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SUBSONIC TRAVELING WAVE IN HYPERBOLIC EQUATION

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Abstract. We consider a subsonic traveling wave for a semilinear hyperbolic equation. The main attention is paid to the stability issue, due to which we study a spectral problem for a non–self–adjoint operator. This serves as a base for the description of a wide class of initial perturbations, with respect to which the wave is asymptotically stable in the linear approximation.

Keywords: nonlinear differential equation, traveling wave, spectrum, stability.

Mathematics Subject Classification: 34D20, 35L52

1. Introduction

The initial object is a semilinear hyperbolic equation

$$\delta\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial\varphi}{\partial t}+f(\varphi)=0,\qquad x\in\mathbb{R},\quad t>0;\qquad \delta=\mathrm{const}>0. \tag{1.1}$$

The presence of a first time derivative means that we take into consideration dissipative effects in the corresponding mathematical models. Equation (1.1) is encountered in magnetodynamics while describing the dynamics of domain walls [1], [2], in models of heat conduction and diffusion [7], [11].

The function $f(\varphi)$ is assumed to be continuously differentiable *in a neighbourhood of the interval* $\varphi\in[0,1]$. The main restriction consists in the presence of a pair of zeros

$$f(0)=f(1)=0;\qquad f'(0)>0,\qquad f'(1)<0, \tag{1.2}$$

with the condition $f(\varphi)\neq 0$ for $0<\varphi<1$. The zeros of the function $f(\varphi)$ are trivial solutions (equilibria) of equation (1.1). Under conditions (1.2) one of them, $\varphi\equiv 0$, is dynamically stable, while the other, $\varphi\equiv 1$, is unstable.

A traveling wave is understood as a function depending on a single variable $\Phi(x-S(t))$. A solution to equation (1.1) in the form of a wave with constant velocity $V=S'(t)=\mathrm{const}>0$ is found by the ordinary differential equation

$$(\delta V^2-1)\Phi''-V\Phi'+f(\Phi)=0.$$

The equation is supplemented by boundary conditions at infinity, which single out the solution with a phase trajectory connecting the fixed points $(0,0)$ and $(1,0)$ [10]. The type of the fixed points depends on the coefficient (δV^2-1). In the case $(\delta V^2-1)<0$ the traveling wave corresponds to a trajectory from the saddle $(0,0)$ to the stable node (or focus) $(1,0)$ [10], [3]. In particular, such a situation occurs for $\delta=0$, when (1.1) turns into the parabolic KPP equation (Kolmogorov, Petrovskii, Piskunov). Traveling waves of this type for the KPP equation have been well–studied: there are results on stability and on the asymptotic approach to them of other solutions as $t\to\infty$ [6], [13], [14], [10], [4], [5]. A different situation arises in the hyperbolic

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equation as $\delta > 0$. Here rigorous results on the asymptotics as $t \to \infty$ are absent. Numerical experiments indicated asymptotically such waves as $t \to \infty$ [12].

The value V at which the coefficient turns out to be zero, $(\delta V^2 - 1) = 0$, is sometimes called the sound velocity in the corresponding mathematical models. In the present work we analyse the stability of subsonic waves.

2. Existence

In the considered case, as $\delta V^2 - 1 < 0$, it is convenient to use the renormalized variable

$$s = \frac{x - Vt}{\sqrt{1 - \delta V^2}}.$$

Then the equation for traveling wave equation becomes

$$\frac{d^2\Phi}{ds^2} + \gamma \frac{d\Phi}{ds} - f(\Phi) = 0. \tag{2.1}$$

The equation is supplemented by conditions of stabilization of the solution to the equilibria:

$$\Phi(s) \to 0 \qquad \text{as} \quad s \to -\infty, \qquad \Phi(s) \to 1 \quad \text{as} \quad s \to +\infty. \tag{2.2}$$

Then the wave describes the dynamics of replacement of the unstable equilibrium $\varphi \equiv 1$ by the stable one $\varphi \equiv 0$ ([10], [3]). The coefficient $\gamma > 0$ is related with the velocity by the relation $\gamma = \dfrac{V}{\sqrt{1 - \delta V^2}}$. The subsonic wave condition is equivalent to

$$0 < \gamma < \infty. \tag{2.3}$$

In vicinity of the fixed points the structure of phase portrait for equation (2.1) is determined by the linearized equation

$$\frac{d^2\phi}{ds^2} + \gamma \frac{d\phi}{ds} - f'\phi = 0.$$

Here $f' = f'(0)$, or $f' = f'(1)$. We constructe the solution in the form $\phi = \exp(\lambda\, s)$, which leads to the characteristic equation and determines λ

$$\lambda_0^{\pm} = \frac{1}{2}\Big(-\gamma \pm \sqrt{\gamma^2 + 4f'(0)}\Big), \qquad \lambda_1^{\pm} = \frac{1}{2}\Big(-\gamma \pm \sqrt{\gamma^2 + 4f'(1)}\Big).$$

Since $f'(0) > 0$, the exponents $\lambda_0^- < 0 < \lambda_0^+$ have different signs, and the point $(0, 0)$ is hence a saddle. The type of fixed point $(1, 0)$ depends on two parameters, γ and $f'(1)$. Under the condition $\gamma^2 \geqslant -4f'(1)$ this point is a stable node. The quantity $\gamma_* = 2\sqrt{-f'(1)}$ determines the minimal (critical) value of the parameter γ, at which the point $(1, 0)$ remains a node. For smaller values $\gamma < \gamma_*$ the point $(1, 0)$ is a focus.

Obviously, the solvability of boundary value problem (2.1), (2.2) is equivalent to the existence of a phase trajectory connecting the fixed points.

Lemma 2.1. *For each smooth function $f(\varphi)$ obeying the conditions (1.2), (2.3) there exists a minimal value $\gamma_0 \geqslant \gamma_*$ among those γ for which boundary value problem (2.2) for equation (2.1) has a monotone solution $\Phi(s)$ corresponding to a trajectory from the saddle point $(0, 0)$ to the node $(1, 0)$.*

The proof reduces to finding the minimal value of the coefficient γ, at which the separatrix from the saddle point $(0, 0)$ enters the node $(1, 0)$ as $s \to +\infty$ monotonically (without going around the point $(1, 0)$). A formal proof is obtained by analyzing the vector field; it has been reproduced many times and is presented, for example, in [13], [5].

The mathematical formulations of the original problems are usually restricted to specifying the function $f(\varphi)$ on the interval between the equilibria $\varphi \in [0, 1]$, and therefore non–monotone

waves, the phase trajectory of which goes around the equilibrium point, were not discussed [6], [13]. Moreover, a non–monotone wave is considered unstable and hence physically unrealizable. However, in the context of the problems discussed below such a wave is not distinguished in any way, and the notion of stability should be refined in the direction discussed in [10]. Therefore, we must first point out the possibility of the existence of a non–monotone wave.

If $\gamma < \gamma_0$, then the point $(1,0)$ remains a node for $\gamma_* \leqslant \gamma < \gamma_0$, or becomes a focus for $\gamma < \gamma_*$. In any case, for $\gamma < \gamma_0$ the trajectory goes around the point $(1,0)$, and the solution is non–monotone. Moreover, for small γ the separatrix from the saddle may not enter $(1,0)$ at all, so that boundary value problem (2.2) turns out to be unsolvable. This property is determined by the global behaviour of the right hand side $f(\varphi)$, and no effective conditions for the solvability of the boundary value problem for a nonlinear equation are known. However, if γ is close to γ_0, then the separatrix from the saddle point $(0,0)$ still enters the point $(1,0)$, even if not monotonically (going around the point). This fact follows from the continuity of the separatrix with respect to the parameter γ. One of the ways of a formal proof is based on constructing a Lyapunov function in a neighbourhood of the point $(1,0)$. The main goal of the following assertion is to show that boundary value problem (2.1), (2.2) has non–monotone solutions for some values of the parameter γ.

Theorem 2.1. *For each smooth function $f(\varphi)$ with conditions (1.2) there exists a value $\gamma_1 < \gamma_0$ such that for all $\gamma \in (\gamma_1, \gamma_0)$ boundary value problem (2.2) for equation (2.1) has a (non–monotone) solution $\Phi(s) = \Phi(s;\gamma)$, which corresponds to a trajectory from the saddle to the node (or focus).*

Proof. As the Lyapunov function one can use the first integral of the non–dissipative equation

$$J(\dot\Phi, \Phi) = \frac{1}{2}\dot\Phi^2 - \int\limits_0^\Phi f(\phi)\, d\phi.$$

Along the trajectory of any solution the identity

$$\frac{d}{ds}\Big(\frac{1}{2}(\dot\Phi(s))^2 - \int\limits_{s_0}^{s} f(\Phi(\zeta))\dot\Phi(\zeta)\, d\zeta\Big) = -\gamma(\dot\Phi(s))^2.$$

holds; this identity is implied by the initial equation. Therefore, as the point $(\Phi(s), \dot\Phi(s))$ moves in the direction $s \to +\infty$, the first integral along the trajectory does not increase

$$\frac{d}{ds} J(\dot\Phi, \Phi) = -\gamma(\dot\Phi(s))^2 \leqslant 0.$$

In a sufficiently small neighbourhood $S_R = \{(1+\Phi, \dot\Phi) :\ \dot\Phi^2 + \Phi^2 \leqslant R\}$ we have

$$f(\phi) = f'(1)\phi + \mathcal{O}(\phi^2), \quad f'(1) < 0.$$

Therefore, the integral $J(\dot\Phi, \Phi)$ is bounded from above and below by a quadratic form

$$m(\dot\Phi^2 + \Phi^2) < J(\dot\Phi, \Phi) < M(\dot\Phi^2 + \Phi^2)$$

with constants $0 < m < M < \infty$, which depend on R, the boundary of the neighbourhood. If the quantity R is not too large, then such a neighbourhood contains only one fixed point $(1,0)$. Let us choose a smaller neighbourhood $\dot\Phi^2 + \Phi^2 \leqslant m\,R/M$. On its boundary the first integral satisfies the estimate $J(\dot\Phi, \Phi) < mR$. If at some time s_0 the trajectory crosses the boundary of this neighbourhood, then at subsequent times $s > s_0$ this estimate is preserved by the monotonicity along the trajectory. The lower bound yields $(\Phi^2 + \dot\Phi^2) < R$. Thus, the half–trajectory $(\Phi(s), \dot\Phi(s))$ turns out to be located in the bounded domain S_R for all times $s > s_0$. Equation (2.1) has no limit cycles by the Bendixson criterion. Since the closed domain

S_R contains one fixed point, and it is stable as $s \to +\infty$, the trajectory enters this point as $s \to +\infty$ [8]. Let us show that for $\gamma > \gamma_0$ the separatrix leaving the saddle point $(0,0)$ possesses this property. On the phase plane with coordinates $x = \Phi$, $y = \dot\Phi$ the phase trajectories of autonomous equation (2.1) are tangent to the vector field

$$\dot x = y, \qquad \dot y = f(x) - \gamma y.$$

We consider the family of separatrix trajectories which depend on $\gamma > 0$ and go from the saddle point $(0,0)$ into the upper half–plane. Let us write them in the form of integral curves $y = Y(x;\gamma)$ of the corresponding equation:

$$\frac{dy}{dx} = f(x) - \gamma.$$

The initial point $(0,0)$ is a singular point. Nevertheless, the function $Y(x;\gamma)$ is continuous in the parameter γ, since at the initial point the vector $(1, \lambda_0^+(\gamma))$ tangent to the separatrix is continuous in γ.

For $\gamma = \gamma_0$ the separatrix enters the point $(1,0)$, and therefore crosses the boundary $\Phi^2 + \dot\Phi^2 = r$. Due to the continuous dependence on the parameter γ, this boundary is crossed by the family of separatrices for close values of the parameter: $\gamma \in (\gamma_1, \gamma_0]$, $\gamma_1 = \text{const} < \gamma_0$. The proof is complete. □

Remark 2.1. *The proved statements on the existence of γ_1 and of the boundary value of the parameter γ_0 contain no recipes for computing these quantities. In problems with specific nonlinear functions $f(\varphi)$ the value γ_0 is computed approximately, as it was demonstrated in* [5].

The results obtained for ordinary differential equation (2.1) can be interpreted for the traveling wave in equation (1.1).

Proposition 2.1. *For each smooth function $f(\varphi)$ with conditions* (1.2) *and for each sufficiently large velocity, for which $\gamma > \gamma_1$, there exists a subsonic wave $\Phi(s)$ running from the stable equilibrium $\varphi \equiv 0$ to the unstable one $\varphi \equiv 1$.*

The asymptotics of this wave at the leading and trailing edges corresponds to the structure of the fixed points and is described by exponentials with the exponents $\lambda_0^\pm$, $\lambda_1^\pm$, which are roots of the characteristic equation [8]:

$$\Phi(s) = \begin{cases} 1 + \big(c_1^+ \exp(\lambda_1^+ s) + c_1^- \exp(\lambda_1^- s)\big)\big(1 + o(1)\big), & s \to +\infty, \\ c_0^+ \exp(\lambda_0^+ s)\big(1 + \mathcal{O}\big(\exp(\lambda_1^+ s)\big)\big), & s \to -\infty, \quad \lambda_1^+ > 0, \end{cases} \qquad c_{0,1}^\pm = \text{const}\,.$$

The value of the parameter γ may turn out to be either above or below the critical γ_*, which affects the structure of the asymptotics as $s \to +\infty$. If $\gamma < \gamma_*$ (the focus case), the imaginary parts $\operatorname{Im}\lambda_1^\pm \neq 0$, and the asymptotics contain oscillating functions. In the exceptional case $\gamma = \gamma_*$ the exponents $\lambda_1^+ = \lambda_1^- < 0$ coincide and a linear factor s appears in one of the exponentials. Such situations are excluded for $\gamma > \gamma_0$.

In what follows the traveling wave $\Phi(s)$ is considered as a solution to boundary value problem (2.1), (2.2) for the parameter $\gamma \in (\gamma_1, \infty)$. Under conditions (1.2) this wave corresponds to a trajectory from the saddle to the node (or focus). The monotonicity for the wave is not assumed.

3. Stability in linear approximation

The central question of the theory of traveling waves is to find out the conditions on the original initial boundary value problem under which the exact solution approaches the wave in the asymptotics as $t \to \infty$. For the KPP equation this question is considered to be solved [13].

For the hyperbolic equation there are no similar results. This problem can be attempted to be solved in another way, widely used in applied research.

The stability of the wave in the linear approximation is understood as the stability in time t (in the sense of Lyapunov) of the exact solution $\varphi(x,t) \equiv \Phi(s)$ of the initial boundary value problem for the original partial differential equation. The linear approximation means that while analyzing the perturbed solution

$$\varphi(x,t) = \Phi(s) + \varepsilon u(x,t) + \mathcal{O}(\varepsilon^2), \quad \varepsilon \to 0, \qquad s = \frac{(x - Vt)}{\sqrt{\delta V^2 - 1}},$$

the correction $u(x,t)$ is determined from the linearized equation

$$\delta \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + f'(\Phi(s))u = 0, \tag{3.1}$$

with zero conditions at infinity

$$u(x,t) \to 0, \quad x \to \pm\infty \tag{3.2}$$

and with nonzero initial data

$$u(x,t)|_{t=0} = u_0(x), \qquad \partial_t u(x,t)|_{t=0} = u_1(x). \tag{3.3}$$

Statement of the problem. *Indicate the class of initial data $u_0(x)$, $u_1(x)$ (initial perturbations), for which the solution of the linearized problem tends to zero $u(x,t) \to 0$ as $t \to \infty$ uniformly in $x \in \mathbb{R}$.*

Below we provide a set of solutions to equation (3.1) with homogeneous boundary conditions (3.2). These solutions, taken at $t = 0$, determine the class of initial data. Here the pair of functions $u_0(x)$, $u_1(x)$ turns out to be consistent, as is the case for the solution of a hyperbolic equation when a wave in one direction is singled out.

In the present work the first (formal) stage of the stability analysis is considered. To obtain results comparable with the known results for the KPP equation, the second stage will require estimates of the difference $\varphi(x,t) - \Phi(s) - \varepsilon u(x,t)$, as is done in the theory of Lyapunov stability. The second stage is beyond the scope of this article.

3.1. Passage to spectral problem. Bearing in mind the method of separation of variables, in the formal approach it suffices to analyze corrections of a special form in the traveling wave variable

$$u(x,t;\mu) = \phi(s;\mu)\exp(\mu t), \quad \mu = \text{const} \in \mathbb{C}, \qquad s = \frac{x - Vt}{\sqrt{1 - \delta V^2}}.$$

In this way, we obtains the differential equation for the function $\phi(s;\mu)$

$$\frac{d^2\phi}{ds^2} + \gamma(1 + 2\delta\mu)\frac{d\phi}{ds} - (f'(\Phi(s)) + \delta\,\mu^2 + \mu)\phi = 0, \qquad \gamma = \frac{V}{\sqrt{1 - \delta V^2}} \tag{3.4}$$

with homogeneous conditions at infinity

$$\phi(s;\mu) \to 0, \quad s \to \pm\infty. \tag{3.5}$$

The resulting spectral problem[1] is considered for the parameter $\mu \in \mathbb{C}$. Asymptotic stability is associated with the exponential decay of the correction $u(x,t;\mu)$ as $t \to \infty$, which corresponds to the requirement $\mathrm{Re}(\mu) < 0$. Such a restriction on the parameter μ singles out a certain class of functions $\phi(s;\mu)$ with respect to which the wave $\Phi(s)$ under consideration is stable. It should be borne in mind that the set of solutions $\phi(s;\mu)$ singled out in this way depends on the requirements on the stabilization rate as $s \to \pm\infty$. An increase in the rate of stabilization leads to a narrowing of the class of stable perturbations. On the other hand, a weakening of this requirement leads not only to an expansion of the set of stable perturbations, but also to

[1]Such problems are sometimes associated with the notion of the operator pencil.

the appearance of unstable perturbations. Such properties of the spectral problem (3.4), (3.5) are due to its non–self–adjointness.

Equation (3.4) has two linearly independent solutions for any complex constant μ. The structure of these solutions in the asymptotics at infinity depends on μ and is determined by the roots of the characteristic equations near the equilibria [9]:

$$\phi(s;\mu)\approx\exp(\lambda_0^{\pm}(\mu)s),\quad s\to-\infty,\qquad \phi(s;\mu)\approx\exp(\lambda_1^{\pm}(\mu)s),\quad s\to+\infty.$$

The exponents $\lambda_{0,1}^{\pm}(\mu)$ are determined by the equations

$$\lambda^2+\gamma(1+2\delta\mu)\lambda-(f'+\delta\,\mu^2+\mu)=0;\qquad f'=f'(0),\quad \text{or}\quad f'=f'(1). \tag{3.6}$$

The corresponding formula

$$\lambda=-\frac{1}{2}\gamma(1+2\delta\mu)\pm\frac{1}{2}\sqrt{\gamma^2(1+2\delta\mu)^2+4(\delta\,\mu^2+\mu)+4f'}$$

reduces to

$$\begin{aligned}\lambda_0^{\pm}(\mu)&=-\frac{1}{2}\gamma(1+2\delta\mu)\pm\frac{1}{2}\sqrt{(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(0)-\frac{1}{\delta}\Big)},\quad f'(0)>0,\\ \lambda_1^{\pm}(\mu)&=-\frac{1}{2}\gamma(1+2\delta\mu)\pm\frac{1}{2}\sqrt{(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(1)-\frac{1}{\delta}\Big)},\quad f'(1)<0.\end{aligned} \tag{3.7}$$

The exponents $\lambda_{0,1}^{\pm}(\mu)$ depend on the parameters of the original equation δ, $f'(0)$, $f'(1)$ and on the wave parameter γ.

Examples of graphs of the functions $\lambda=\lambda_{0,1}^{\pm}(\mu)$ on the real axis $\mu\in\mathbb{R}$ for different $f'(0)$, $f'(1)$ are given in Figure 1. The segments of the straight line $\lambda=1+2\delta\mu$ connect the points of multiple roots and correspond to the graphs of the real part $\operatorname{Re}\lambda_{0,1}^{\pm}(\mu)$ in the case of complex exponents. The zeros of these functions $\mu_1^-<\mu_0^-<\mu_*<\mu_0^+<\mu_1^+$ are marked by dots.

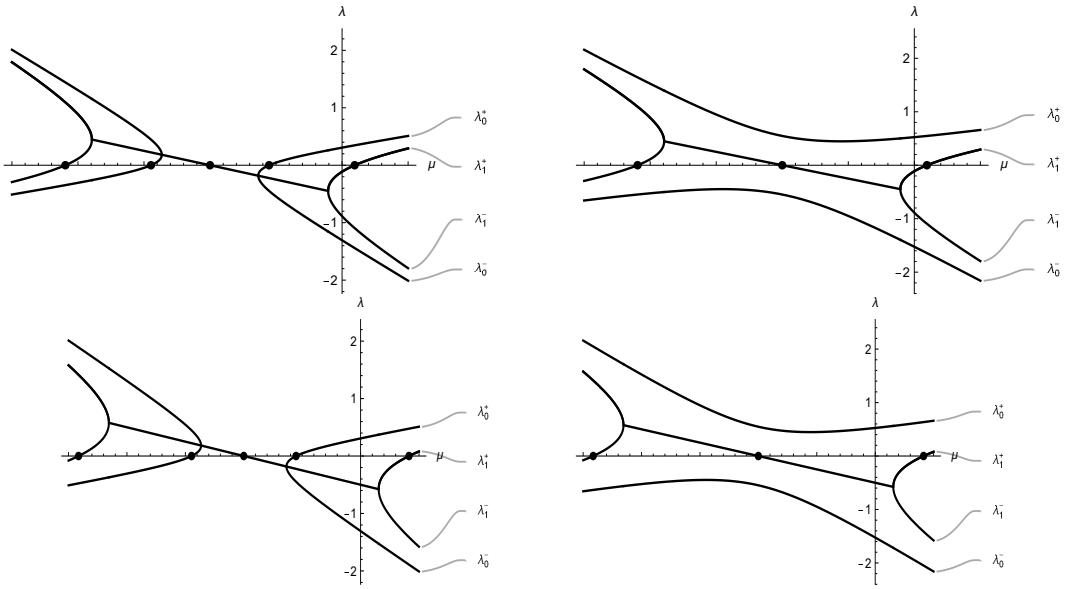


FIGURE 1. Variants of the graphs of the exponents $\lambda_{0,1}^{\pm}(\mu)$ as functions of the real parameter $\mu\in\mathbb{R}$

3.2. Point spectrum on the real axis. Let us consider problem (3.4), (3.5) on the real axis $\mu\in\mathbb{R}$. A suitable solution is fixed by the decaying asymptotics as $s\to-\infty$. If the largest exponent $\lambda_0^+(\mu)>0$ is positive, then the boundary condition at minus infinity allows one to single out (up to a factor) one of the two independent solutions exponentially decaying

as $s \to -\infty$; let us denote it by $\phi_-(s;\mu)$. This function is expressed as a linear combination of the solutions with exponents $\lambda_1^{\pm}(\mu)$ that determine the asymptotics at the other infinity.

The necessary condition for the solvability of spectral problem (3.4), (3.5) is formulated in the form of restrictions on the signs of the exponents: *at least one of* $\operatorname{Re}\lambda_0^{\pm}(\mu) > 0$ *and at least one of* $\operatorname{Re}\lambda_1^{\pm}(\mu) < 0$. By the graphs we see easily that on the real axis there exist intervals for the spectral parameter $\mu \in \mathbb{R}$, on which the problem is either unconditionally solvable or unconditionally unsolvable. The boundaries of such intervals are determined by the zeros of the functions $\lambda_{0,1}^{\pm}(\mu)$, arranged successively $\mu_1^- < \mu_0^- < \mu_0^+ < \mu_1^+$, and by the zero of their real parts $\mu_* = -\frac{1}{2}\delta$. For example, in the case when there exists a zero of the function $\lambda_0^+(\mu)$, the spectrum fills the interval $\mu_0^+ < \mu < \mu_1^+$. If $\lambda_0^+(\mu) \neq 0$, then the spectrum fills the interval $\mu_* < \mu < \mu_1^+$. The corresponding eigenfunctions decay exponentially as $s \to +\infty$ with exponent $\operatorname{Re}\lambda_1^+(\mu) < 0$, or $\operatorname{Re}\lambda_1^-(\mu) < 0$. There are no points of the spectrum in the interval $\mu_1^- < \mu < \mu_*$, since here $\operatorname{Re}\lambda_1^{\pm}(\mu) > 0$ and each solution grows as $s \to +\infty$.

Conclusion: The function $\phi_-(s;\mu)\exp(\mu t)$ is a solution of the boundary value problem for the correction in linear approximation (3.1), (3.2). Linear combinations of solutions (or integrals) over all possible $\mu \in \mathcal{L} \subset (\mu_0^+, \mu_1^+)$ inside the interval possess the same property. If we restrict ourselves to the interval on the left half axis $\mu \in \mathcal{L} \subset (\mu_0^+, 0)$, then such combinations with weight functions $\hat{u}_0(\mu)$

$$u(s,t) = \int\limits_{\mathcal{L}} \hat{u}_0(\mu)\phi_-(s;\mu)\exp(\mu t)\,d\mu$$

and their derivatives, taken at $t = 0$, determine the class of initial perturbations with respect to which the wave is stable. At the same time, unstable perturbations $\phi_-(s;\mu)\exp(\mu t)$ always exist for the parameter $\mu \in (0, \mu_1^+)$.

3.3. Version with self–adjoint operator. There exists another approach to spectral problem (3.4), (3.5). The change of function

$$\phi(s;\mu) = \tilde{\phi}(s;\mu)\exp\Big(-\frac{\gamma}{2}(1+2\delta\mu)s\Big)$$

allows one to eliminate the first derivative from the differential equation (3.4). For the new function with the condition $\tilde{\phi}(s;\mu) \to 0$, $s \to \pm\infty$, one obtains a self-adjoint problem, the spectrum of which is real and semibounded. However, the resulting spectral problem is not equivalent to original one (3.4), (3.5). While passing to $\tilde{\phi}(s;\mu)$, we exclude the solutions $\phi_+(s;\mu)$, which decay as $s \to +\infty$ not too rapidly, namely with the exponent $\lambda_1^+(\mu)$. The set of stable perturbations found in this way turns out to be poor. It consists of rapidly decaying solutions (with exponent $\lambda_1^-(\mu)$) which exist for exceptional (eigen) values of the parameter μ and correspond to the eigenfunctions of the self–adjoint problem. Such isolated points of the discrete spectrum may exist on the real axis to the left of zero $\mu < \mu_1^+$; for example, for $\mu = 0$ there is an eigenfunction $\phi(s;0) = \Phi'(s)$. If the wave is monotone, then there is no spectrum on the right half axis $\mu > 0$, and consequently there is never instability with respect to rapidly decaying perturbations [5].

Thus, the reduction of spectral problem (3.4), (3.5) to the self–adjoint form narrows the class of stable perturbations. If, however, one admits solutions $\phi(s;\mu)$ decaying not so rapidly as $s \to +\infty$, then spectral problem (3.4), (3.5) remains non–self–adjoint. Exactly this problem should be employed for identifying a wider class of initial perturbations with respect to which the wave $\Phi(s)$ is stable. Indications of such stability for the KPP equation can be seen in the analytical results [13], and for the hyperbolic equation in the numerical experiments [12]. This problem was discussed in [10].

3.4. Point spectrum in the complex plane. The class of initial perturbations can be extended by adding the solutions $\phi_-(s;\mu)\exp(\mu t)$ with complex values of μ. In this case, for the unambiguous identification of the exponents $\lambda^{\pm}_{0,1}(\mu)$ in the form of formulas (3.7) (as roots of the algebraic equation (3.6)) on the complex plane, one should indicate the cuts that connect the branch points. The coordinates of the branch points $\mu^{\pm}_{0,1}$ on the complex plane correspond to the zeros of the radicand in (3.7):

$$\hat{\mu}^{\pm}_{0,1} = \frac{1}{2\delta}\left(-1 \pm \sqrt{\frac{1-4\delta f'}{\gamma^2\delta+1}}\right). \tag{3.8}$$

Here $f' = f'(0) > 0$, or $f' = f'(1) < 0$. If $4f'(0)\delta < 1$, then both pairs of points $\hat{\mu}^{\pm}_{0,1}$ are located on the real axis. If $4f'(0)\delta > 1$, then the pair of points

$$\hat{\mu}^{\pm}_0 = -\frac{1}{2}\delta \pm i\,\mu_*, \qquad \mu_* = \frac{1}{2\delta}\sqrt{\frac{4\delta f'(0)-1}{\gamma^2\delta+1}}$$

is located on the line Γ_*, which plays an important role in what follows

$$\Gamma_* = \Big\{\mu\in\mathbb{C}:\ \operatorname{Re}\mu = -\frac{1}{2}\delta\Big\}.$$

It is convenient to draw the cuts through the point at infinity in the form of two pairs of rays on these lines.

Now the exponential asymptotics as $s\to\pm\infty$ is characterized by the real part of the corresponding pair of exponents $\operatorname{Re}\lambda^{\pm}_{0,1}(\mu)$. The condition of definite sign of these functions singles out domains on the complex plane

$$D^{\pm}_0 = \{\mu\in\mathbb{C}:\ \operatorname{Re}\lambda^{\pm}_0(\mu) > 0\}, \qquad D^{\pm}_1 = \{\mu\in\mathbb{C}:\ \operatorname{Re}\lambda^{\pm}_1(\mu) < 0\}.$$

They depend on the parameters of the equation δ, $f'(0)$, $f'(1)$ and on the wave parameter γ. The intersection of the domains $D^+_0\bigcap D^+_1$ determines the point spectrum of the non–self–adjoint problem. It is not difficult to obtain the description of the boundary of such a domain in an analytic form.

Let us consider the more general problem on finding the lines of zeros of the real parts of the exponents on the complex plane

$$\Gamma_0 = \{\mu\in\mathbb{C}:\ \operatorname{Re}\lambda^{\pm}_0(\mu) = 0\}, \qquad \Gamma_1 = \{\mu\in\mathbb{C}:\ \operatorname{Re}\lambda^{\pm}_1(\mu) = 0\}.$$

Lemma 3.1. *Let conditions (1.2) and (2.3) be satisfied. If $4\delta f'(0) < 1$, then in the complex plane $\mu\in\mathbb{C}$ each line of zeros consists of two components: $\Gamma_{0,1} = \Gamma_*\cup\mathcal{E}_{0,1}$, including the line Γ_* and the ellipses*

$$\mathcal{E}_{0,1} = \Big\{\mu\in\mathbb{C}:\ \delta(\operatorname{Re}\mu)^2 + \Big(\delta+\frac{1}{\gamma^2}\Big)(\operatorname{Im}\mu)^2 + f' + \operatorname{Re}\mu = 0\Big\}.$$

In the case $4\delta f'(0)\geqslant 1$ the line of zeros Γ_0 changes: the ellipse $\mathcal{E}_0$ is absent, the line Γ_0 consists of two rays on the line Γ_* and passes along the chosen cuts

$$\Gamma_0 = \Big\{\mu\in\mathbb{C}:\ \operatorname{Re}\mu = -\frac{1}{2}\delta, \quad |\operatorname{Im}\mu|\geqslant\mu_*\Big\}.$$

Proof. The analytic description of the boundary is extracted from characteristic equation (3.6). This leads to a pair of equations in terms of the complex variable $\mu = x+iy$ and the imaginary part $\operatorname{Im}\lambda^{\pm}_{0,1}(\mu) = Y$ (not to be confused with the variables x, y, Y in Section 2):

$$Y^2 + 2\gamma\delta yY + f' + \delta(x^2-y^2) + x = 0, \qquad (\gamma Y - y)(1+2\delta x) = 0.$$

As we see, the second equation splits into two ones.

One part of the boundary is determined by the relation $\gamma Y = y$ and leads to the equation of the ellipse

$$\mathcal{E}_{0,1} = \Big\{ x + iy \in \mathbb{C} : \ \delta x^2 + \Big(\delta + \frac{1}{\gamma^2}\Big) y^2 + f' + x = 0 \Big\}.$$

The values of the derivatives are taken as $f' = f'(0) > 0$, or $f' = f'(1) < 0$. They determine two nested ellipses: the ellipse $\mathcal{E}_0$ corresponds to $\operatorname{Re}\lambda_0^{\pm}(\mu) = 0$ and is embedded into the ellipse $\mathcal{E}_1$, corresponding to the pair $\operatorname{Re}\lambda_1^{\pm}(\mu) = 0$. In each pair the branch of the ellipse (the right one $\mathcal{E}_{0,1}^{+}$ or the left one $\mathcal{E}_{0,1}^{-}$) intersects the real axis at the points $\mu_{0,1}^{\pm}$, which are the zeros of the corresponding exponent $\lambda_{0,1}^{\pm}(\mu)$ (see Fig. 2). Since the zeros are determined by the expressions

$$\frac{1}{2\delta}\big(-1 \pm \sqrt{1 - 4\delta f'}\,\big),$$

by comparing with (3.8) we see that for $4f'(0)\delta < 1$ the branch points lie on the real axis inside the ellipses. In the case $4f'(0)\delta = 1$ the ellipse $\mathcal{E}_0$ degenerates into a point and disappears for $4\delta f'(0) > 1$.

The second component of the boundary is singled out by the equation $(1 + 2\delta x) = 0$, which determines the line $\Gamma_* : \ x = -\dfrac{1}{2}\delta$, $y \in \mathbb{R}$, parallel to the imaginary axis. Here the remaining quadratic equation for Y is solvable in real numbers only under the condition

$$(\delta\gamma^2 - 1)y^2 + f' - \frac{1}{4\delta} \geqslant 0.$$

This relation for $f'(1) < 0$ and for $4\delta f'(0) < 1$ holds for all $y \in \mathbb{R}$. In the case $4f'(0)\delta > 1$ the restrictions arise

$$|y| > \sqrt{\frac{f'(0) - \frac{1}{4}\delta}{(\gamma^2\delta + 1)\delta}} = \frac{1}{2\delta}\sqrt{\frac{4\delta f'(0) - 1}{\gamma^2\delta + 1}},$$

which corresponds to singling out two rays with boundary at the branch points from the line. The proof is complete. □

Remark 3.1. *The interior of each ellipse $\mathcal{E}_{0,1}$ is a domain on the complex plane in which the real parts of the corresponding exponents $\operatorname{Re}\lambda_0^{\pm}(\mu)$, or $\operatorname{Re}\lambda_1^{\pm}(\mu)$, have different signs, see Fig. 2. The sign changes upon crossing the line Γ_*, which divides the ellipse into two parts. The sign of each of the functions is preserved on half of the corresponding ellipse, or in the remaining part of the plane. Outside the ellipse the sign does not change upon crossing the line Γ_*.*

The main object for the stability analysis is half of the interior of the ellipse $\mathcal{E}_1^{+}$ located in the half plane $\{\operatorname{Re}\mu > -\frac{1}{2}\delta\}$, in which $\operatorname{Re}\lambda_1^{+}(\mu) < 0$. The second domain consists of the exterior of the other half–ellipse $\mathcal{E}_0^{+}$, where $\operatorname{Re}\lambda_0^{+}(\mu) > 0$, see Fig. 3. The intersection of these domains represents the layer between the two half–ellipses, if $4\delta f'(0) < 1$, see Fig. 4. In the case $4\delta f'(0) > 1$ this domain coincides with the interior of the half–ellipse $\mathcal{E}_1^{+}$.

Remark 3.2. *In the case of the KPP equation, when $\delta = 0$, the lines $\Gamma_{0,1}$ consist of parabolas*

$$\frac{1}{\gamma^2}(\operatorname{Im}\mu)^2 + f' + \operatorname{Re}\mu = 0.$$

For a formal proof of the solvability of the spectral problem it is necessary to establish some relations between the real parts of different exponents. These relations are established in the general case on the complex plane $\mu \in \mathbb{C}$. They are analogous to those indicated by the graphs in Fig. 1 for the case of real μ.

Lemma 3.2. *Let conditions (1.2) and (2.3) be satisfied. The exponents possess the property $\operatorname{Re}\lambda_{0,1}^{+}(\mu) \geqslant \operatorname{Re}\lambda_{0,1}^{-}(\mu)$ everywhere in the complex plane $\mu \in \mathbb{C}$.*

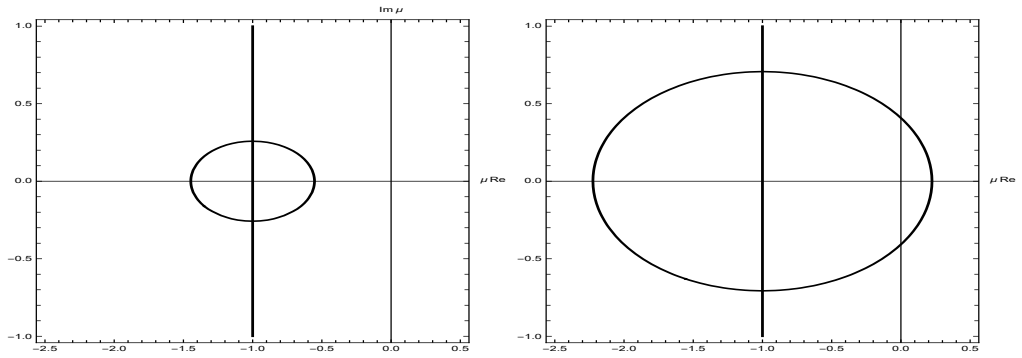


FIGURE 2. Lines of zeros of the real parts of exponents $\operatorname{Re}\lambda_0^{\pm}(\mu)$, and $\operatorname{Re}\lambda_1^{\pm}(\mu)$ on the complex plane $\mu \in \mathbb{C}$ in the case $4\delta f'(0) < 1$

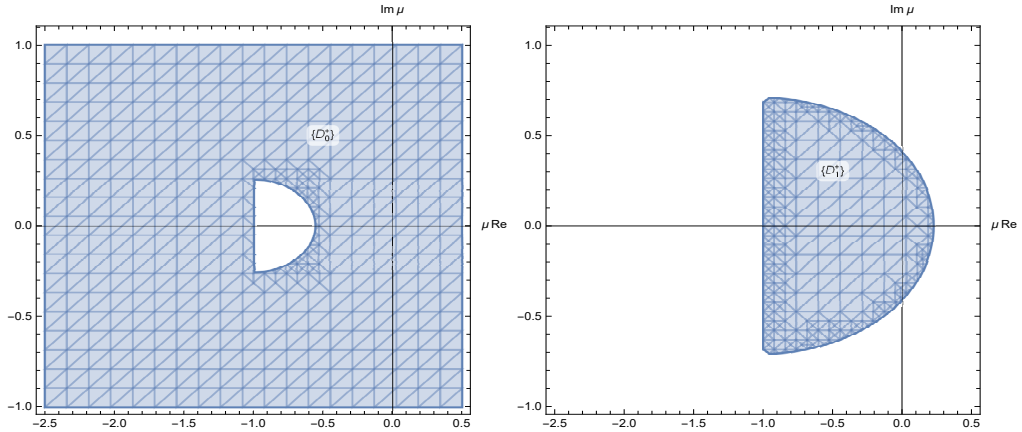


FIGURE 3. Domains of definite sign of the real part of exponents: $\operatorname{Re}\mu_1^{+}(\mu) < 0$, $\operatorname{Re}\mu_0^{+}(\mu) > 0$ in the complex plane $\mu \in \mathbb{C}$

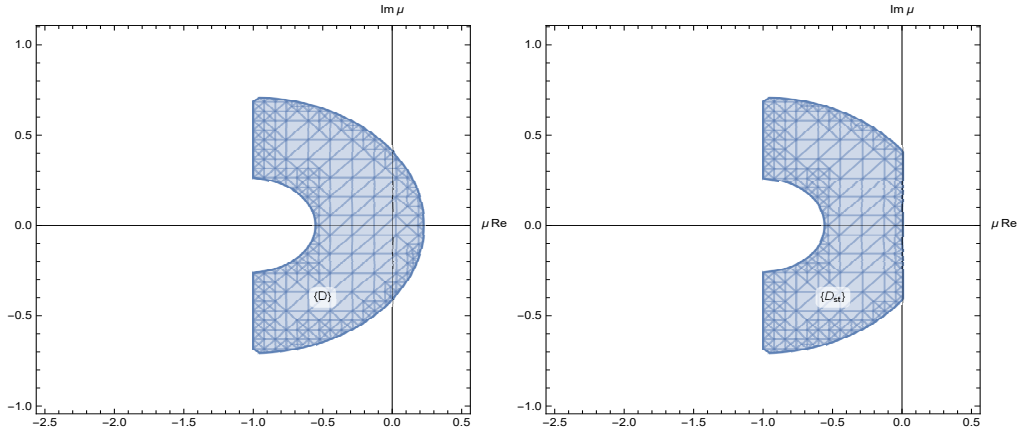


FIGURE 4. The domain of the point spectrum and the domain of spectrum of stable perturbations in the complex plane $\mu \in \mathbb{C}$

Proof. For $\mu \in \mathbb{R}$ on the real axis the inequality is obvious in view of formulas (3.7). In the remaining part of the complex plane, for the difference of the exponents we have the expression

$$\lambda_{0,1}^{+}(\mu) - \lambda_{0,1}^{-}(\mu) = \sqrt{(1+2\delta\mu)^2\Big(\gamma^2 + \frac{1}{\delta}\Big) + \Big(4f'(0) - \frac{1}{\delta}\Big)}.$$

Suppose that for some $\mu \in \mathbb{C}$ the real part of the difference vanishes $\operatorname{Re}(\lambda_{0,1}^{+}(\mu) - \lambda_{0,1}^{-}(\mu)) = 0$. Then this difference is a purely imaginary quantity. This is possible if the radicand is a real

negative quantity:

$$(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(0)-\frac{1}{\delta}\Big)<0.$$

But this is possible only if $1+2\delta\,\mathrm{Re}\,\mu=0$, or if $\mathrm{Im}\,\mu=0$. However, the first relation corresponds to the boundary points $\Gamma_{0,1}$, at which $\mathrm{Re}\,\lambda^{\pm}_{0,1}(\mu)=0$. The second relation corresponds to the above considered case of the real axis. Thus, the inequality $\mathrm{Re}\,\lambda^{+}_{0,1}(\mu)\geqslant\mathrm{Re}\,\lambda^{-}_{0,1}(\mu)$, which holds on the real axis in Fig.1, is preserved in the domain $\mu\in D^{-}_{0}$. The proof is complete. □

Lemma 3.3. *Let conditions (1.2) and (2.3) be satisfied. Then everywhere in the complex plane $\mu\in\mathbb{C}$ the exponents possess the property:*

$$\mathrm{Re}\,\lambda^{+}_{1}(\mu)\leqslant\mathrm{Re}\,\lambda^{+}_{0}(\mu),\qquad \mathrm{Re}\,\lambda^{-}_{0}(\mu)\leqslant\mathrm{Re}\,\lambda^{-}_{1}(\mu).$$

Proof. The formulas for exponents (3.7) differ only in the real part of the radicand

$$(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(0)-\frac{1}{\delta}\Big)$$

because of the differences in the derivatives: $f'=f'(0)$ or $f'=f'(1)$. Since $f'(1)<0<f'(0)$, the corresponding inequality holds for the real parts. A similar inequality is preserved for the real part after calculating the square root (on a suitable branch)

$$\mathrm{Re}\,\sqrt{(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(1)-\frac{1}{\delta}\Big)}\leqslant\mathrm{Re}\,\sqrt{(1+2\delta\mu)^2\Big(\gamma^2+\frac{1}{\delta}\Big)+\Big(4f'(0)-\frac{1}{\delta}\Big)}.$$

Taking into consideration formulas (3.7), we complete the proof. □

Theorem 3.1. *Let conditions (1.2) and (2.3) be satisfied. If the parameter μ belongs to the nonempty intersection of the domains*

$$\mu\in D=D^{+}_{0}\cap D^{+}_{1},$$

then for equation (3.4) boundary value problem (3.5) is solvable. The domain D, hereinafter called the domain of the point spectrum, in the case $4\delta f'(0)<1$ is a layer enclosed between the two ellipses $\mathcal{E}_{0,1}$ in the half–plane $\mathrm{Re}(\mu)>-\frac{1}{2}\delta$. In the case $4\delta f'(0)\geqslant 1$ the ellipse $\mathcal{E}_0$ is absent.

Proof. The domain of the parameter μ under consideration corresponds to the condition $\mu\in D^{+}_{0}$. Therefore, $\mathrm{Re}\,\lambda^{+}_{0}(\mu)>0$. Moreover, μ belongs to the domain D^{+}_{1}. Hence, $\mathrm{Re}\,\lambda^{+}_{1}(\mu)<0$, see Fig. 3. Equation (3.4) has two linearly independent solutions which can be identified by the exponents $\lambda^{\pm}_{0,1}(\mu)$. For our purposes the solution determined by the asymptotics as $s\to-\infty$ with the exponent $\lambda^{+}_{0}(\mu)>0$ is suitable. Let us denote it by $\phi_{-}(s;\mu)$; it tends to zero as $s\to-\infty$. Such a solution can be represented as a linear combination of the solutions which are determined by the asymptotics at the positive infinity with the exponents $\lambda^{\pm}_{1}(\mu)$. Since the parameter μ belongs to the domain D^{+}_{1}, Fig. 3, then, in view of Lemma 3.3, the real part of both exponents $\mathrm{Re}\,\lambda^{\pm}_{1}(\mu)<0$ is negative. Consequently each of these solutions tends to zero as $s\to+\infty$. Therefore, their linear combination also tends to zero, and it is this that represents the function $\phi_{-}(s;\mu)\to 0$, $s\to+\infty$.

It can turn out that there exist values of μ, for which both exponents $\mathrm{Re}\,\lambda^{\pm}_{0}(\mu)>0$ are positive, see Fig.1. In such a case any solution tends to zero as $s\to-\infty$. However, by Lemmas 3.2 and 3.3

$$0<\mathrm{Re}\,\lambda^{-}_{0}(\mu)\leqslant\mathrm{Re}\,\lambda^{-}_{1}(\mu)\leqslant\mathrm{Re}\,\lambda^{+}_{1}(\mu)$$

both exponents in the asymptotics at the positive infinity are positive. Such a solution grows as $s\to+\infty$, and therefore these values of μ do not belong to the spectrum. The proof is complete. □

Remark 3.3. *In the case of the KPP equation, when $\delta = 0$, the domain of the point spectrum D is the layer between two parabolas*

$$-f'(0) < \frac{1}{\gamma^2}(\operatorname{Im}\mu)^2 + \operatorname{Re}\mu < -f'(1).$$

Remark 3.4. *For exceptional μ the asymptotics of the solution $\phi_-(s;\mu)$ as $s \to +\infty$ contains only the rapidly decaying exponential, whose exponent is $\lambda_1^-(\mu)$. In such a case the problem reduces to a self–adjoint one, and therefore μ must be a real eigenvalue.*

3.5. Continuous spectrum. If the spectral parameter lies on the boundary of the domain of point spectrum $\mu \in \partial D$, then for the solutions $\phi(s;\mu)$ of equation (3.4) do not satisfy the boundary conditions (3.5) as $s \to \pm\infty$. The reason for this is the vanishing of the real part of one of the exponents which ensured decay at infinity. For such $\mu \in \partial D$ we have either $\operatorname{Re}\lambda_0^+(\mu) = 0$, or $\operatorname{Re}\lambda_1^+(\mu) = 0$. Instead of the decay at infinity as $s \to \pm\infty$, oscillations are detected, due to the nonzero imaginary part of the exponents. Such a solution of linearized equation (3.1) does not satisfy boundary condition (3.2) and therefore is not associated with the stability of the original wave $\Phi(s)$. Nevertheless, the integral over the set of such functions (with respect to the parameter μ) may decay as $s \to \pm\infty$. Such an effect of averaging oscillating functions underlies the use of the Fourier transform. The corresponding set of the parameter μ is associated with the notion of the continuous spectrum.

In the considered problem, by the continuous spectrum we mean the set $\Gamma \subset \mathbb{C}$ which consists of the boundary of the domain of point spectrum ∂D and the rays adjacent to it on the line $\Gamma_* = \left\{\mu :\ \operatorname{Re}\mu = -\frac{1}{2}\delta\right\}$. This line can be defined by the relation

$$\Gamma = \{\mu \in \mathbb{C} :\ \operatorname{Re}\lambda_0^+(\mu) = 0\} \cup \{\mu \in \mathbb{C} :\ \operatorname{Re}\lambda_1^+(\mu) = 0\}.$$

It consists of the right branches of the ellipses (let us denote them by $\mathcal{E}_{0,1}^+$) and the line Γ_*, see Fig. 5. Recall that on this line the exponents possess the property $\operatorname{Re}\lambda_{0,1}^{\pm}(\mu) = 0$ if $4\delta f'(0) < 1$. In the case $4\delta f'(0) \geqslant 1$ there is an interval $I_* = \{\mu \in \Gamma_* : |\operatorname{Im}\mu| < \mu_*\}$ on which $\operatorname{Re}\lambda_0^+(\mu) > 0$

$$\mu_* = \frac{1}{2\delta}\sqrt{\frac{4\delta f'(0) - 1}{\gamma^2\delta + 1}}.$$

On different components of the continuous spectrum spectral problem (3.4), (3.5) has solutions with different asymptotics at infinity.

Theorem 3.2. *Let conditions (1.2) and (2.3) be satisfied.*

1. *In the case $4\delta f'(0) < 1$ for $\mu \in \Gamma_*$ equation (3.4) has two linearly independent solutions $\phi_\pm(s;\mu)$, each of which oscillates at both infinities $s \to \pm\infty$.*
2. *In the case $4\delta f'(0) \geqslant 1$ for μ on the interval $|\operatorname{Im}\mu| < \mu_*$ there exists at most one bounded solution, which decays exponentially as $s \to -\infty$ and oscillates as $s \to +\infty$.*
3. *On the remaining part of the continuous spectrum (on the right branches of the ellipses) for $\mu \in \mathcal{E}_{0,1}^+$ there exists one bounded solution which oscillates at one infinity (as $s \to \mp\infty$), and decays exponentially at the other infinity (as $s \to \pm\infty$).*
4. *Exponential decay at both infinities can occur only for $\mu = \mu_{0,1}^+$ on the real axis, at the zeros of the exponents $\lambda_{0,1}^+(\mu)$, when these points belong to the discrete spectrum of the problem reduced to the self–adjoint form.*

Proof. If $4\delta f'(0) < 1$, then for $\mu \in \Gamma_*$ all exponents $\lambda_{0,1}^{\pm}(\mu)$ are purely imaginary, since the line Γ_* is part of the lines of zeros $\Gamma_{0,1}$. Therefore any pair of linearly independent solutions of equation (3.4) oscillates at the infinities $s \to \pm\infty$.

In the case $4\delta f'(0) \geqslant 1$ the line of zeros Γ_0 changes. Now the interval I_* does not intersect Γ_0 and enters the domain of definite sign $I_* \subset D_0^+$. Therefore one of the exponents has $\operatorname{Re}\lambda_0^+(\mu) > 0$. The second exponent either has $\operatorname{Re}\lambda_0^-(\mu) < 0$, or $\operatorname{Re}\lambda_0^-(\mu) > 0$ and then the spectral problem is not solvable, as shown in the previous theorem. Therefore there exists no more than one linearly independent bounded solution $\phi_-(s;\mu)$ which decays exponentially as $s \to -\infty$. In the asymptotics at the other infinity, as $s \to +\infty$, the interval I_* remains on the line of zeros $I_* \subset \Gamma_* \subset \Gamma_1$, so that both exponents have $\operatorname{Re}\lambda_1^\pm(\mu) = 0$. Therefore the solution $\phi_-(s;\mu)$ oscillates as $s \to +\infty$.

A similar situation occurs on the right branch of the ellipse $\mu \in \mathcal{E}_1^+$. Here the solution $\phi_-(s;\mu)$ decays exponentially as $s \to -\infty$, since this branch lies in the domain D_0^+, where $\operatorname{Re}\lambda_0^+(\mu) > 0$. In the asymptotics at the other infinity one exponent has $\operatorname{Re}\lambda_1^+(\mu) = 0$, the second satisfies the estimate $\operatorname{Re}\lambda_1^+(\mu) \leqslant 0$ by Lemma 3.3. Therefore, in the general case the solution $\phi_-(s;\mu)$ oscillates as $s \to +\infty$. Exponential decay of this (or a linearly independent) solution as $s \to +\infty$ with exponent $\operatorname{Re}\lambda_1^-(\mu) < 0$ is possible for some μ. But then the problem reduces to the self–adjoint one, so that the parameter μ (taken on the ellipse) must be real $\mu = \mu_1^+ \in \mathbb{R}$ and coincide with an eigenvalue.

When the parameter belongs to the right branch of the other ellipse $\mu \in \mathcal{E}_0^+$, we choose the solution $\phi_-(s;\mu)$ oscillating as $s \to -\infty$. At the other infinity, as $s \to -\infty$, this solution decays exponentially since the ellipse $\mathcal{E}_0^+ \subset D_1^+$ lies in the domain where $\operatorname{Re}\lambda_1^+(\mu) < 0$. Just as in the previous item, for $\mu = \mu_0^+ \in \mathbb{R}$ an additional solution may exist if μ_0^+ is an eigenvalue of the problem reduced to the self–adjoint form. The proof is complete. □

Remark 3.5. *In the case of the KPP equation, when $\delta = 0$, the continuous spectrum consists of two parabolas*

$$\frac{1}{\gamma^2}(\operatorname{Im}\mu)^2 + \operatorname{Re}\mu + f' = 0; \qquad f' = f'(0), \quad \text{or} \quad f' = f'(1).$$

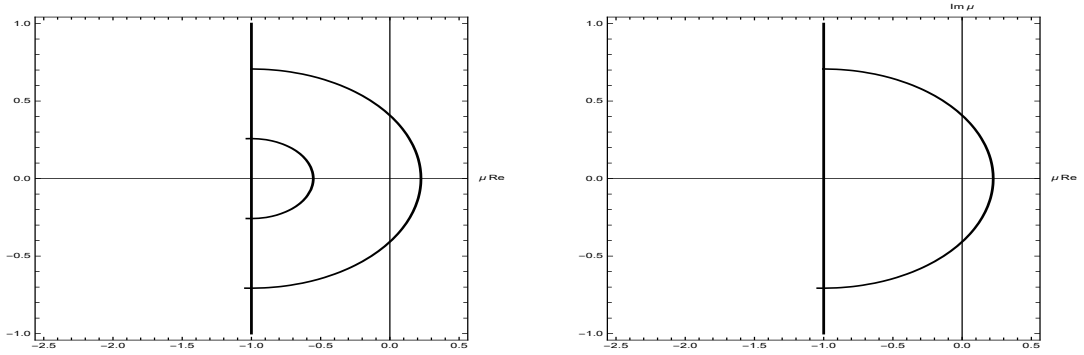


FIGURE 5. The line of the continuous spectrum on the complex domain $\mu \in \mathbb{C}$ in the case $4f'(0)\delta < 1$ and in the case $4f'(0)\delta \geqslant 1$

4. Application to stability of traveling wave

The function constructed byf the solution of spectral problem (3.4), (3.5)

$$u(x,t;\mu) = \phi_-(s;\mu)\exp(\mu t), \qquad \mu \in D, \qquad s = \frac{x - Vt}{\sqrt{\delta V^2 - 1}}$$

solves boundary value problem (3.1), (3.2) for the correction in the linear approximation. The solution is determined up to a multiplicative constant. If the parameter μ belongs to the subdomain of the point spectrum in the left half–plane $\mu \in D_{st} := D \cap \{\mu < 0\}$, Fig. 4, then

the correction is exponentially small as $t \to \infty$. This determines a particular set of initial perturbations (3.3) in the form of a pair of mutually consistent functions:

$$u_0(x;\mu) = \phi_-(x;\mu), \qquad u_1(x;\mu) = \mu\,\phi_-(x;\mu) - \gamma\,\partial_x\phi_-(x;\mu),$$

with respect to which the original wave $\Phi(s)$ is stable. These functions decay exponentially as $x \to \pm\infty$.

In view of the linearity of problem (3.1), (3.2), a linear combination of solutions taken at different μ possesses the same property. Such combinations are represented by series and integrals

$$u(x,t) = \sum_n \hat{u}(\mu_n)\,\phi_-(s;\mu_n)\exp(\mu_n\,t); \qquad u(x,t) = \int\limits_{\mathcal{L}} \hat{u}(\mu)\,\phi_-(s;\mu)\exp(\mu\,t)\,d\mu.$$

The coefficients $\hat{u}(\mu)$ are chosen sufficiently smooth to ensure the smoothness and decay at infinity of the resulting solutions. By narrowing the set $\mathcal{L}$ one can control the rate of decay as $t \to \infty$.

Thus, the initial perturbations with respect to which the wave is stable are described by series or integrals

$$u_0(x) = \int\limits_{\mathcal{L}} \hat{u}(\mu)\,\phi_-(x;\mu)\,d\mu, \qquad u_1(x) = \int\limits_{\mathcal{L}} \hat{u}(\mu)(\mu\,\phi_-(x;\mu) - \gamma\,\partial_x\phi_-(x;\mu))\,d\mu. \tag{4.1}$$

As $\mathcal{L}$ one can take a smooth line (or subdomain) in the stability domain of the point spectrum D_{st}.

Let us introduce the class of functions $\mathcal{N}_\gamma$ which have expansion (4.1). Note that both the original wave $\Phi(s)$ and the basis $\phi_-(s;\mu)$ depend on the parameter γ. The outcome of these constructions is the following statement.

Proposition 4.1. *Let conditions (1.2) and (2.3) be satisfied. Then the traveling wave with parameter γ is stable with respect to initial perturbations from the class $\mathcal{N}_\gamma$.*

Similar constructions are made on the continuous spectrum. Here the class of functions $\mathcal{M}_\gamma$ is introduced by integrals over the line Γ. As the base one takes the pair of linearly independent solutions of the spectral problem described in Theorem 3.2. One of these functions $\phi_-(s;\mu)$ is used on the whole line $\mu \in \Gamma$. The second function $\phi_+(s;\mu)$ is used only on the line Γ_*. The solution of the linearized problem (3.1), (3.2) is written through the integrals

$$u(x,t) = \int\limits_{\Gamma} \hat{u}^-(\mu)\phi_-(s;\mu)\exp(\mu t)\,d\mu + \int\limits_{\Gamma_*} \hat{u}^+(\mu)\phi_+(s;\mu)\exp(\mu t)\,d\mu. \tag{4.2}$$

with weight functions $\hat{u}^\pm(\mu)$. If the integral is taken over the part of the spectrum $\Gamma_{st} = \{\mu \in \Gamma:\ \operatorname{Re}\mu < 0\}$, Fig. 5, then the solution tends to zero as $t \to \infty$. The values at $t = 0$ determine the class of initial data $\mathcal{M}_\gamma$ with respect to which the wave is stable. For this class an analogue of Proposition 4.1 holds.

Remark 4.1. *The use of the basis $\phi_\pm(s;\mu)$ on the base of the solutions of equation (3.4) provides an exhaustive description of the set of initial perturbations with respect to which the traveling wave is stable. However, this description remains rather ineffective. This is related with the origination of the basis from the spectral problem for the non–self–adjoint equation. It cannot be used for inverting expansion formulas (4.2), unlike, for example, the Fourier expansion. For the functions in the classes $\mathcal{N}_\gamma$, $\mathcal{M}_\gamma$, there is no description in the usual terms of smoothness and asymptotics at infinity. Some idea of the class of stable perturbations can be obtained from the numerical experiments* [12].

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