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EXAMPLES OF HYPERCYCLIC SHIFTS IN WEIGHTED SPACES ON GRAPHS

M.A. MATVEEV

Abstract. Shift operators play a special role in linear dynamics since very often they serve as examples of operators with chaotic behaviour (hypercyclicity, weak mixing, mixing). For the case of ordinary weighted sequence spaces $\ell^p(\mathbb{N},\mu)$ or $\ell^p(\mathbb{Z},\mu)$, hypercyclic shift operators were described by G. Salas in 1995. Recently the dynamics of operators on graphs has attracted a great interest. Hypercyclic shift operators on trees were described by K. Grosse–Erdmann and D. Papathanasiou in 2023. The first results for graphs that are not trees were obtained in a recent preprint by A.D. Baranov, A.A. Lishanskii and D. Papathanasiou. In this article we consider new examples of hypercyclic shifts on graphs.

Keywords: linear dynamics, hypercyclic operator, weakly mixing operator, shift operator on a graph.

Mathematics Subject Classification: 47B37, 47A16

1. Introduction

Let T be a continuous operator acting in some separable Banach space or Fréchet space X. The operator T is called *hypercyclic* if there exists a vector $x \in X$ (a *hypercyclic vector*) such that

$$\overline{\{T^n x : n = 0, 1, \ldots\}} = X,$$

that is, the orbit of the element x is dense in X. It is well–known that this property is equivalent to the following one: for each pair U, V of non–empty open subsets of X there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. The operator is called *weakly mixing* if the operator $T \oplus T$ is hypercyclic on $X \oplus X$. Finally, the *mixing* operator T is such that for each pair U, V of nonempty open subsets of X there exists some N such that $T^n(U) \cap V \neq \emptyset$ for all $n \geqslant N$. The mixing property implies the weak mixing property, and the weak mixing property obviously implies hypercyclicity. All these properties express the fact that the behaviour of the orbits of elements under the action of the operator T is extremely chaotic. A detailed exposition of linear dynamics and its applications can be found in monographs [3], [5].

Various forms of shift operators often arise as examples of operators with chaotic behaviour. The first example of a hypercyclic operator in a Banach space was found by Rolewicz [10]: this is the operator λB in the space $\ell^p(\mathbb{N})$, $1 \leqslant p < \infty$, where B is the backward shift operator and $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Abstract shift operators with respect to a chosen basis were considered in [4]. For the case of ordinary weighted sequence spaces $\ell^p(\mathbb{N},\mu)$ or $\ell^p(\mathbb{Z},\mu)$, hypercyclic shift operators were described by Salas [11].

Recently the dynamics of shift operators on graphs has attracted a great interest (see, e.g., [1], [6], [7], [8], [9]). Let $G = (V, E)$ be a connected directed graph consisting of a countable

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set of vertices V, where $E \subset V \times V$ is the set of its edges. For $v, u \in V$ we write $v \to u$ if $(v, u) \in E$. For a given vertex $v \in V$, by $\mathrm{Chi}(v)$ we denote the set of its "children":

$$\mathrm{Chi}(v) = \{u \in V : \ v \to u\}.$$

For any graph G and function $f : V \to \mathbb{C}$ one can define the (unweighted) *backward shift*

$$(Bf)(v) = \sum_{u \in \mathrm{Chi}(v)} f(u), \qquad v \in V.$$

We shall consider the properties of hypercyclicity of the backward shift on the standard spaces $\ell^p(V, \mu)$ and $c_0(V, \mu)$ on V. Let $\mu = (\mu_v)_{v \in V}$ be a family of positive real numbers, called a weight. For $1 \leqslant p < \infty$ we let

$$\ell^p(V, \mu) = \{f : V \to \mathbb{C} : \|f\|^p_{\ell^p(V,\mu)} = \sum_{v \in V} |\mu_v f(v)|^p < \infty\}.$$

The space $c_0(V, \mu)$ is defined as

$$c_0(V, \mu) = \Big\{f : V \to \mathbb{C} : \ \forall \varepsilon > 0 \quad \exists \text{ a finite } F \subset V \text{ such that } |\mu_v f(v)| < \varepsilon, \ v \in V \setminus F\Big\};$$

it is equipped with the norm $\|f\|_{c_0(V,\mu)} = \sup\limits_{v \in V} |\mu_v f(v)|$. The dual space for $\ell^p(V, \mu)$ is the space $\ell^{p^*}(V, \frac{1}{\mu})$, where

$$\frac{1}{p} + \frac{1}{p^*} = 1, \qquad \frac{1}{\mu} = \Big(\frac{1}{\mu_v}\Big)_{v \in V}, \qquad c_0(V, \mu)^* = \ell^1\Big(V, \frac{1}{\mu}\Big).$$

It is obviously that for a graph with uniformly bounded vertex degrees the boundedness of the backward shift operator is equivalent to the condition

$$\mu_v \leqslant C \min_{u \in \mathrm{Chi}(v)} \mu_u, \qquad v \in V,$$

for some constant $C > 0$. In what follows it is always assumed that the weights satisfy this condition.

The problem of hypercyclicity of the shift operator in weighted spaces on trees (with or without a root) was completely solved in the work of Gross–Erdmann and Papathanasiou [6]. Let us cite one of the results of this article relating to the case of rooted trees.

Theorem 1.1. *Let $G = (E, V)$ be a rooted directed tree, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$. The operator B on $\ell^p(V, \mu)$ is hypercyclic if and only if it is weakly mixing, and if and only if there exists an increasing sequence (n_k) of positive integers such that for each $v \in V$*

$$\sum_{u \in \mathrm{Chi}^{n_k}(v)} |\mu_u|^{-p^*} \to \infty, \quad k \to \infty.$$

The first examples of hypercyclic shifts on graphs that are not trees were considered in [2]. Let us cite one of the results of the article [2].

For $m \in \mathbb{N}$ we consider the graph $G_m = (V_m, E_m)$ such that $V_m = (v_{i,j})_{1 \leqslant i \leqslant m, j \geqslant 1}$ and $(v_{i,j}, u) \in E_m \iff u = v_{i,j+1}$ or $u = v_{i+1,j}$, $1 \leqslant i \leqslant m - 1$. Figure 1 shows the graph G_m for $m = 3$.

Theorem 1.2. *Let B be the backward shift on G_m, $m \in \mathbb{N}$, and let X be any of the spaces $\ell^p(V_m, \mu)$, $1 \leqslant p < \infty$, or $c_0(V_m, \mu)$. Suppose that the operator B is bounded on X. Then the following conditions are equivalent:*

1. *B is a hypercyclic operator on X;*
2. *B is a weakly mixing operator on X;*

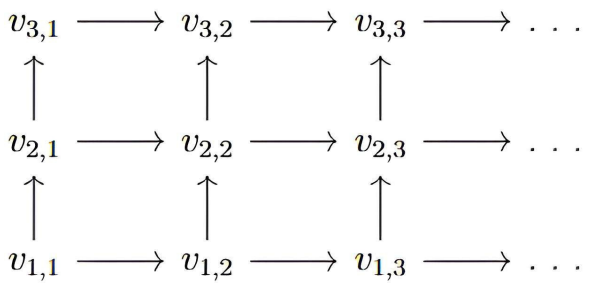


FIGURE 1. The graph G_m for $m = 3$.

3. *There exists an increasing sequence* (n_k) *of positive integers such that for each* $1 \leqslant i \leqslant m$,

$$n_k^{m-i}|\mu_{v_{i,n_k}}| \to 0, \quad k \to \infty.$$

The aim of the present note is, by applying the methods of the article [2], to consider some particular new examples of graphs that are not trees and to study the dynamical properties of the backward shift operator (hypercyclicity, weak mixing, mixing) in weighted spaces on such graphs. Among the examples of graphs we consider lattices with added diagonals, as well as lattices with a periodic alternation of edge directions.

2. Classical results

Here we present some classical results which will be used in what follows (see [5]).

Theorem 2.1 (Hypercyclicity criterion)**.** *Let* X *be a separable Fréchet space,* T *a continuous linear operator on* X*. Let there exist dense subsets* $X_0, Y_0 \subset X$*, (not necessarily linear) mappings* $S_k : Y_0 \to Y_0$*, and a sequence* $n_k \to \infty$ *such that*

1. $\forall x \in X_0 \quad T^{n_k}x \to 0, \quad k \to \infty$;
2. $\forall y \in Y_0 \quad S_{n_k}y \to 0, \quad k \to \infty$;
3. $\forall y \in Y_0 \quad T^{n_k}S_{n_k}y = y, \quad k \to \infty$.

Then T *is a weakly mixing and, in particular, hypercyclic operator.*

Remark 2.1. *In fact, the assumptions of Theorem 2.1 are also necessary for the weak mixing property.*

A convenient sufficient condition for hypercyclicity (and even mixing) was found by Godefroy and Shapiro [4].

Theorem 2.2 (Godefroy — Shapiro criterion)**.** *Let* T *be a continuous linear operator on* X*. If each of the families* $\bigcup\limits_{|\lambda|<1} \operatorname{Ker}(T-\lambda I)$ *and* $\bigcup\limits_{|\lambda|>1} \operatorname{Ker}(T-\lambda I)$ *is complete in* X*, then* T *is mixing, and in particular weakly mixing and hypercyclic operator.*

In what follows, to apply Theorem 2.2, it will be convenient for us to use the following lemma.

Lemma 2.1. *Let there be given a graph* $G = (V, E)$ *and* $V = \{v_{i,j} : (i,j) \in N_1 \times N_2\}$*, where* N_1*,* N_2 *are infinite subsets of* $\mathbb{Z}$*. Let* $U_1, U_2 \subset (0, \infty)$ *be non–empty open intervals such that the functions* $f_{a,b}(v_{i,j}) = a^i b^j$*,* $(i,j) \in N_1 \times N_2$*, belong to the space* X *for all* $(a,b) \in U_1 \times U_2$*, where* X *is any of the spaces* $\ell^p(V,\mu)$*,* $1 \leqslant p < \infty$*, or* $c_0(V,\mu)$*. Then the family* $\mathcal{F} = \{f_{a,b} : (a,b) \in U_1 \times U_2\}$ *is complete in* X*.*

Proof. Let the functional $g = \{g_{i,j}\} \in \ell^{p^*}\left(V, \frac{1}{\mu}\right)$ annihilate all vectors of the family $\mathcal{F}$, that is,

$$\sum_{(i,j)\in N_1\times N_2} g_{i,j}a^i b^j = \sum_{j\in N_2}\left(\sum_{i\in N_1} g_{i,j}a^i\right)b^j = 0, \qquad (a,b)\in U_1\times U_2.$$

Since the series in the left hand side converges absolutely, the series $\sum\limits_{i\in N_1} g_{i,j}a^i = 0$ converges absolutely for all $j \in N_2$ and

$$\sum_{j\in N_2} c_j b^j = 0, \quad b\in U_2, \qquad c_j = \sum_{i\in N_1} g_{i,j}a^i, \quad a\in U_1.$$

By the uniqueness theorem for power series (or Laurent series) $c_j = \sum\limits_{i\in N_1} g_{i,j}a^i = 0$, $a \in U_1$, for each j, which by the same uniqueness theorem implies $g_{i,j} \equiv 0$. The proof is complete. □

Let $S : Y \to Y$ and $T : X \to X$ be linear operators. The operator T is called *quasisimilar to* S (quasi–conjugate with S) if there exists a continuous mapping $\phi : Y \to X$ with dense image such that $T \circ \phi = \phi \circ S$, that is, the diagram in Figure 2 is commutative. If the operator T is quasi–similar to the operator S, then T inherits all dynamical properties of S (mixing, weak mixing, hypercyclicity).

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{T} & X \end{array}$$

FIGURE 2. Commutative diagram for ϕ, T and S.

3. Analogue of Theorem 1.2

We consider a directed graph $F_m = (V_m, \overline{E_m})$, where $\overline{E_m} = E_m \cup (v_{i,j}, v_{i+1,j+1})$, $1 \leqslant i \leqslant m-1$, and the backward shift operator on it (see Figure 3).

Thus, the graph F_m differs from the graph G_m of Theorem 1.2 by the additional diagonal edges. In this case the description of the weights for which the shift operator is weakly mixing will be the same as in Theorem 1.2.

Theorem 3.1. *Let B be the backward shift on F_m, $m \in \mathbb{N}$, and let X be any of the spaces $\ell^p(V_m, \mu)$, $1 \leqslant p < \infty$, or $c_0(V_m, \mu)$. Suppose that the operator B is bounded on X. Then the following conditions are equivalent:*

1. *B is a hypercyclic operator on X.*
2. *B is a weakly mixing operator on X.*
3. *There exists an increasing sequence (n_k) of positive integers such that for each $1 \leqslant i \leqslant m$*

$$n_k^{m-i}\mu_{v_{i,n_k}} \to 0, \quad k \to \infty.$$

Proof. 3) $\Longrightarrow$ 2).

First of all, we observe that condition 3) implies a stronger statement: there exists a sequence (n_k) such that for all $1 \leqslant i \leqslant m$ and $j \geqslant 1$ we have $n_k^{m-i}\mu_{v_{i,j+n_k}} \to 0$, $k \to \infty$. Indeed, if $n_k^{m-i}\mu_{v_{i,n_k}} \to 0$, then the boundedness of operator B it imply that $(n_k - l)^{m-i}|\mu_{v_{i,n_k-l}}| \to 0$ for

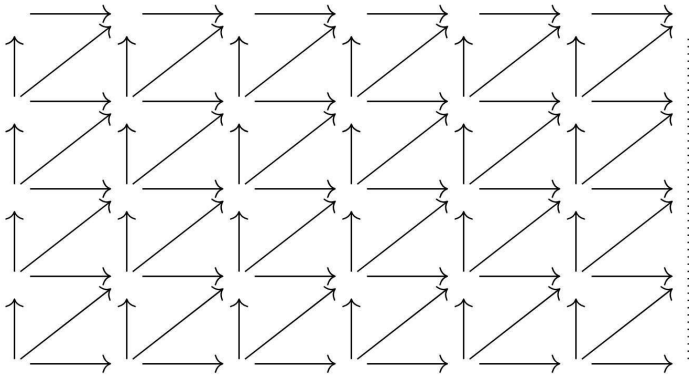


FIGURE 3. The directed graph F_m

each $l \geqslant 0$. Now it is not difficult to show (see [5, Lm. 4.2]) that there exists an increasing sequence m_k such that $(m_k + j)^{m-i}|\mu_{v_{i,m_k+j}}| \to 0$ for all $j \geqslant 0$.

By e_v we denote the function on V_m such that $e_v(v) = 1$ and $e_v(u) = 0$, $u \neq v$. The operator B can be represented as the sum of two commuting operators A and D:

$$(Af)(v_{i,j}) = f(v_{i+1,j}) + f(v_{i,j+1}),$$
$$(Df)(v_{i,j}) = f(v_{i+1,j+1}).$$

Using this representation and the identity $(A^k f)(v_{i,j}) = \sum\limits_{l=0}^{k} C_k^l f(v_{i+l,j+k-l})$, it is easy to show that

$$B^n(e_{v_{i,j}}) = (A+D)^n(e_{v_{i,j}}) = \sum_{k=0}^{n} C_n^k A^k D^{n-k}(e_{v_{i,j}})$$
$$= \sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l e_{v_{i-(n-k)-l,j-(n-k)-(k-l)}},$$

where we let $e_{v_{a,b}} = 0$ if $a \leqslant 0$ or $b \leqslant 0$.

The set of compactly supported functions is dense in X and $B^n f = 0$ for each compactly supported function f and sufficiently large n. Let us construct operators R_{n_k} so that $R_{n_k} f \to 0$ and $B^{n_k} R_{n_k} f = f$ on compactly supported functions. We let

$$R_{n_k}(e_{v_{1,j}}) = e_{v_{1,j+n_k}}, \qquad R_{n_k}(e_{v_{2,j}}) = e_{v_{2,j+n_k}} - n e_{v_{1,j+1+n_k}} - n e_{v_{1,j+n_k}},$$

and, supposing that R_{n_k} is already defined on the layers with numbers $1, \ldots, i-1$, we let

$$R_{n_k}(e_{v_{i,j}}) = e_{v_{i,j+n_k}} - \sum_{l=1}^{i-1} C_{n_k}^l R_{n_k}(e_{v_{i-l,j+l}}) - \sum_{p=1}^{i-1} C_{n_k}^p \sum_{l=0}^{i-1-p} C_{n_k-p}^l R_{n_k}(e_{v_{i-p-l,j+l}}).$$

Since

$$B^{n_k}(e_{v_{i,j+n_k}}) = \sum_{p=0}^{i-1} C_{n_k}^p \sum_{l=0}^{i-1-p} C_{n_k-p}^l e_{v_{i-p-l,j+l}},$$

we obtain that $B^{n_k} R_{n_k}(e_{v_{i,j}}) = e_{v_{i,j}}$.

By induction one can prove that

$$R_{n_k}(e_{v_{i,j}}) = \sum_{p=0}^{i-1} \sum_{s=1}^{i-p} \alpha_{i,s,p} e_{v_{s,i+j-p-s+n_k}} = \sum_{s=1}^{i} \sum_{p=0}^{i-s} \alpha_{i,s,p} e_{v_{s,i+j-p-s+n_k}}$$

with coefficients $\alpha_{i,s,p}$ independent of j. Let us substitute this formula into the recurrence formula for R_{n_k}:

$$\sum_{s=1}^{i}\sum_{p=0}^{i-s}\alpha_{i,s,p}e_{v_{s,i+j-p-s+n_k}} = e_{v_{i,j+n_k}} - \sum_{l=1}^{i-1}C_{n_k}^{l}\sum_{s=1}^{i-l}\sum_{p=0}^{i-l-s}\alpha_{i-l,s,p}e_{v_{s,i+j-p-s+n_k}} - \sum_{p=1}^{i-1}C_{n_k}^{p}\sum_{l=0}^{i-1-p}C_{n_k-p}^{l}\sum_{s=1}^{i-p-l}\sum_{p_1=0}^{i-p-l-s}\alpha_{i-p-l,s,p_1}e_{v_{s,i+j-p-p_1-s+n_k}}.$$

Taking into consideration that $e_{v_{i,j}}$ is a basis and comparing the coefficients at the corresponding basis vectors, it is easy to prove by induction that $|\alpha_{i,s,p}| < Cn_k^{i-s}$, where the constant C depends only on m. In particular, $|\alpha_{i,s,p}| = O(n_k^{m-s})$, $1 \leqslant i \leqslant m$. By the condition $n_k^{m-s}\mu_{v_{s,i+j-s-p+n_k}} \to 0$ for all i, j, s, p, we conclude that $R_{n_k}f \to 0$ for any finitely supported f. Thus, the operator B is weakly mixing.

1) $\Longrightarrow$ 3).

If B is hypercyclic, then for all $\varepsilon, \delta > 0$ and $K \in \mathbb{N}$ we can choose an arbitrarily large n and $f \in X$ such that $\|f\|_X < \varepsilon$ and

$$\|B^n f - \sum_{j=1}^{K} e_{v_{m,j}}\|_X < \delta \min_{v\in F}\mu_v.$$

For $n > m$

$$(B^n f)(v_{i,j}) = \sum_{p=0}^{m-i}C_n^p\sum_{k=0}^{m-i-p}C_{n-p}^k f(v_{i+p+k,j+n-k}),$$

$$(B^n f)(v_{m-l,j}) = \sum_{p=0}^{l}C_n^p\sum_{k=0}^{l-p}C_{n-p}^k f(v_{m-l+p+k,j+n-k}).$$

In particular, $(B^n f)(v_{m,j}) = f(v_{m,j+n})$, and hence $|f(v_{m,j+n}) - 1| < \delta$, $1 \leqslant j \leqslant K$. We have

$$|(B^n f)(v_{m-1,j})| = |f(v_{m-1,j+n}) + nf(v_{m,j+n-1}) + nf(v_{m,j+n})| < \delta,$$

and therefore

$$|f(v_{m-1,j+n}) + 2n| < \delta(1+2n), \quad 2 \leqslant j \leqslant K.$$

In the general case

$$|(B^n f)(v_{m-l,j})| = |\sum_{p=0}^{l}C_n^p\sum_{k=0}^{l-p}C_{n-p}^k f(v_{m-l+p+k,j+n-k})| < \delta, \qquad l+1 \leqslant j \leqslant K.$$

Using the inequalities for smaller l, this inequality can be rewritten in the form

$$|f(v_{m-l,j+n}) + P_l(n)| < C\delta n^l,$$

where C is some numerical constant, and the polynomials $P_l(n)$ in the variable n are given by the recurrence formula

$$P_l(n) = -\sum_{k=1}^{l}C_n^k P_{l-k}(n) - \sum_{p=1}^{l}C_n^p\sum_{k=0}^{l-p}C_{n-p}^k P_{l-p-k}(n),$$

$$P_0(n) = -1, \qquad P_1(n) = 2n,$$

which we rewrite it in a simpler form

$$P_l(n) = -\sum_{k=1}^{l}\Big(\sum_{p=0}^{k}C_n^p C_{n-p}^{k-p}\Big)P_{l-k}(n).$$

We denote by a_l the coefficient at the l-th power in the polynomial P_l. The recurrence formula for these polynomials implies

$$a_l = -\sum_{k=1}^{l}\Big(\sum_{p=0}^{k}\frac{1}{p!}\frac{1}{(k-p)!}\Big)a_{l-k} = -\sum_{k=1}^{l}\Big(\frac{1}{k!}\sum_{p=0}^{k}C_k^p\Big)a_{l-k}$$
$$-\sum_{k=1}^{l}\frac{2^k}{k!}a_{l-k} \qquad \Longrightarrow \qquad \frac{a_l}{2^l} = -\sum_{k=1}^{l}\frac{1}{k!}\frac{a_{l-k}}{2^{l-k}}.$$

Let us consider the function $g = \sum_{l=0}^{\infty}\frac{a_l}{2^l}x^l$. By the recurrence formula we can obtain the following relation for g:

$$(e^x-1)g = -g+1 \quad \Longrightarrow \qquad g = -e^{-x}.$$

This relation shows that P_l is a polynomial of degree exactly l.

Thus, if we choose $\delta > 0$ sufficiently small, we obtain

$$|f(v_{i,j+n})| \geqslant Cn^{m-i}, \qquad 1 \leqslant i \leqslant m, \qquad m \leqslant j \leqslant K,$$

where $C > 0$ is a constant depending only on m.

Replacing n by $n+m-1$, we obtain

$$|f(v_{i,j+n})| \geqslant Cn^{m-i}$$

for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant K-m+1$. Recall that $\|f\|_X < \varepsilon$. Since $|\mu_v f(v)| \leqslant \|f\|_X$, we conclude that

$$n^{m-i}|\mu_{v_{i,j+n}}| \leqslant C^{-1}\varepsilon, \qquad 1 \leqslant i \leqslant m, \qquad 1 \leqslant j \leqslant K-m+1.$$

Finally, since $\varepsilon > 0$ and $K \in \mathbb{N}$ are arbitrary, we repeat this procedure for the sequences $\varepsilon_k \to 0$ and $K_k \to \infty$ and find the required sequence $n_k \to \infty$. The proof is complete. □

Now we consider the case of infinitely many horizontal layers, that is, the directed graph $F_\infty = (V_\infty,\ E_\infty)$, where

$$V_\infty = (v_{i,j})_{i\geqslant 0, j\geqslant 0}, \qquad E_\infty = \{(v_{i,j}, v_{i,j+1}), (v_{i,j}, v_{i+1,j}), (v_{i,j}, v_{i+1,j+1}),\ i \geqslant 0,\ j \geqslant 0\},$$

and the shift operator on it.

We see that as the number of rows in the graph F_m grows, the conditions on the weight ensuring hypercyclicity become ever more strict. Therefore it is somewhat surprising that in the case of the lattice $\mathbb{Z}_+ \times \mathbb{Z}_+$ infinite in both directions with diagonal edges the backward shift is weakly mixing under much milder conditions on the weight. The following result gives a sufficient condition for hypercyclicity/mixing, which is sharp on the rough exponential scale.

Theorem 3.2. *Let B be the backward shift on F_∞, and let X be any of the spaces $\ell^p(V_\infty, \mu)$, $1 \leqslant p < \infty$, or $c_0(V_\infty, \mu)$. Suppose that the operator B is bounded on X. Then*

1. *If $\varlimsup_{i+j\to\infty} \mu_{i,j}^{\frac{1}{i+j}} < \sqrt{2}+1$, then the operator B is mixing on X.*
2. *Suppose that $\mu_{i,j} = p_i q_j$, $p_i, q_j > 0$. Then under the conditions*

$$\varlimsup_{i\to\infty} p_i^{\frac{1}{i}} = C > 1, \qquad \varlimsup_{j\to\infty} q_j^{\frac{1}{j}} < \frac{C+1}{C-1}, \tag{3.1}$$

or

$$\varlimsup_{i\to\infty} p_i^{\frac{1}{i}} = C \leqslant 1, \qquad \varlimsup_{j\to\infty} q_j^{\frac{1}{j}} < \infty$$

(in particular, for $p_i \equiv 1$) the operator B is mixing on X.

3. *Let for some $C > 1$ the inequalities*

$$p_i \geqslant \gamma C^i, \quad q_j \geqslant \gamma \Big(\frac{C+1}{C-1}\Big)^j \quad \forall i,\ j \in \mathbb{Z}_+,$$

hold, where $\gamma > 0$. Then the operator B is not hypercyclic on X.

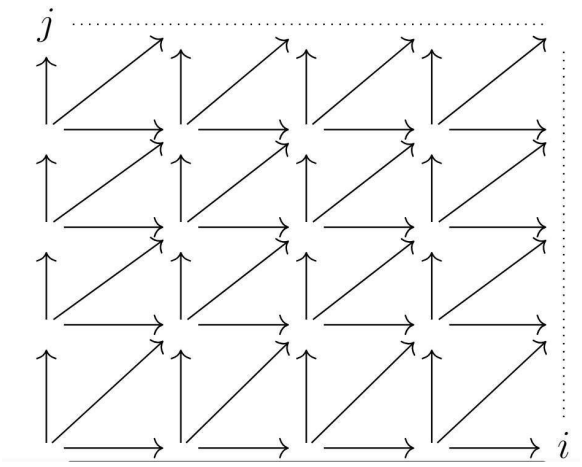


FIGURE 4. The graph F_∞.

Proof. To prove the mixing property for the shift operator B, we apply the Godefroy — Shapiro criterion (Theorem 2.2). Note that the function $f_{a,b} = f_{a,b}(v_{i,j}) = a^i b^j$ is an eigenvector of the operator B with the eigenvalue $a + b + ab$.

It is obvious that the condition

$$\varlimsup_{i+j\to\infty} \mu_{i,j}^{\frac{1}{i+j}} < \min\Big\{\frac{1}{|a|}, \frac{1}{|b|}\Big\}. \tag{3.2}$$

is sufficient for the vector $f_{a,b}$ to belong to the space X. To apply Theorem 2.2, we need sufficiently many eigenvectors with eigenvalues greater than 1. A similar statement for eigenvalues less than 1 is obvious since $f_{a,b} \in X$ if a and b are sufficiently small. Note that $2x + x^2 > 1$ for $x > \sqrt{2} - 1$. Since $\varlimsup\limits_{i+j\to\infty} \mu_{i,j}^{\frac{1}{i+j}} < \sqrt{2} + 1$, there exist open sets $U_1, U_2 \subset (\sqrt{2} - 1, \infty)$ such that condition (3.2) holds for $(a, b) \in U_1 \times U_2$. Moreover, $a + b + ab > 1$. By Lemma 2.1 the family $\{f_{a,b} : (a, b) \in U_1 \times U_2\}$ is complete in X. Thus, the operator B is mixing by Theorem 2.2.

Now suppose that the weight reads $\mu_{i,j} = p_i q_j$. The eigenvector $a^i b^j$ lies in the space if

$$\varlimsup_{i\to\infty} p_i^{\frac{1}{i}} < \frac{1}{|a|}, \qquad \varlimsup_{j\to\infty} q_j^{\frac{1}{j}} < \frac{1}{|b|}.$$

Condition (3.1) implies that these inequalities hold for

$$a \in \Big(\frac{1}{C} - \varepsilon, \frac{1}{C}\Big), \qquad b \in \Big(\frac{C-1}{C+1} + \delta, \frac{C-1}{C+1} + 2\delta\Big)$$

for all sufficiently small $\varepsilon, \delta > 0$. At the same time $a + b + ab > 1$ if ε is much smaller than δ since

$$\frac{1}{C} + \frac{C-1}{C+1} + \frac{1}{C}\frac{C-1}{C+1} = 1.$$

Thus, the family of eigenvectors with eigenvalues greater than 1 (and, of course, with eigenvalues less than 1) is complete in X, and hence the operator B is mixing.

Now let $C = 1$ and $U_1 = (1 - 2\varepsilon, 1 - \varepsilon)$ and $U_2 = (2\varepsilon, 3\varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Then $f_{a,b} \in X$ and $a + b + ab > 1$. Moreover, the modulus of b can be arbitrarily small.

The operator B can be represented as the sum of two commuting operators A and D

$$(Af)(v_{i,j}) = f(v_{i+1,j}) + f(v_{i,j+1}),$$
$$(Df)(v_{i,j}) = f(v_{i+1,j+1}).$$

Accordingly,

$$(A+D)^n = \sum_{k=0}^{n} C_n^k A^k D^{n-k}, \qquad (A^k f)(v_{i,j}) = \sum_{l=0}^{k} C_k^l f(v_{i+l,j+k-l}).$$

If the operator B is hypercyclic, we can approximate the vector $e_{v_{0,0}}$ with any accuracy, namely, for all ε, δ there exist an arbitrarily large n and a vector f such that $\|(B^n f)(v_{0,0}) - e_{v_{0,0}}\|_X < \varepsilon$ and $\|f\|_X < \delta$. As f one can choose a vector proportional to a hypercyclic vector. Thus,

$$\Big|\sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l f(v_{n-k+l,n-k+k-l}) - 1\Big| < \varepsilon, \tag{3.3}$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\mu_{i,j} f(v_{i,j})|^p < \delta^p.$$

Therefore, if

$$p_i \geqslant \gamma C^i, \qquad q_j \geqslant \gamma\Big(\frac{C+1}{C-1}\Big)^j,$$

we see that

$$|f(v_{i,j})| \leqslant \frac{\delta}{\gamma^2}\Big(\frac{1}{C}\Big)^i\Big(\frac{C-1}{C+1}\Big)^j$$

for all i, j. And since

$$\sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l \Big(\frac{1}{C}\Big)^{n-k+l}\Big(\frac{C-1}{C+1}\Big)^{n-l} = \sum_{k=0}^{n} C_n^k \Big(\frac{1}{C}\Big)^{n-k}\Big(\frac{C-1}{C+1}\Big)^{n-k}\Big(\frac{1}{C}+\frac{C-1}{C+1}\Big)^k = 1,$$

we arrive at a contradiction with condition (3.3) for sufficiently small δ. The proof is complete. □

4. Shift operator on "ladder" graph

Here we study the hypercyclicity of operator B for a special graph $L_2 = (V_2, E)$ obtained by modifying the graph G_2 (see the description before Theorem 1.2): all its edges are the same as those of G_2, except for edges $(v_{1,2j}, v_{2,2j})$, the direction of which is reversed, see Figure 5. In the figure, for simplicity of the further exposition, a special numbering of the vertices is introduced, namely, we write n instead of $v_{1,n}$ and $\overline{n}$ instead of $v_{2,n}$.

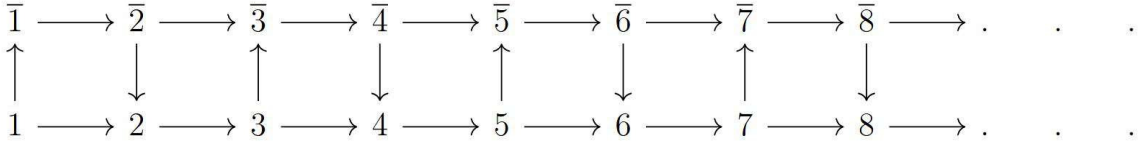


Figure 5. The graph L_2.

The following result shows that such a change of the edges directions (compared with the graph G_2) changes significantly the hypercyclicity sufficient conditions on the weight.

Theorem 4.1. *Let B be the backward shift on L_2, and let X be any of the spaces $\ell^p(V_2,\mu)$, $1 \leqslant p < \infty$, or $c_0(V_2,\mu)$. Suppose that the operator B is bounded on X. Then*

1. *If*
$$\overline{\lim_{n\to\infty}} \mu_n^{\frac{1}{n}} < \frac{\sqrt{5}+1}{2}, \qquad \overline{\lim_{n\to\infty}} \mu_{\overline{n}}^{\frac{1}{n}} < \frac{\sqrt{5}+1}{2},$$
then the operator B is weakly mixing.

2. *If*
$$\mu_n \geqslant C\Big(\frac{\sqrt{5}+1}{2}\Big)^n, \qquad \mu_{\overline{n}} \geqslant C\Big(\frac{\sqrt{5}+1}{2}\Big)^n$$
for all $n \in \mathbb{N}$, where $C > 0$, then the operator B is not hypercyclic.

Proof. We begin with observing that the square of operator B has a simpler structure:
$$B^2 f(n) = f(\overline{n+1}) + f(n+2),$$
$$B^2 f(\overline{n}) = f(n+1) + f(\overline{n+2}).$$
Let us show that under assumptions of Statement 1 the operator B^2 (and hence also B) is weakly mixing.

The analysis of the action of B^2 shows that the graph can be partitioned into two subgraphs with the vertices $N_1 = \{1, \bar{2}, 3, \dots, 2k-1, \overline{2k}, \dots\}$ and $N_2 = \{\bar{1}, 2, \bar{3}, \dots, \overline{2k-1}, 2k, \dots\}$, on which the operator acts independently. Straightening these subgraphs, we obtain a representation of the space as a direct sum of ordinary weighted spaces $\ell^p(N_1, \mu|_{N_1})$ and $\ell^p(N_2, \mu|_{N_2})$ (or $c_0(N_1, \mu|_{N_1})$ and $c_0(N_2, \mu|_{N_2})$), and of the operator B^2 as the direct sum $B_0 + B_0^2$ with itself, where B_0 denotes the ordinary backward shift in a space of sequences (formally, we are dealing with the backward shift on different spaces, but we use the same notation).

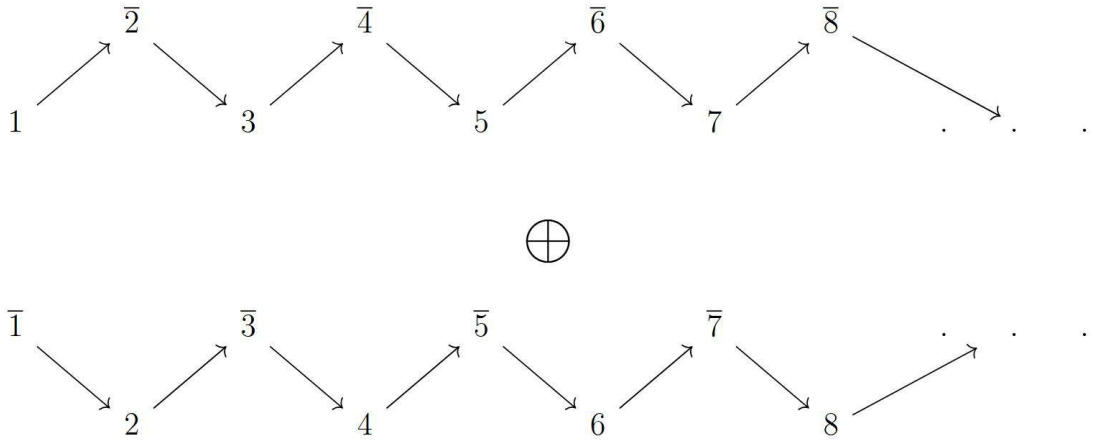


FIGURE 6. Partition of the graph L_2.

Let us show that under the assumptions of theorem the operator $B_0 + (B_0)^2$ is mixing on each of the aforementioned spaces. For the sake of definiteness, we consider the space $\ell^p(N_1, \mu')$, where $\mu' = \mu|_{N_1} = (\mu_1, \mu_{\overline{2}}, \mu_3, \mu_{\overline{4}}, \dots)$.

It is easy to see that the vectors
$$f_\lambda = \Big(1, \frac{\sqrt{1+4\lambda}-1}{2}, \Big(\frac{\sqrt{1+4\lambda}-1}{2}\Big)^2, \Big(\frac{\sqrt{1+4\lambda}-1}{2}\Big)^3, \dots\Big)$$
are eigenvectors of the operator $B_0 + (B_0)^2$ associated with the eigenvalue λ. Since
$$\overline{\lim_{n\to\infty}} |\mu'_n|^{\frac{1}{n}} < \frac{\sqrt{5}+1}{2},$$

there exist open intervals $U_1 \subset (0,1)$ and $U_2 \subset (1,\infty)$ such that the vectors f_λ, $\lambda \in U_1 \cup U_2$, lie in the space X. Similarly to Lemma 2.1, the families $\{f_\lambda : \lambda \in U_1\}$ and $\{f_\lambda : \lambda \in U_2\}$ are complete in X. Hence, the operator $B_0 + B_0^2$ in $\ell^p(N_1, \mu')$ is mixing. The same is true for the operator $B_0 + B_0^2$ on $\ell^p(N_2, \mu|_{N_2})$. Since the direct sum of mixing operators is mixing, we conclude that B^2 is mixing and, in particular, weakly mixing.

Suppose that the operator B is hypercyclic. Then the operator $B_0+B_0^2$ on the space $\ell^p(N_1, \mu')$, where $\mu' = \mu|_{N_1} = (\mu_1, \mu_{\overline{2}}, \mu_3, \mu_{\overline{4}}, \ldots)$, is also hypercyclic (the case of the spaces c_0 is analogous).

We consider the vector $e_1 = (1, 0, 0, \ldots)$. If the operator $B_0 + B_0^2$ is hypercyclic on $\ell^p(N_1, \mu')$, then for all $\varepsilon, \delta > 0$ there exist $n \in \mathbb{N}$ and $f = (a_k)_{k\in\mathbb{N}}$ such that

$$\|(B_0 + B_0^2)^n(f) - e_1\|_X < \varepsilon, \qquad \sum_{k=1}^{\infty} |a_k \mu'_k|^p < \delta^p.$$

By assumption,

$$\mu'_k \geqslant C\Big(\frac{\sqrt{5}+1}{2}\Big)^k,$$

hence

$$|a_k| \leqslant \frac{\delta\Big(\frac{\sqrt{5}-1}{2}\Big)^k}{C}.$$

We note that

$$(B_0 + B_0^2)^n = \sum_{i=0}^{n} C_n^i B_0^{n+i}.$$

Therefore,

$$\begin{aligned} |(B_0 + B_0^2)^n(f)(1)| &\leqslant \Big|\sum_{i=0}^{n} C_n^i a_{n+i+1}\Big| \\ &\leqslant \frac{\delta}{C}\Big(\frac{\sqrt{5}-1}{2}\Big)^{n+1} \sum_{i=0}^{n} C_n^i \Big(\frac{\sqrt{5}-1}{2}\Big)^i = \frac{\delta}{C}\frac{\sqrt{5}-1}{2}, \end{aligned}$$

which contradicts the condition

$$|(B_0 + B_0^2)^n(f)(1) - 1| \leqslant \|(B_0 + B_0^2)^n(f) - e_1\|_X < \varepsilon$$

provided ε, δ are small enough. The proof is complete. □

5. Two lattices of form $\mathbb{Z}_+ \times \mathbb{Z}$

In this section we consider two types of lattices of the form $\mathbb{Z}_+ \times \mathbb{Z}$ and find sufficient conditions for hypercyclicity/mixing by using the Godefroy — Shapiro criterion. We note that in this case there is no chance to find exact conditions on the exponential scale as in Theorem 3.2, since to ensure the belonging of the vectors $f_{a,b}$ to the space X, the weights must decay exponentially in one of the directions.

We consider the graph $G_\infty = (V_\infty, E_\infty)$, where

$$V_\infty = \{v_{i,j}\}_{i\geqslant 0,\ j\in\mathbb{Z}}, \qquad E_\infty = \{(v_{i,j}, v_{i,j+1}), (v_{i,j}, v_{i+1,j}) :\ i \geqslant 0,\ j \in \mathbb{Z}\}.$$

The graph G_∞ represents the Cartesian product of the graphs $\mathbb{Z}_+$ and $\mathbb{Z}$ with the natural orientation of the edges.

Theorem 5.1. *Let B be the backward shift on G_∞, and let X be any of the spaces $\ell^p(V_\infty, \mu)$, $1 \leqslant p < \infty$, or $c_0(V_\infty, \mu)$. Suppose that the operator B is bounded on X. Let $\mu_{i,j} = p_i q_j$, $p_i, q_j > 0$.*

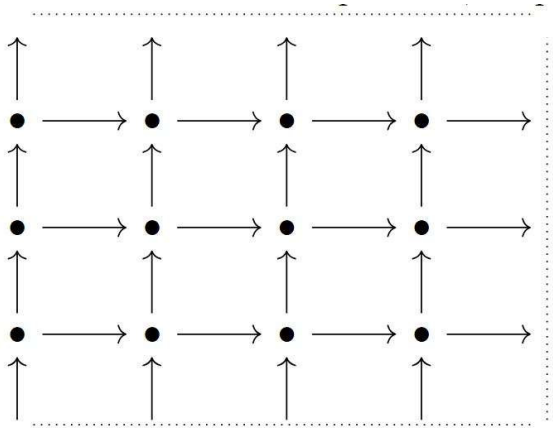


FIGURE 7. The graph G_∞.

1. *If*
$$\overline{\lim_{i\to\infty}}\, p_i^{\frac{1}{i}} = C, \qquad \overline{\lim_{j\to\infty}}\, q_j^{\frac{1}{j}} \leqslant H, \qquad \overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} < \frac{1}{H},$$
where
$$C > 0, \qquad H \geqslant 1, \qquad \frac{1}{H} + \frac{1}{C} > 1,$$
then the operator B on X is mixing. In the case $H = 1$ the assertion holds for any $C > 0$.
2. *If*
$$C > 1, \qquad p_i \geqslant \gamma C^i, \qquad q_j \geqslant \gamma\Big(\frac{C}{C-1}\Big)^j, \quad i, j \geqslant 0,$$
then the operator B is not hypercyclic on X.

Proof. Choose $\varepsilon > 0$ so that
$$\overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} < \frac{1}{H+\varepsilon}$$
and let $f_{a,b}(v_{i,j}) = a^i b^j$, $a, b > 0$. These are eigenvectors of the operator B associated with the eigenvalues $a + b$. If
$$a < \frac{1}{C}, \qquad \frac{1}{H+\varepsilon} < b < \frac{1}{H},$$
then $f_{a,b} \in X$. Moreover, there exist open intervals of the parameters a and b for which, respectively, $a+b < 1$ or $a+b > 1$. It is obvious that $a+b < 1$ if $a < 1 - \dfrac{1}{H}$. At the same time, the condition $\dfrac{1}{H} + \dfrac{1}{C} > 1$ implies $a + b > 1$ if a is sufficiently close to $\dfrac{1}{C}$ and b is sufficiently close to $\dfrac{1}{H}$.

By Lemma 2.1, the families of eigenvectors of operator B associated with the eigenvalues being greater and less than one are complete, and therefore the operator B is mixing by Theorem 2.2.

It is easy to see that
$$(B^n f)(v_{i,j}) = \sum_{k=0}^{n} C_n^k f(v_{i+k,j+n-k}).$$
If the operator B is hypercyclic, then for all ε, δ there exist $f \in X$ and $n \in \mathbb{N}$ such that
$$\|(B^n f)(v_{0,0}) - e_{v_{0,0}}\|_X < \varepsilon, \qquad \|f\|_X < \delta.$$

In particular,

$$|\sum_{k=0}^{n} C_n^k f(e_{v_{k,n-k}}) - 1| < \varepsilon, \qquad |f(v_{i,j})\mu_{i,j}| \leqslant \|f\|_X < \delta. \tag{5.1}$$

Since

$$\mu_{i,j} \geqslant \gamma^2 C^i \Big(\frac{C}{C-1}\Big)^j,$$

we obtain

$$\Big|\sum_{k=0}^{n} C_n^k f(e_{v_{k,n-k}})\Big| \leqslant \frac{\delta}{\gamma^2} \sum_{k=0}^{n} C_n^k \frac{1}{C^k} \Big(\frac{C-1}{C}\Big)^{n-k} = \frac{\delta}{\gamma^2}.$$

This contradicts the first inequality in (5.1) once δ is sufficiently small. The proof is complete. □

Remark 5.1. *As it was shown in* [2, Thm. 2.4], *in the case* $q_j \equiv 1$ *the operator* B *on* G_∞ *is always mixing once it is bounded. In Theorem 5.1 one must additionally require an exponential decay in one of the vertical directions, but in the other direction an exponential growth is allowed.*

We consider the graph $W = (V_W, E_W)$, where

$$W = (v_{i,j})_{i\geqslant 0, j\in\mathbb{Z}, i+j\in 2\mathbb{Z}}, \qquad E_W = \{(v_{i,j}, v_{i+1,j+1}), (v_{i,j}, v_{i+1,j-1}), (v_{i,j}, v_{i+2,j}),\ i \geqslant 0,\ j \in \mathbb{Z}\},$$

and the shift operator on it.

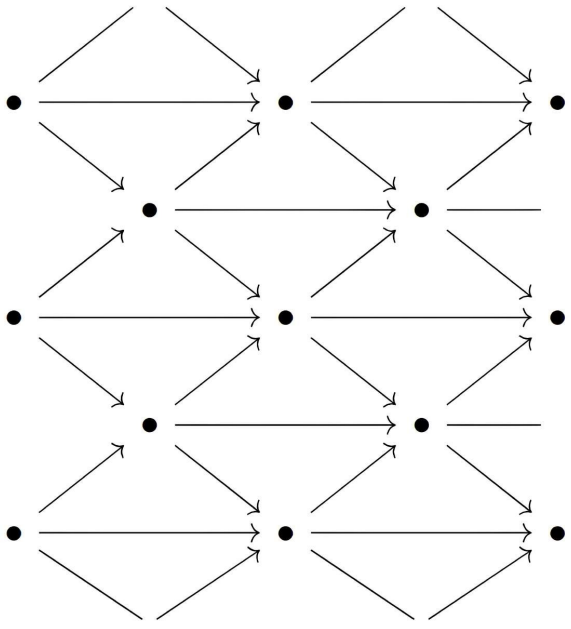


FIGURE 8. The graph W.

Theorem 5.2. *Let* B *be the backward shift on* W, *and let* X *be any of the spaces* $\ell^p(V_W, \mu)$, $1 \leqslant p < \infty$, *or* $c_0(V_W, \mu)$. *Suppose that the operator* B *is bounded on* X. *Let* $\mu_{i,j} = p_i q_j$, $p_i, q_j > 0$. *Then*

1. *Under the conditions*

$$\overline{\lim_{i\to\infty}}\, p_i^{\frac{1}{i}} = C,$$
$$\overline{\lim_{j\to\infty}}\, q_j^{\frac{1}{j}} < \frac{2C}{C^2-1+\sqrt{C^4-6C^2+1}}, \tag{5.2}$$
$$\overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} \leqslant \frac{C^2-1+\sqrt{C^4-6C^2+1}}{2C}, \tag{5.3}$$

where $C > \sqrt{2}+1$, the operator B is mixing on X. If, however, $1 \leqslant C \leqslant \sqrt{2}+1$, then the operator B on X is mixing under the conditions

$$\overline{\lim_{i\to\infty}}\, p_i^{\frac{1}{i}} = C, \qquad \overline{\lim_{j\to\infty}}\, q_j^{\frac{1}{j}} \leqslant H, \qquad \overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} < \frac{1}{H},$$

where $H > 0$.

2. *Let $q_j = 1$ and $p_i \geqslant c(\sqrt{2}+1)^i$, $\forall i \geqslant 0$, $j \in \mathbb{Z}$, $i+j \in 2\mathbb{Z}$, where $c > 0$. Then the operator B is not hypercyclic on X.*

Remark 5.2. 1. *By virtue of the symmetry of the graph we can interchange the conditions on the growth of the weights upwards and downwards, respectively, and also interchange the strict and non–strict inequalities.*

2. *The operator B can be mixing under a rapid growth of the weights along the horizontals, but for this one needs rapid decay of the weights along the verticals in one of the directions (at the same time, in the opposite direction the weights may grow rapidly).*

Proof. We consider the vectors $f_{a,b}(v_{i,j}) = a^i b^j$, $a, b > 0$. These are eigenvectors of the operator B associated with the eigenvalues $ab + \dfrac{a}{b} + a^2$. The vectors $f_{a,b}$ belong to the space X if the conditions

$$\overline{\lim_{i\to\infty}}\, p_i^{\frac{1}{i}} < \frac{1}{a}, \qquad \overline{\lim_{j\to\infty}}\, q_j^{\frac{1}{j}} < \frac{1}{b}, \qquad \overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} < b \tag{5.4}$$

are satisfied.

Thus, a must satisfy the condition $a < \dfrac{1}{C}$, while b should range in the interval $\Big(\overline{\lim\limits_{j\to-\infty}}\, q_j^{\frac{1}{|j|}}, \big(\overline{\lim\limits_{j\to\infty}}\, q_j^{\frac{1}{j}}\big)^{-1}\Big)$. It is obvious that under the assumptions of the theorem there exist open intervals of the parameters a and b for which $f_{a,b} \in X$ and $0 < ab + \dfrac{a}{b} + a^2 < 1$ (if one chooses a sufficiently small). Let us show that we can find intervals, which satisfy the condition

$$ab + \frac{a}{b} + a^2 > 1.$$

Let $C > \sqrt{2}+1$. Then for

$$b > \frac{C^2-1+\sqrt{C^4-6C^2+1}}{2C}$$

the inequality

$$\frac{b}{C} + \frac{1}{bC} + \frac{1}{C^2} > 1$$

holds, and hence also the inequality

$$ab + \frac{a}{b} + a^2 > 1$$

is true for a close to $\dfrac{1}{C}$. Moreover, by conditions (5.2) and (5.3), there exists a non–empty open interval of parameters

$$b > \frac{C^2 - 1 + \sqrt{C^4 - 6C^2 + 1}}{2C}$$

lying in the interval $\Big(\varlimsup\limits_{j\to-\infty} q_j^{\frac{1}{|j|}}, \big(\varlimsup\limits_{j\to\infty} q_j^{\frac{1}{j}} \big)^{-1} \Big)$.

Now let $1 \leqslant C \leqslant \sqrt{2} + 1$. Then the inequality

$$\frac{b}{C} + \frac{1}{bC} + \frac{1}{C^2} > 1$$

holds for all $b > 0$, $b \neq 1$, and it sufficient to choose an interval of values of the parameter b lying in $\Big(\varlimsup\limits_{j\to-\infty} q_j^{\frac{1}{|j|}}, \big(\varlimsup\limits_{j\to\infty} q_j^{\frac{1}{j}} \big)^{-1} \Big)$, and an interval of values of a sufficiently close to $\dfrac{1}{C}$.

In each of the cases there are open intervals of the parameters corresponding to eigenvalues greater and less than one. By Lemma 2.1 the corresponding families of eigenvectors are complete, and hence the operator B is mixing by Theorem 2.2.

We observe that in the case $q_j = 1$ that the operator $B_0^2 + 2B_0$ on $Y = \ell^p(\mathbb{Z}_+, \{p_i\})$ (where B_0 is the ordinary backward shift operator on $\ell^p(\mathbb{Z}_+)$) is quasisimilar to the studied operator B. The corresponding mapping of the spaces is $\phi : X \to Y$,

$$\phi(f)(k) = \sum_{j\in\mathbb{Z}:\, j+k\in 2\mathbb{Z}} f(v_{k,j}), \quad k \geqslant 0,$$

this mapping assigns to $f \in X$ a sequence from Y, the k-th coordinate of which is equal to the sum of all numbers standing on the vertical with number k. It is easy to verify, by the literal reproducing the arguing in the proof of Statement 2 in Theorem 4.1, that for $p_i \geqslant c(\sqrt{2}+1)^i$ the operator $B_0^2 + 2B_0$ on $Y = \ell^p(\mathbb{Z}_+, \{p_i\})$ is not hypercyclic. Therefore the operator B is also not hypercyclic, since this holds for an operator quasisimilar to it. The proof is complete. □

6. Infinite version of "ladder"

We consider the directed graph $L_\infty = (V'_\infty, E'_\infty)$, where $V'_\infty = (v_{i,j})_{i\geqslant 0, j\in\mathbb{Z}}$ and

$$\begin{aligned} E'_\infty = \big\{ &(v_{i,j}, v_{i+1,j}),\ (v_{2i,2j}, v_{2i,2j+1}),\ (v_{2i,2j}, v_{2i,2j-1}),\\ &(v_{2i+1,2j+1}, v_{2i+1,2j}),\ (v_{2i+1,2j+1}, v_{2i+1,2j+2}),\ \ j \in \mathbb{Z},\ i \geqslant 0 \big\} \end{aligned}$$

and the shift operator B on it.

Theorem 6.1. *Let B be the backward shift on L_∞, and let X be any of the spaces $\ell^p(V'_\infty, \mu)$, $1 \leqslant p < \infty$, or $c_0(V'_\infty, \mu)$. Then*

1. *Let $\mu_{i,j} = p_i q_j$, $p_i, q_j > 0$. If*

$$\begin{aligned} &\varlimsup_{i\to\infty} p_i^{\frac{1}{i}} < C,\\ &\varlimsup_{j\to\infty} q_j^{\frac{1}{j}} < \frac{2C}{C^2 - 1 + \sqrt{C^4 - 6C^2 + 1}},\\ &\varlimsup_{j\to-\infty} q_j^{\frac{1}{|j|}} \leqslant \frac{C^2 - 1 + \sqrt{C^4 - 6C^2 + 1}}{2C}, \end{aligned}$$

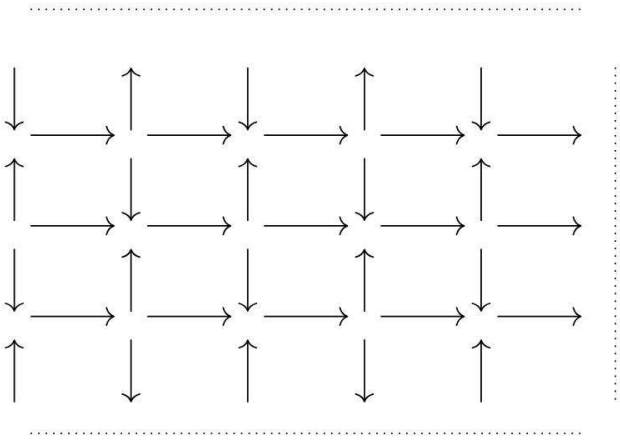


FIGURE 9. The graph L_∞.

where $C > \sqrt{2}+1$, *then the operator* B *is weakly mixing on* X. *If, however,* $1 \leqslant C \leqslant \sqrt{2}+1$, *then the operator* B *on* X *is weakly mixing under the conditions*

$$\overline{\lim_{i\to\infty}}\, p_i^{\frac{1}{i}} < C, \qquad \overline{\lim_{j\to\infty}}\, q_j^{\frac{1}{j}} \leqslant \frac{1}{H}, \qquad \overline{\lim_{j\to-\infty}}\, q_j^{\frac{1}{|j|}} < H,$$

where $H > 0$.

2. *Suppose that* $\inf\limits_j \mu_{i,j} \geqslant c(\sqrt{2}+1)^i$, $\forall i \in \mathbb{Z}_+$, *where* $c > 0$. *Then the operator* B *on* X *is not hypercyclic.*

Proof. We consider the square of the operator B

$$(B^2 f)(v_{i,j}) = f(v_{i+1,j+1}) + f(v_{i+1,j-1}) + f(v_{i+2,j}).$$

The action of this operator is shown on Figure 10.

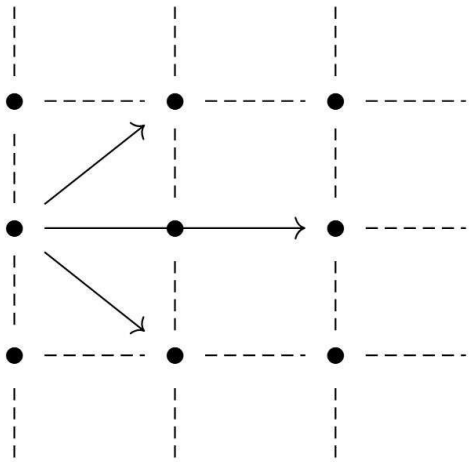


FIGURE 10. The action of operator B^2.

It is easy to see that, as in the case of the graph L_2, the graph L_∞ partitions into two identical subgraphs, each of which is the graph W from Theorem 5.2. The space X also decomposes into a direct sum of spaces, on each of which the same operator acts, namely, the backward shift operator on the graph W. Under the assumptions of the theorem the backward shift operators on each of the summands are mixing, and hence, the operator B^2, as their direct sum, is also mixing. In particular, the operator B^2, and hence also the operator B, is weakly mixing.

If one of the summands in the decomposition of the original operator is not hypercyclic, then the original operator is not hypercyclic either. Therefore, by Theorem 5.2 we also obtain necessary conditions on the weights. However, now we consider one more proof of the necessity of the hypercyclicity conditions for the shift operator on the graph W (for convenience of the computations a new numbering of the vertices is introduced: $i = \dfrac{i' - j'}{2}$, $j = \dfrac{i' + j'}{2}$, where the primed symbols denote the old vertex numbers), see Figure 11.

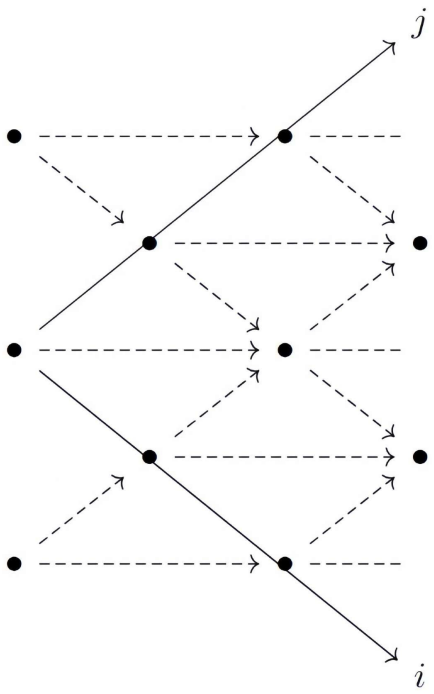


FIGURE 11. The graph W.

Similar to the proof of Theorem 3.2, the operator B^2 can be represented as the sum of two commuting operators A and D

$$(Af)(v_{i,j}) = f(v_{i+1,j}) + f(v_{i,j+1}), \qquad (Df)(v_{i,j}) = f(v_{i+1,j+1}).$$

Hence,

$$(A + D)^n = \sum_{k=0}^{n} C_n^k A^k D^{n-k}, \qquad (A^k f)(v_{i,j}) = \sum_{l=0}^{k} C_k^l f(v_{i+l,j+k-l}).$$

If the operator B^2 is hypercyclic, then we can approximate the vector $e_{v_{0,0}}$ with an arbitrary accuracy. Therefore, for all ε, δ there exist $f \in X$ and $n \in \mathbb{N}$ such that

$$|\sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l f(e_{v_{n-k+l,n-l}}) - 1| < \varepsilon, \qquad |f(v_{i,j})\mu_{i,j}| \leqslant \|f\|_X < \delta.$$

In the new coordinates the condition from Statement 2 becomes $\mu_{i,j} \geqslant c(\sqrt{2}+1)^{i+j}$ for all i, j, and hence, $|f(v_{i,j})| \leqslant \frac{\delta}{c}(\sqrt{2}+1)^{-i-j}$. Therefore,

$$\Big|\sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l f(e_{v_{n-k+l,n-l}})\Big| \leqslant \frac{\delta}{c} \sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_k^l (\sqrt{2}+1)^{k-2n}$$
$$= \frac{\delta}{c}(\sqrt{2}+1)^{-2n} \sum_{k=0}^{n} C_n^k (2\sqrt{2}+2)^k = \frac{\delta}{c},$$

which for sufficiently small δ contradicts the fact that this sum is close to 1. Thus, the operator B^2 is not hypercyclic. The proof is complete. □

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BIBLIOGRAPHY

1. E. Abakumov, A. Abbar. *On several dynamical properties of shifts acting on directed trees* // Canadian J. Math. **78**:2, 375–404 (2026). https://doi.org/10.4153/S0008414X24000981
2. A. Baranov, A. Lishanskii, D. Papathanasiou. *Hypercyclic shifts on lattice graphs* // Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **120**, 78 (2026). https://doi.org/10.1007/s13398-026-01868-8
3. F. Bayart, É. Matheron. *Dynamics of Linear Operators*. Cambridge Univ. Press, Cambridge (2009).
4. G. Godefroy, J. H. Shapiro. *Operators with dense, invariant cyclic vector manifolds* // J. Funct. Anal. **98**:2, 229–269 (1991). https://doi.org/10.1016/0022-1236(91)90078-J
5. K.–G. Grosse–Erdmann, A.P. Manguillot. *Linear Chaos*. Springer, Berlin (2011). https://doi.org/10.1007/978-1-4471-2170-1
6. K.–G. Grosse–Erdmann, D. Papathanasiou. *Dynamics of weighted shifts on directed trees* // Indiana Univ. Math. J. **72**:1, 263–299 (2023). https://doi.org/10.1512/iumj.2023.72.9315
7. K.–G. Grosse–Erdmann, D. Papathanasiou. *Chaotic weighted shifts on directed trees* // Preprint, arXiv:2303.03980 [math.FA] (2023).
8. A. López–Martinez, D. Papathanasiou. *Shifts on trees versus classical shifts in chain recurrence* // J. Differ. Equations **433**, 113230 (2025). https://doi.org/10.1016/j.jde.2025.113230
9. R.A. Martínez–Avendaño. *Hypercyclicity of shifts on weighted L^p spaces of directed trees* // J. Math. Anal. Appl. **446**:1, 823–842 (2017). https://doi.org/10.1016/j.jmaa.2016.08.066
10. S. Rolewicz. *On orbits of elements* // Studia Math. **32**:1, 17–22 (1969). https://doi.org/10.4064/sm-32-1-17-22
11. H. N. Salas. *Hypercyclic weighted shifts* // Trans. Amer. Math. Soc. **347**:3, 993–1004 (1995). https://doi.org/10.2307/2154883

Maxim Andreevich Matveev,
Chebyshev Laboratory,
Vasilyevsky Island, 14th Line 29B, room 122,
199178, Saint–Petersburg, Russia,

Saint–Petersburg State University,
Faculty of Mathematics and Mechanics,
Universitetskii av. 28, litera D,
198504, Saint–Petersburg, Peterhof, Russia
E-mail: st087207@student.spbu.ru