\rho\left(A_{ji}f^{(j)}\right) < n$ and $\rho(B) < n$. We also have $a_{0,n} \neq b_{0,n}$ and $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}$, $c_j > 1$, $j = 1, \ldots, k-1$,. Hence, $a_{j,n} \neq b_{j,n}$ and $\deg(P_j - P_0) = \deg(Q_j - Q_0) = n$. Since $A_{0,1}(z)f \not\equiv 0, A_{0,2}(z)f \not\equiv 0$, by Lemma 2.1 we find that the order of growth of the left hand side of equation (4.11) is n. This contradicts the inequality $\rho(B) < n$. Therefore, each meromorphic solution f to equation (1.5) is transcendental and is of order $\rho(f) \geqslant n$.

Let $z = re^{i\theta}$, $a_{0,n} = |a_{0,n}| e^{i\theta_1}$, $b_{0,n} = |b_{0,n}| e^{i\theta_2}$, $\theta_1, \theta_2 \in [0, 2\pi)$. Then

$$\delta(P_0, \theta) = |a_{0,n}| \cos(n\theta + \theta_1), \delta(Q_0, \theta) = |b_{0,n}| \cos(n\theta + \theta_2). \tag{4.12}$$

Since $a_{j,n} = c_j a_{0,n}$, $b_{j,n} = c_j b_{0,n}, c_j > 1$, $j = 1, \ldots, k-1$, and c_j are distinct numbers, we have

$$\delta(P_j, \theta) = c_j \delta(P_0, \theta), \ \delta(Q_j, \theta) = c_j \delta(Q_0, \theta), \tag{4.13}$$

and there exists exactly one c_s such that $c_s = \max\{c_j, j = 0, 1, \ldots, k-1\}$. Let $c_0 = 1$, $\delta_1 = \max\{\delta(P_0, \theta), \delta(Q_0, \theta)\}$. We split our proof into two cases:

Case 1. Assume that $\delta_1 > 0$. By Lemma 2.3, for any given

$$0 < \varepsilon < \min\left\{n - \rho_1, \frac{1}{2}\left(\frac{c_s - c_j}{c_s + c_j}\right), \ j \neq s\right\},$$

there exists a set $E_2 \subset [1, +\infty)$ of a finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, where

$$H_3 = \{\theta \in [0, 2\pi) : \delta(P_0, \theta) = \delta(Q_0, \theta)\}$$

is a finite set, we have

$$\begin{aligned} |A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| &\geqslant \left|A_{s,1}(z)e^{P_s(z)}\right| - \left|A_{s,2}(z)e^{Q_s(z)}\right| \\ &\geqslant \exp\{(1-\varepsilon) c_s \delta(P_0, \theta) r^n\} - \exp\{(1-\varepsilon) c_s \delta(Q_0, \theta) r^n\} \\ &\geqslant \frac{1}{2} \exp\{(1-\varepsilon) c_s \delta_1 r^n\}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right| &\leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ &\leqslant \exp\{(1+\varepsilon) c_j \delta(P_0, \theta) r^n\} + \exp\{(1+\varepsilon) c_j \delta(Q_0, \theta) r^n\} \\ &\leqslant 2 \exp\{(1+\varepsilon) c_j \delta_1 r^n\}, \ j = 0, 1, \ldots, k-1, \ j \neq s. \end{aligned} \tag{4.15}$$

By (1.5) we have

$$\begin{aligned} &|A_{s,1}(z)e^{P_s(z)} + A_{s,2}(z)e^{Q_s(z)}| \\ &\leqslant \left|\frac{f}{f^{(s)}}\right| \left\{\left|\frac{F(z)}{f}\right| + \left|\frac{f^{(k)}}{f}\right| + \sum_{j=0, j\neq s}^{k-1} \left\{\left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right| \left|\frac{f^{(j)}}{f}\right|\right\}\right\}. \end{aligned} \tag{4.16}$$

Since f is transcendental, from Lemma 2.2, there exists a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ and constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1$, we have (3.5) holds and by Lemma 2.11, there exists a set E_8 of finite logarithmic measure such that $|z| = r \notin E_8$,

$|g(z)| = M(r,g)$ and for r sufficiently large inequality (3.6) holds. We know that f is transcendental with $\rho(f) \geqslant n$, and by the assumptions, the poles of f are of uniformly bounded multiplicities. By Hadamard factorization theorem, we can express f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) < n, \quad \rho(g) = \rho(f) \geqslant n.$$

Let $\rho_1 = \max\{\rho(F), \rho(d)\} < n$. Since $|g(z)| = M(r,g) \geqslant 1$, then, by Lemma 2.4 we obtain

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)F(z)}{g(z)}\right| = \frac{|d(z)F(z)|}{M(r,g)} \leqslant \exp\left(r^{\rho_1+\varepsilon}\right)\exp\left(r^{\rho_1+\varepsilon}\right) = \exp\left(2r^{\rho_1+\varepsilon}\right) \tag{4.17}$$

as $|z| = r \notin [0,1] \cup E_3$, $r \to +\infty$.

By (3.5), (3.6), (4.14),(4.15), (4.16) and (4.17), for all z satisfying $|z| = r_m \notin$ $\notin [0,1] \cup E_1 \cup E_3 \cup E_8$, $r_m \to +\infty$, $|g(z)| = M(r_m,g)$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta_1 r_m^n\} \leqslant & r_m^{2s}\Bigg\{\exp\left(2r_m^{\rho_1+\varepsilon}\right) + B\left[T(2r_m,f)\right]^{k+1} \\ & + B\left[T(2r_m,f)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1} 2\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}\Bigg\} \\ \leqslant & 4r_m^{2s}\exp\left(2r_m^{\rho_1+\varepsilon}\right)B\left[T(2r_m,f)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}\end{aligned}$$

which gives

$$\exp\{(1-\varepsilon)c_s\delta_1 r_m^n\} \leqslant 8r_m^{2s}\exp\left(2r_m^{\rho_1+\varepsilon}\right)B\left[T(2r_m,f)\right]^{k+1}\sum_{j=0,j\neq s}^{k-1}\exp\{(1+\varepsilon)c_j\delta_1 r_m^n\}. \tag{4.18}$$

Since $\varepsilon < \min\left\{n-\rho_1, \frac{1}{2}\left(\frac{c_s-c_j}{c_s+c_j}\right), \ j \neq s\right\}$ is arbitrary, so by Lemma 2.6 and (4.18) we obtain

$$\rho(f) = \limsup_{r_m\to+\infty}\frac{\log^+ T(r_m,f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m\to+\infty}\frac{\log\log T(r_m,f)}{\log r_m} \geqslant n.$$

In addition, by Lemma 2.7 and equation (1.5), we have $\rho_2(f) \leqslant n$, so $\rho_2(f) = n$. Then, each meromorphic solution to (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

Case 2. Assume that $\delta_1 < 0$. By Lemma 2.3, for any given $\varepsilon > 0$ we obtain

$$\begin{aligned}|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}| & \leqslant \left|A_{j,1}(z)e^{P_j(z)}\right| + \left|A_{j,2}(z)e^{Q_j(z)}\right| \\ & \leqslant \exp\{(1-\varepsilon)c_j\delta(P_0,\theta)r^n\} + \exp\{(1-\varepsilon)c_j\delta(Q_0,\theta)r^n\} \\ & \leqslant 2\exp\{(1-\varepsilon)c_j\delta_1 r^n\}, \ j = 0,1,2,\ldots,k-1.\end{aligned} \tag{4.19}$$

By (1.5) we get

$$1 \leqslant \left|\frac{f}{f^{(k)}}\right|\left(\left|\frac{F(z)}{f(z)}\right| + \sum_{j=0}^{k-1}\left\{\left|A_{j,1}(z)e^{P_j(z)} + A_{j,2}(z)e^{Q_j(z)}\right|\left|\frac{f^{(j)}}{f}\right|\right\}\right). \tag{4.20}$$

As in Case 1, by (3.5), (3.6), (4.17),(4.19) and (4.20), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_3 \cup E_8$, $r_m \to +\infty$, at which $|g(z)| = M(r_m, g)$, and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_3$, we have

$$1 \leqslant r_m^{2k} \left(\exp\left(2r_m^{\rho_1+\varepsilon}\right) + \sum_{j=0}^{k-1} B\left[T\left(2r_m, f\right)\right]^{k+1} 2\exp\left\{(1-\varepsilon)\, c_j \delta_1 r_m^n\right\} \right). \tag{4.21}$$

Since $c_j \geqslant 1$, $j = 0, \ldots, k-1$, $r_m > R_1 > 1$ and $\delta_1 < 0$, we obtain

$$\exp\left\{(1-\varepsilon)\, c_j \delta_1 r_m^n\right\} \leqslant \exp\left\{(1-\varepsilon)\, \delta_1 r_m^n\right\},\ j = 0, \ldots, k-1$$

so (4.21) becomes

$$1 \leqslant 2r_m^{2k}\,(k+1)\exp\left(r_m^{\rho_1+\varepsilon}\right) B\left[T\left(2r_m, f\right)\right]^{k+1} \exp\left\{(1-\varepsilon)\, \delta_1 r_m^n\right\}$$

which gives

$$\exp\left\{(\varepsilon-1)\,\delta_1 r_m^n - r_m^{\rho_1+\varepsilon}\right\} \leqslant 2r_m^{2k}\,(k+1)\, B\left[T\left(2r_m, f\right)\right]^{k+1}. \tag{4.22}$$

By Lemma 2.6 and (4.22) we obtain

$$\rho(f) = \limsup_{r_m \to +\infty} \frac{\log T(r_m, f)}{\log r_m} = +\infty$$

and

$$\rho_2(f) = \limsup_{r_m \to +\infty} \frac{\log\log T(r_m, f)}{\log r_m} \geqslant n.$$

In addition, by Lemma 2.7 and equation (1.5) we get $\rho_2(f) \leqslant n$ and hence, $\rho_2(f) = n$. Then, each meromorphic solution to (1.5) with poles of uniformly bounded multiplicities is of infinite order and satisfies $\rho_2(f) = n$.

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