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ON GROUND STATES ASYMPTOTICS FOR BOUNDARY VALUE PROBLEM FOR EQUATION $-\varepsilon\Delta_p u = a|u|^{q-2}u - b|u|^{\gamma-2}u$

Ya.Sh. IL'YASOV, E.I. TUR'YANOVA

Abstract. We study a singularly perturbed Dirichlet boundary value problem for the equation

$$-\varepsilon\Delta_p u = a(x)|u|^{q-2}u - b(x)|u|^{\gamma-2}u,$$

where Δ_p is the p-Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < q < \gamma < p^*$, $a, b \in L^\infty(\Omega)$, $a(x) \geqslant 0$, $b(x) \geqslant \sigma_b > 0$, and $\varepsilon > 0$ is a small parameter.

Using nonlinear Rayleigh quotient method, we introduce two critical values of the parameter, ε^* and ε_e^*, associated respectively with the Nehari manifold and the zero energy level. We prove that for $\varepsilon > \varepsilon^*$ the problem has no nontrivial weak solutions, while for $0 < \varepsilon < \varepsilon_e^*$ there exist at least two positive weak solutions; one of them is the ground state.

The main asymptotic result describes the behaviour of the ground states in the singular limit $\varepsilon \to 0^+$. Under the additional condition $a(x) \geqslant \sigma_a > 0$ we prove that each family of ground states u_ε converges in measure in Ω to the explicitly given limit function

$$\bar{u}_0(x) = \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{\gamma-q}}.$$

Moreover, $u_\varepsilon \to \bar{u}_0$ strongly in $L^r(\Omega)$, $1 \leqslant r < \gamma$, and $u_\varepsilon \rightharpoonup \bar{u}_0$ weakly in $L^\gamma(\Omega)$.

Keywords: Rayleigh quotient, Nehari manifold, ground state, singular perturbation, p-Laplacian.

Mathematics Subject Classification: 35J61, 35J20, 35J70, 35D30

1. Introduction

In this paper we consider the Dirichlet boundary value problem with a small parameter $\varepsilon > 0$

$$\begin{cases} -\varepsilon\Delta_p u = a(x)|u|^{q-2}u - b(x)|u|^{\gamma-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is the p-Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < q < \gamma < p^*$, where

$$p^* = \begin{cases} \dfrac{pN}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } p \geqslant N. \end{cases}$$

We suppose that

$$a, b \in L^\infty(\Omega), \qquad a(x) \geqslant 0, \qquad b(x) \geqslant \sigma_b > 0 \quad \text{for a.e. } x \in \Omega, \tag{1.2}$$

where $\sigma_b > 0$ is some constant. Moreover, in what follows we suppose that $a \not\equiv 0$.

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A function $u \in W_0^{1,p}(\Omega)$ is called the *weak solution* of problem (1.1) if it is a critical point of the energy functional

$$\Phi_\varepsilon(u) = \frac{\varepsilon}{p}\int\limits_\Omega |\nabla u|^p\,dx - \frac{1}{q}\int\limits_\Omega a|u|^q\,dx + \frac{1}{\gamma}\int\limits_\Omega b|u|^\gamma\,dx. \tag{1.3}$$

Here $W_0^{1,p} := W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$. A weak solution $u \in W_0^{1,p}(\Omega)$ is called the *ground state* of functional Φ_ε if

$$\Phi_\varepsilon(u) \leqslant \Phi_\varepsilon(w)$$

for each weak solution $w \in W_0^{1,p}(\Omega)$ to problem (1.1).

Problem (1.1) is singularly perturbed: as $\varepsilon \to 0^+$, in the limit the order of equation lessens, and this can produce boundary layers in a neighbourhood of $\partial\Omega$. Indeed, in the one–dimensional case for $p = 2$, $a = b = 1$ the problem becomes

$$\begin{cases} -\varepsilon u''(x) = u^{q-1}(x) - u^{\gamma-1}(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.4}$$

The formal method of matched asymptotic expansions (see, e.g., [1], [28]) leads to the approximation

$$u(x) \sim U\left(\frac{x}{\sqrt{\varepsilon}}\right) + U\left(\frac{1-x}{\sqrt{\varepsilon}}\right) - 1, \qquad 0 \leqslant x \leqslant 1,$$

where the boundary layer profile U is determined by the problem

$$-U'' = U^{q-1} - U^{\gamma-1}, \qquad U(0) = 0, \qquad U(+\infty) = 1.$$

This profile admits the integral representation

$$\int\limits_0^{U(\xi)} \frac{dt}{\sqrt{2\left(\frac{t^\gamma}{\gamma} - \frac{t^q}{q} + \frac{1}{q} - \frac{1}{\gamma}\right)}} = \xi.$$

Such an approximation agrees with the boundary conditions in the leading order: inside the domain both arguments $\dfrac{x}{\sqrt{\varepsilon}}$ and $\dfrac{(1-x)}{\sqrt{\varepsilon}}$ are large, so that $U \approx 1$ and $u(x) \approx 1$, whereas near the endpoints of the interval the corresponding boundary profile ensures the transition to the zero value at the boundary.

The generalization of such a description to multidimensional domains, especially in the case of asymmetric geometry or variable coefficients, represents a substantial mathematical difficulty. At the same time, singularly perturbed nonlinear boundary value problems remain an important object of contemporary analysis in connection with numerous applications. They arise, in particular, in population theory, models of the spread of infections, control theory, mechanics of materials with memory, optics, physiological models, and also in "predator–prey" type systems; see, e.g., [25], [27], [31], [35]. Related classes of parametric elliptic problems, including problems with concentration of solutions at critical values of the parameters, were actively studied in [4], [5], [7], [14], [15], [19], [26], [29], [33], [24].

Parametric elliptic problems with competing power nonlinearities have been studied in several directions. The classical line of research goes back to work by Ambrosetti, Brezis and Cerami [3], where there were obtained threshold results on existence, multiplicity and nonexistence of positive solutions to semilinear problems with concave–convex nonlinearity. For quasilinear p-Laplacian problems analogous questions were studied, in particular, in [10], [18], [8]; see also [11], [2], [30] for related variational and bifurcation approaches. The problem we consider has,

however, a different character. In contrast to the standard concave–convex setting $1 < q < p < r$ with right hand side $\lambda u^{q-1} + u^{r-1}$, here it is assumed that

$$1 < p < q < \gamma < p^*$$

and the right hand side contains two competing superlinear terms with different signs:

$$a(x)|u|^{q-2}u - b(x)|u|^{\gamma-2}u.$$

Therefore, the natural limiting regime is connected not with the concentration of positive solutions, but with the balance of the two nonlinear terms.

The main aim of the present work is to describe this singular limit for the ground states and to show that for $a \geqslant \sigma_a > 0$ they select an explicit distributed limit profile

$$\bar{u}_0(x) = \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{(\gamma-q)}}.$$

In the present work problem (1.1) is investigated by means of the nonlinear Rayleigh quotient method developed in [21], [23]. Application of this method to the parameter ε leads to the following nonlinear generalized Rayleigh quotient:

$$\Upsilon(u) := \frac{\left(\int\limits_\Omega a|u|^q\,dx\right)^{\frac{\gamma-p}{\gamma-q}}}{\int\limits_\Omega |\nabla u|^p\,dx \left(\int\limits_\Omega b|u|^\gamma\,dx\right)^{\frac{q-p}{\gamma-q}}}.$$

Since in the general setting we only assume that $a \geqslant 0$, there can be nonzero functions $u \in W_0^{1,p}(\Omega)$ such that

$$\int\limits_\Omega a|u|^q\,dx = 0.$$

Therefore, the extremal quantities are naturally defined on the set

$$\mathcal{D} := \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : \int\limits_\Omega a|u|^q\,dx > 0 \right\}.$$

In what follows Υ is considered on $\mathcal{D}$.

There are two extremal values of the parameter associated with this quotient

$$\varepsilon^* = c_{p,q,\gamma} \sup_{u\in\mathcal{D}} \Upsilon(u), \qquad \varepsilon_e^* = c_{e,p,q,\gamma} \sup_{u\in\mathcal{D}} \Upsilon(u),$$

where

$$c_{p,q,\gamma} = \frac{\gamma-q}{\gamma-p}\left(\frac{q-p}{\gamma-p}\right)^{\frac{q-p}{\gamma-q}}, \qquad c_{e,p,q,\gamma} = \frac{p(\gamma-q)}{q(\gamma-p)}\left(\frac{\gamma(q-p)}{q(\gamma-p)}\right)^{\frac{q-p}{\gamma-q}}. \tag{1.5}$$

Moreover,

$$0 < \varepsilon_e^* < \varepsilon^* < +\infty.$$

Our first main result describes the existence and nonexistence of positive solutions to problem (1.1) depending on the parameter ε.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < q < \gamma < p^*$, and let conditions (1.2) be satisfied. Then the following statements hold.*

(1) *If $\varepsilon > \varepsilon^*$, then problem (1.1) has no nontrivial weak solutions in $W_0^{1,p}(\Omega)$.*

(2) *For each* $0 < \varepsilon < \varepsilon_e^*$ *problem* (1.1) *has a weak positive solution*

$$u_\varepsilon \in W_0^{1,p}(\Omega).$$

Moreover,

(i) u_ε *is a ground state;*

(ii)

$$\Phi_\varepsilon(u_\varepsilon) < 0, \qquad \frac{d^2}{dt^2}\Phi_\varepsilon(tu_\varepsilon)\Big|_{t=1} > 0;$$

(iii) $u_\varepsilon \in C^{1,\kappa}_{\mathrm{loc}}(\Omega)$ *for some* $\kappa \in (0,1)$.

(3) *For each* $0 < \varepsilon < \varepsilon_e^*$ *problem* (1.1) *has a second weak positive solution*

$$v_\varepsilon \in W_0^{1,p}(\Omega).$$

Moreover,

$$\Phi_\varepsilon(v_\varepsilon) > 0, \qquad v_\varepsilon \in C^{1,\kappa}_{\mathrm{loc}}(\Omega)$$

for some $\kappa \in (0,1)$.

Corollary 1.1 (The classical case $p = 2$)**.** *Let* $p = 2$, $\partial\Omega$ *have the smoothness* $C^{2,\alpha}$, *and* $a, b \in C^{0,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$, *and let conditions* (1.2) *be satisfied. Then the positive weak solutions* u_ε *and* v_ε *obtained in Theorem 1.1 are classical solutions to problem* (1.1). *In particular,*

$$u_\varepsilon,\ v_\varepsilon \in C^2(\Omega) \cap C(\overline{\Omega}).$$

The next result describes the asymptotic behaviour of ground states as $\varepsilon \to 0^+$.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain,* $1 < p < q < \gamma < p^*$, *let conditions* (1.2) *be satisfied, and let there exist a constant* $\sigma_a > 0$ *such that*

$$a(x) \geqslant \sigma_a \quad \text{for a.e. } x \in \Omega.$$

Let u_ε *be a family of ground states of problem* (1.1), $0 < \varepsilon < \varepsilon_e^*$. *Then, as* $\varepsilon \to 0^+$,

$$u_\varepsilon \to \bar{u}_0 \quad \text{in measure in} \quad \Omega,$$

where

$$\bar{u}_0(x) = \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{(\gamma-q)}} \quad \text{for a.e. } x \in \Omega. \tag{1.6}$$

Moreover,

$$u_\varepsilon - \bar{u}_0 \to 0 \quad \text{strongly in} \quad L^r(\Omega), \qquad 1 \leqslant r < \gamma, \tag{1.7}$$

and

$$u_\varepsilon - \bar{u}_0 \rightharpoonup 0 \quad \text{weakly in} \quad L^\gamma(\Omega).$$

The limit function $\bar{u}_0$ *satisfies the limit equation*

$$a(x)\bar{u}_0^{q-1}(x) - b(x)\bar{u}_0^{\gamma-1}(x) = 0 \quad \text{for a.e.} \quad x \in \Omega.$$

By a simple scaling $u \mapsto tu$ problem (1.1) for $\varepsilon > 0$ is reduced to one of the following forms:

$$\begin{cases} -\Delta_p u = \lambda a|u|^{q-2}u - b|u|^{\gamma-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}$$

and

$$\begin{cases} -\Delta_p u = a|u|^{q-2}u - \nu b|u|^{\gamma-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

where $\lambda, \nu > 0$.

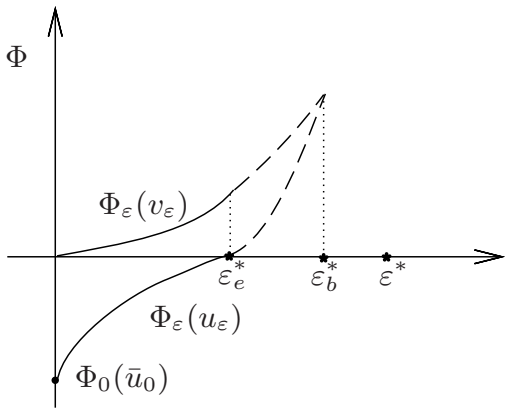


FIGURE 1. Sketch of the graph of energy levels for the branches of positive solutions up to the point ε_e^*, with the addition of the expected behaviour (dashed lines) and the indication of the presumed bifurcation point ε_b^*

These problems are in one–to–one correspondence with the problem (1.1) for positive values of the parameters. Namely, if $u_\varepsilon \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1), then the functions

$$u_\lambda := \varepsilon^{-\frac{1}{(\gamma-p)}} u_\varepsilon, \qquad \lambda = \varepsilon^{-\frac{(\gamma-q)}{(\gamma-p)}}, \tag{1.10}$$

$$u_\nu := \varepsilon^{-\frac{1}{(q-p)}} u_\varepsilon, \qquad \nu = \varepsilon^{\frac{(\gamma-q)}{(q-p)}} \tag{1.11}$$

are weak solutions to problems (1.8) and (1.9), respectively. The inverse transformations read

$$u_\varepsilon = \lambda^{-\frac{1}{(\gamma-q)}} u_\lambda, \qquad u_\varepsilon = \nu^{\frac{1}{(\gamma-q)}} u_\nu.$$

Together with Theorems 1.1 and 1.2 this implies the following statement.

Corollary 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Then*

(1) *Problems (1.8) and (1.9) possess ground states u_λ and u_ν for*

$$\lambda \in (\lambda_e^*, +\infty), \qquad \nu \in (0, \nu_e^*),$$

respectively, where

$$\lambda_e^* = (\varepsilon_e^*)^{-\frac{(\gamma-q)}{(\gamma-p)}}, \qquad \nu_e^* = (\varepsilon_e^*)^{\frac{(\gamma-q)}{(q-p)}}.$$

(2) *If in addition the assumptions of Theorem 1.2 are satisfied, then as $\lambda \to +\infty$,*

$$\lambda^{-\frac{1}{(\gamma-q)}} u_\lambda \to \bar{u}_0 \quad \text{in measure in} \quad \Omega,$$

$$\lambda^{-\frac{1}{(\gamma-q)}} u_\lambda - \bar{u}_0 \to 0 \quad \text{strongly in} \quad L^r(\Omega), \qquad 1 \leqslant r < \gamma,$$

and

$$\lambda^{-\frac{1}{(\gamma-q)}} u_\lambda - \bar{u}_0 \rightharpoonup 0 \quad \text{weakly in} \quad L^\gamma(\Omega).$$

Similarly, as $\nu \to 0^+$,

$$\nu^{\frac{1}{(\gamma-q)}} u_\nu \to \bar{u}_0 \quad \text{in measure in} \quad \Omega,$$

$$\nu^{\frac{1}{(\gamma-q)}} u_\nu - \bar{u}_0 \to 0 \quad \text{strongly in } L^r(\Omega), \qquad 1 \leqslant r < \gamma,$$

and

$$\nu^{\frac{1}{(\gamma-q)}} u_\nu - \bar{u}_0 \rightharpoonup 0 \quad \text{weakly in} \quad L^\gamma(\Omega).$$

Remark 1.1. *Corollary 1.2 shows that the same singular limit can be written in two standard parametrizations: as the limit $\lambda \to +\infty$ in problem (1.8) and as the limit $\nu \to 0^+$ in problem (1.9). In both forms, after the natural normalization, the ground state tends to the same profile*

$$\bar{u}_0(x) = \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{(\gamma-q)}}.$$

Thus, the asymptotics is determined by the balance of the two competing nonlinear terms, and not by the specific way of introducing the parameter.

2. Preliminaries

In what follows, by $L^r := L^r(\Omega)$, $1 \leqslant r < +\infty$, we denote the space of measurable functions with a finite norm

$$\|u\|_r = \left(\int\limits_\Omega |u|^r \, dx\right)^{\frac{1}{r}}.$$

By the Poincaré — Friedrichs inequality, the norm in $W_0^{1,p}(\Omega)$ can be defined by the equivalent formula

$$\|u\|_{1,p} = \left(\int\limits_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

We let

$$T(u) := \int\limits_\Omega |\nabla u|^p \, dx, \qquad A(u) := \int\limits_\Omega a(x)|u|^q \, dx, \qquad B(u) := \int\limits_\Omega b(x)|u|^\gamma \, dx.$$

Since $b(x) \geqslant \sigma_b > 0$ for a.e. $x \in \Omega$, we have

$$B(u) > 0 \quad \text{for all} \quad u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Since $a \not\equiv 0$, the set

$$\mathcal{D} := \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : A(u) > 0\right\}$$

is non–empty.

The Nehari manifold corresponding to problem (1.1) reads

$$\mathcal{N}_\varepsilon := \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : \varepsilon T(u) - A(u) + B(u) = 0\right\}.$$

Then for each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$

$$D_u\Phi_\varepsilon(u)(u) = 0 \quad \Longleftrightarrow \quad u \in \mathcal{N}_\varepsilon.$$

We note that for $\varepsilon > 0$ each function $u \in \mathcal{N}_\varepsilon$ satisfies

$$A(u) = \varepsilon T(u) + B(u) > 0.$$

Therefore, while studying positive values of the parameter, it is natural to restrict oneself to the directions $u \in \mathcal{D}$.

Following the approach based on nonlinear generalized Rayleigh quotients [23], we introduce two Rayleigh quotients associated respectively with the Nehari manifold and the zero energy level. The first quotient is given by the formula

$$R_{\mathcal{N}}(u) := \frac{A(u) - B(u)}{T(u)}, \qquad u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Then

$$\mathcal{N}_\varepsilon = \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : R_{\mathcal{N}}(u) = \varepsilon\right\}.$$

We choose $u \in \mathcal{D}$ and consider the function

$$s \mapsto R_{\mathcal{N}}(su), \qquad s > 0.$$

We have

$$R_{\mathcal{N}}(su) = \frac{s^q A(u) - s^\gamma B(u)}{s^p T(u)} = \frac{A(u)s^{q-p} - B(u)s^{\gamma-p}}{T(u)}.$$

Since $1 < p < q < \gamma$, this function has a unique positive critical point, and moreover it is a point of global maximum. Direct computations give

$$s_{\mathcal{N}}(u) = \left(\frac{(q-p)A(u)}{(\gamma-p)B(u)}\right)^{\frac{1}{(\gamma-q)}}. \tag{2.1}$$

Substituting $s = s_{\mathcal{N}}(u)$ into $R_{\mathcal{N}}(su)$, we obtain the *nonlinear generalized Rayleigh quotient*

$$\varepsilon(u) := R_{\mathcal{N}}(s_{\mathcal{N}}(u)u) = c_{p,q,\gamma}\frac{A(u)^{\frac{\gamma-p}{\gamma-q}}}{T(u)B(u)^{\frac{q-p}{\gamma-q}}} = c_{p,q,\gamma}\Upsilon(u), \qquad u \in \mathcal{D}, \tag{2.2}$$

where

$$c_{p,q,\gamma} = \frac{\gamma-q}{\gamma-p}\left(\frac{q-p}{\gamma-p}\right)^{\frac{q-p}{\gamma-q}}. \tag{2.3}$$

We consider the extremal value

$$\varepsilon^* := \sup_{u\in\mathcal{D}} \varepsilon(u). \tag{2.4}$$

Since $\mathcal{D}$ is non–empty, we have

$$\varepsilon^* > 0.$$

Let us show that $\varepsilon^* < +\infty$. By the Hölder, Poincaré — Friedrichs and Sobolev inequalities we see that there exists a constant $C > 0$, independent of u, such that

$$A(u) = \int\limits_\Omega a(x)|u|^q\,dx \leqslant CT(u)^{\frac{\gamma-q}{\gamma-p}}B(u)^{\frac{q-p}{\gamma-p}}.$$

Indeed, since $q \in (p,\gamma)$, the L^q–norm is interpolated between the L^p– and L^γ–norms, and the L^p–norm is controlled by the norm $\|\nabla u\|_p$. The condition $b(x) \geqslant \sigma_b > 0$ allows one to replace the L^γ–norm by the corresponding $B(u)$–term. Substituting this estimate into (2.2), we obtain

$$0 < \varepsilon^* < +\infty. \tag{2.5}$$

The second Rayleigh quotient corresponds to the zero energy level (cf. [12], [13], [21], [23]):

$$R_e(u) := \frac{\frac{1}{q}A(u) - \frac{1}{\gamma}B(u)}{\frac{1}{p}T(u)}, \qquad u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Then

$$R_e(u) = \varepsilon \quad \Longleftrightarrow \quad \Phi_\varepsilon(u) = 0.$$

For a fixed $u \in \mathcal{D}$ the function

$$s \mapsto R_e(su), \qquad s > 0,$$

has a unique positive critical point. It is given by the formula

$$s_e(u) = \left(\frac{\gamma(q-p)A(u)}{q(\gamma-p)B(u)}\right)^{\frac{1}{(\gamma-q)}}. \tag{2.6}$$

Substituting $s = s_e(u)$ into $R_e(su)$, we obtain

$$\varepsilon_e(u) := R_e(s_e(u)u) = c_{e,p,q,\gamma}\frac{A(u)^{\frac{\gamma-p}{\gamma-q}}}{T(u)B(u)^{\frac{q-p}{\gamma-q}}} = c_{e,p,q,\gamma}\Upsilon(u), \qquad u \in \mathcal{D}, \tag{2.7}$$

where

$$c_{e,p,q,\gamma} = \frac{p(\gamma-q)}{q(\gamma-p)}\left(\frac{\gamma(q-p)}{q(\gamma-p)}\right)^{\frac{q-p}{\gamma-q}}. \tag{2.8}$$

We let

$$\varepsilon_e^* := \sup_{u\in\mathcal{D}} \varepsilon_e(u). \tag{2.9}$$

The same estimate as above yields

$$0 < \varepsilon_e^* < +\infty.$$

Lemma 2.1. *For each $u \in \mathcal{D}$ and all $s > 0$ the equivalence holds*

$$R_e(su) = R_{\mathcal{N}}(su) \quad \Longleftrightarrow \quad s = s_e(u).$$

Proof. We choose $u \in \mathcal{D}$ and let

$$\psi(s) := R_e(su), \qquad s > 0.$$

Then

$$\psi(s) = \frac{p}{T(u)}\left(\frac{A(u)}{q}s^{q-p} - \frac{B(u)}{\gamma}s^{\gamma-p}\right),$$

and

$$\psi'(s) = \frac{p}{T(u)}\left(\frac{q-p}{q}A(u)s^{q-p-1} - \frac{\gamma-p}{\gamma}B(u)s^{\gamma-p-1}\right).$$

On the other hand,

$$R_{\mathcal{N}}(su) - R_e(su) = \frac{1}{T(u)}\left(\frac{q-p}{q}A(u)s^{q-p} - \frac{\gamma-p}{\gamma}B(u)s^{\gamma-p}\right).$$

Hence,

$$\psi'(s) = \frac{p}{s}\left(R_{\mathcal{N}}(su) - R_e(su)\right).$$

Therefore,

$$R_e(su) = R_{\mathcal{N}}(su) \quad \Longleftrightarrow \quad \psi'(s) = 0.$$

The equation $\psi'(s) = 0$ has a unique positive solution $s = s_e(u)$ and this completes the proof. □

Lemma 2.2. *For each $u \in \mathcal{D}$ the strict inequality*

$$\varepsilon(u) > \varepsilon_e(u)$$

holds. Moreover,

$$\varepsilon^* > \varepsilon_e^*.$$

Proof. By formulas (2.2) and (2.7) we see that $\varepsilon(u)$ and $\varepsilon_e(u)$ differ only by multiplicative constants. Therefore, it is sufficient to prove that

$$c_{p,q,\gamma} > c_{e,p,q,\gamma}.$$

We have

$$\frac{c_{p,q,\gamma}}{c_{e,p,q,\gamma}} = \frac{q}{p}\left(\frac{q}{\gamma}\right)^{\frac{q-p}{\gamma-q}}.$$

Let us show that this quantity is greater than 1. Denote

$$\theta := \frac{p}{q} \in (0,1), \qquad x := \frac{\gamma}{q} > 1.$$

Then the required inequality is equivalent to

$$x^{\frac{1-\theta}{x-1}} < \frac{1}{\theta}.$$

Taking the logarithms, we see that it is sufficient to verify

$$\frac{1-\theta}{x-1}\ln x < -\ln\theta.$$

Since $x > 1$, we have $\ln x < x - 1$, and since $0 < \theta < 1$, we have $1 - \theta < -\ln\theta$. Therefore,

$$\frac{1-\theta}{x-1}\ln x < 1 - \theta < -\ln\theta.$$

Hence,

$$c_{p,q,\gamma} > c_{e,p,q,\gamma}.$$

Therefore,

$$\varepsilon(u) > \varepsilon_e(u) \quad \text{for all} \quad u \in \mathcal{D}.$$

Since ε^* and ε_e^* are defined by formulas (2.4) and (2.9) with positive factors $c_{p,q,\gamma}$ and $c_{e,p,q,\gamma}$ at the same positive finite supremum, we obtain

$$\varepsilon^* > \varepsilon_e^*.$$

The proof is complete. □

3. Proof of Theorem 1.1

We begin with proving Statement (1). We argue by contradiction and suppose that for $\varepsilon > \varepsilon^*$ problem (1.1) has a nontrivial weak solution $\bar{u} \in W_0^{1,p}(\Omega)$, $\bar{u} \not\equiv 0$. Then

$$D_u\Phi_\varepsilon(\bar{u})(\bar{u}) = 0,$$

that is,

$$\varepsilon\int\limits_\Omega |\nabla\bar{u}|^p\,dx - \int\limits_\Omega a|\bar{u}|^q\,dx + \int\limits_\Omega b|\bar{u}|^\gamma\,dx = 0.$$

Since $\varepsilon > 0$, $\bar{u} \not\equiv 0$, and $b \geqslant \sigma_b > 0$, it follows from this that

$$\int\limits_\Omega a|\bar{u}|^q\,dx > 0.$$

Hence, $\bar{u} \in \mathcal{D}$. Moreover,

$$R_{\mathcal{N}}(\bar{u}) = \varepsilon.$$

Therefore, taking into consideration (2.2),

$$\varepsilon = R_{\mathcal{N}}(\bar{u}) \leqslant \varepsilon(\bar{u}) \leqslant \varepsilon^* < \varepsilon,$$

which is impossible. Hence, for $\varepsilon > \varepsilon^*$ problem (1.1) has no nontrivial weak solutions.

We proceed to proving Statement (2).

Lemma 3.1. *For each $\varepsilon > 0$ the functional Φ_ε is coercive on $W_0^{1,p}(\Omega)$.*

Proof. By the conditions on a and b, as well as the Hölder and Sobolev inequalities, there exist constants $C_1, C_2 > 0$, independent of u, such that

$$\int\limits_\Omega a|u|^q\,dx \leqslant C_1\|u\|_\gamma^q, \qquad \int\limits_\Omega b|u|^\gamma\,dx \geqslant C_2\|u\|_\gamma^\gamma.$$

Therefore,

$$\Phi_\varepsilon(u) \geqslant \frac{\varepsilon}{p}\|\nabla u\|_p^p - C_1\|u\|_\gamma^q + \frac{C_2}{\gamma}\|u\|_\gamma^\gamma.$$

Let $\|u_n\|_{W_0^{1,p}} \to +\infty$. If $\|u_n\|_\gamma$ is bounded, then, due to the first term, the right hand side tends to $+\infty$. If, however, $\|u_n\|_\gamma \to +\infty$, then, since $q < \gamma$, the positive term of order $\|u_n\|_\gamma^\gamma$ dominates the negative term of order $\|u_n\|_\gamma^q$. In both cases

$$\Phi_\varepsilon(u_n) \to +\infty.$$

The proof is complete. □

Now let $0 < \varepsilon < \varepsilon_e^*$ and consider

$$\hat{\Phi}_\varepsilon := \inf_{u\in W_0^{1,p}(\Omega)} \Phi_\varepsilon(u).$$

It follows from the definition of ε_e^* that there exists $u_0 \in \mathcal{D}$ such that

$$\varepsilon < \varepsilon_e(u_0).$$

We let

$$w_0 := s_e(u_0)u_0.$$

Then by the definition of $\varepsilon_e(u_0)$

$$R_e(w_0) = \varepsilon_e(u_0) > \varepsilon.$$

Since

$$R_e(w_0) = \frac{\frac{1}{q}\int\limits_\Omega a|w_0|^q\,dx - \frac{1}{\gamma}\int\limits_\Omega b|w_0|^\gamma\,dx}{\frac{1}{p}\int\limits_\Omega |\nabla w_0|^p\,dx},$$

the inequality $R_e(w_0) > \varepsilon$ is equivalent to

$$\Phi_\varepsilon(w_0) < 0.$$

Therefore,

$$\hat{\Phi}_\varepsilon < 0. \tag{3.1}$$

Let $(u_m) \subset W_0^{1,p}(\Omega)$ be a minimizing sequence

$$\Phi_\varepsilon(u_m) \to \hat{\Phi}_\varepsilon.$$

By Lemma 3.1 the sequence (u_m) is bounded in $W_0^{1,p}(\Omega)$. Therefore, passing to a subsequence, we can assume that there exists $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$u_m \rightharpoonup u_\varepsilon \quad \text{weakly in} \quad W_0^{1,p}(\Omega),$$

and

$$u_m \to u_\varepsilon \quad \text{strongly in} \quad L^r(\Omega), \qquad 1 < r < p^*.$$

In particular,

$$u_m \to u_\varepsilon \quad \text{strongly in} \quad L^q(\Omega) \quad \text{and} \quad u_m \to u_\varepsilon \quad \text{strongly in} \quad L^\gamma(\Omega).$$

Using the weak lower semicontinuity of the norm $\|\nabla u\|_p$ and the strong convergence of the nonlinear terms, we obtain

$$\Phi_\varepsilon(u_\varepsilon) \leqslant \liminf_{m\to+\infty} \Phi_\varepsilon(u_m) = \hat{\Phi}_\varepsilon.$$

Therefore,

$$\Phi_\varepsilon(u_\varepsilon) = \hat{\Phi}_\varepsilon.$$

In particular, by (3.1),

$$\Phi_\varepsilon(u_\varepsilon) < 0 = \Phi_\varepsilon(0),$$

and hence $u_\varepsilon \not\equiv 0$.

Since

$$\Phi_\varepsilon(|u|) = \Phi_\varepsilon(u), \qquad |u| \in W_0^{1,p}(\Omega),$$

we may replace u_ε by $|u_\varepsilon|$ and assume that

$$u_\varepsilon \geqslant 0 \quad \text{in} \quad \Omega.$$

Since u_ε is a point of global minimum of Φ_ε on $W_0^{1,p}(\Omega)$, we have

$$D_u\Phi_\varepsilon(u_\varepsilon) = 0.$$

Therefore, u_ε is a weak solution to problem (1.1).

We are going to show that this solution is positive. The standard apriori estimates for weak solutions to quasilinear elliptic equations with subcritical growth imply

$$u_\varepsilon \in L^\infty(\Omega).$$

Indeed, since

$$1 < p < q < \gamma < p^*,$$

the right hand side of equation (1.1) has the subcritical growth, and the standard Moser iteration scheme (see, e.g., [16]) gives the boundedness of u_ε. Therefore,

$$a(x)u_\varepsilon^{q-1} - b(x)u_\varepsilon^{\gamma-1} \in L^\infty(\Omega).$$

By the interior regularity results for degenerate quasilinear elliptic equations [15], [32] we obtain

$$u_\varepsilon \in C^{1,\kappa}_{\mathrm{loc}}(\Omega)$$

for some $\kappa \in (0,1)$.

Let us show that $u_\varepsilon > 0$ in Ω. Since $u_\varepsilon \geqslant 0$, $u_\varepsilon \not\equiv 0$, and $u_\varepsilon \in L^\infty(\Omega)$, there exists $M > 0$ such that

$$0 \leqslant u_\varepsilon(x) \leqslant M \quad \text{for a.e. } x \in \Omega.$$

Since $\gamma > p$, we have

$$u_\varepsilon^{\gamma-1} = u_\varepsilon^{\gamma-p}u_\varepsilon^{p-1} \leqslant M^{\gamma-p}u_\varepsilon^{p-1}.$$

Therefore,

$$a(x)u_\varepsilon^{q-1} - b(x)u_\varepsilon^{\gamma-1} \geqslant -\|b\|_\infty M^{\gamma-p}u_\varepsilon^{p-1} \quad \text{for a.e.} \quad x \in \Omega.$$

In view of the equation for u_ε this yields that, with some constant $C > 0$,

$$-\varepsilon\Delta_p u_\varepsilon + Cu_\varepsilon^{p-1} \geqslant 0 \quad \text{in the weak sense in} \quad \Omega.$$

By the strong maximum principle for the p–Laplacian (see, e.g., [34]) and since Ω is a domain, we conclude that

$$u_\varepsilon > 0 \quad \text{in} \quad \Omega.$$

Thus, u_ε is a positive weak solution of problem (1.1). Moreover, since u_ε is a global minimum of the functional Φ_ε, this solution is a ground state.

It remains to check the sign of the second derivative of function $t \mapsto \Phi_\varepsilon(tu_\varepsilon)$ at $t = 1$. We let

$$T_\varepsilon := \int\limits_\Omega |\nabla u_\varepsilon|^p \, dx, \qquad A_\varepsilon := \int\limits_\Omega a u_\varepsilon^q \, dx, \qquad B_\varepsilon := \int\limits_\Omega b u_\varepsilon^\gamma \, dx.$$

Since u_ε is a critical point, we have

$$\varepsilon T_\varepsilon - A_\varepsilon + B_\varepsilon = 0.$$

Moreover,

$$\Phi_\varepsilon(u_\varepsilon) < 0.$$

Using $A_\varepsilon = \varepsilon T_\varepsilon + B_\varepsilon$, we obtain

$$0 > \Phi_\varepsilon(u_\varepsilon) = \varepsilon T_\varepsilon \left(\frac{1}{p} - \frac{1}{q}\right) + B_\varepsilon \left(\frac{1}{\gamma} - \frac{1}{q}\right).$$

Hence,

$$B_\varepsilon > \frac{\gamma(q-p)}{p(\gamma-q)}\,\varepsilon T_\varepsilon.$$

Therefore,

$$\frac{d^2}{dt^2}\Phi_\varepsilon(tu_\varepsilon)\bigg|_{t=1} = (p-q)\varepsilon T_\varepsilon + (\gamma-q)B_\varepsilon > 0.$$

Statement (2) is proven.

We proceed to the proof of Statement (3). We introduce the truncated functional

$$\Phi_\varepsilon^+(u) = \frac{\varepsilon}{p}\int\limits_\Omega |\nabla u|^p\,dx - \frac{1}{q}\int\limits_\Omega a(u^+)^q\,dx + \frac{1}{\gamma}\int\limits_\Omega b(u^+)^\gamma\,dx, \qquad u^+ := \max\{u, 0\}.$$

The functional Φ_ε^+ has the same coercivity estimates as Φ_ε and satisfies the Palais — Smale condition.

Indeed, let $(u_n) \subset W_0^{1,p}(\Omega)$ be a Palais — Smale sequence for Φ_ε^+, that is,

$$\Phi_\varepsilon^+(u_n) \text{ is bounded}, \qquad D\Phi_\varepsilon^+(u_n) \to 0 \quad \text{in} \quad (W_0^{1,p}(\Omega))^*.$$

The coercivity of Φ_ε^+ implies that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Therefore, passing to a subsequence, we can suppose that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p}(\Omega),$$

and

$$u_n \to u \quad \text{strongly in} \quad L^r(\Omega), \qquad 1 < r < p^*.$$

In particular,

$$u_n \to u \quad \text{strongly in} \quad L^q(\Omega) \quad \text{and} \quad u_n \to u \quad \text{strongly in} \quad L^\gamma(\Omega).$$

Since the mapping $\mathbb{R} \ni z \mapsto z^+$ is Lipschitz, we have

$$u_n^+ \to u^+ \quad \text{strongly in} \quad L^q(\Omega) \quad \text{and} \quad u_n^+ \to u^+ \quad \text{strongly in} \quad L^\gamma(\Omega).$$

Substituting $u_n - u$ into the derivative $D\Phi_\varepsilon^+(u_n)$, we obtain

$$\begin{aligned} o(1) &= D\Phi_\varepsilon^+(u_n)(u_n - u) \\ &= \varepsilon \int\limits_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx \\ &\quad - \int\limits_\Omega a(u_n^+)^{q-1}(u_n - u)\,dx + \int\limits_\Omega b(u_n^+)^{\gamma-1}(u_n - u)\,dx. \end{aligned}$$

The last two integrals tend to zero. For example,

$$\left|\int\limits_\Omega a(u_n^+)^{q-1}(u_n - u)\,dx\right| \leqslant \|a\|_\infty \|u_n^+\|_q^{q-1}\|u_n - u\|_q \to 0,$$

and similarly

$$\int\limits_\Omega b(u_n^+)^{\gamma-1}(u_n - u)\,dx \to 0.$$

Therefore,

$$\int\limits_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \cdot (\nabla u_n - \nabla u)\, dx \to 0.$$

We use the standard S^+-type property of the p-Laplacian operator: if

$$u_n \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,p}(\Omega)$$

and

$$\limsup_{n\to+\infty} \int\limits_{\Omega} |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\, dx \leqslant 0,$$

then

$$u_n \to u \quad \text{strongly in} \quad W_0^{1,p}(\Omega).$$

This property is a standard consequence of the monotonicity of the p–Laplacian operator and the uniform convexity of space $W_0^{1,p}(\Omega)$; see, e.g., the results on monotone operators of S^+ type in [17]. Applying it to (u_n), we obtain

$$u_n \to u \quad \text{strongly in} \quad W_0^{1,p}(\Omega).$$

Consequently, Φ_ε^+ satisfies the Palais — Smale condition.

We are going to show that Φ_ε^+ has the mountain pass geometry. The Sobolev embedding implies

$$\Phi_\varepsilon^+(u) \geqslant \left(\frac{\varepsilon}{p} - C\|u\|_{W_0^{1,p}}^{q-p}\right)\|u\|_{W_0^{1,p}}^p.$$

Choosing $\rho > 0$ sufficiently small, we obtain

$$\Phi_\varepsilon^+(u) > 0 \quad \text{for} \quad 0 < \|u\|_{W_0^{1,p}} \leqslant \rho.$$

In particular, there exists $\delta > 0$ such that

$$\Phi_\varepsilon^+(u) \geqslant \delta \quad \text{for} \quad \|u\|_{W_0^{1,p}} = \rho.$$

On the other hand, the ground state u_ε found above is non–negative and satisfies

$$\Phi_\varepsilon^+(u_\varepsilon) = \Phi_\varepsilon(u_\varepsilon) < 0.$$

Therefore, $\|u_\varepsilon\|_{W_0^{1,p}} > \rho$, and the functional Φ_ε^+ has the mountain pass geometry.

By the mountain pass theorem [6] there exists a critical point $v_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\Phi_\varepsilon^+(v_\varepsilon) = c_\varepsilon := \inf_{\xi\in\Gamma}\max_{t\in[0,1]} \Phi_\varepsilon^+(\xi(t)) \geqslant \delta > 0,$$

where

$$\Gamma = \left\{\xi \in C([0,1]; W_0^{1,p}(\Omega)) : \xi(0) = 0,\ \xi(1) = u_\varepsilon\right\}.$$

In particular,

$$v_\varepsilon \not\equiv 0.$$

Let us show that $v_\varepsilon \geqslant 0$. Since

$$D\Phi_\varepsilon^+(v_\varepsilon) = 0,$$

we take $-v_\varepsilon^-$ as the test function, where

$$v_\varepsilon^- := \max\{-v_\varepsilon, 0\}.$$

Then the nonlinear terms containing v_ε^+ vanish, and we obtain

$$\varepsilon \int\limits_{\Omega} |\nabla v_\varepsilon^-|^p\, dx = 0.$$

Hence,

$$v_\varepsilon^- = 0, \qquad v_\varepsilon \geqslant 0 \quad \text{in} \quad \Omega.$$

Therefore, $v_\varepsilon^+ = v_\varepsilon$, and v_ε is a critical point of the original functional Φ_ε. Hence, v_ε is a weak solution to problem (1.1).

Since

$$\Phi_\varepsilon(v_\varepsilon) = \Phi_\varepsilon^+(v_\varepsilon) = c_\varepsilon > 0,$$

this solution does not coincide with the ground state u_ε since

$$\Phi_\varepsilon(u_\varepsilon) < 0.$$

As above, the apriori estimates and interior regularity results for the p–Laplacian yield that

$$v_\varepsilon \in C^{1,\kappa}_{\mathrm{loc}}(\Omega)$$

for some $\kappa \in (0,1)$. Finally, since $v_\varepsilon \geqslant 0$, $v_\varepsilon \not\equiv 0$, and

$$-\varepsilon\Delta_p v_\varepsilon + C v_\varepsilon^{p-1} \geqslant 0 \quad \text{in the weak sense in} \quad \Omega$$

with some constant $C > 0$, by the strong maximum principle for the p–Laplacian (see, e.g., [34]) we obtain

$$v_\varepsilon > 0 \quad \text{in} \quad \Omega.$$

The proof is complete.

4. Proof of Theorem 1.2

We let

$$J(u) := -\frac{1}{q}\int\limits_\Omega a(x)|u|^q\,dx + \frac{1}{\gamma}\int\limits_\Omega b(x)|u|^\gamma\,dx.$$

For $s \geqslant 0$ we denote

$$j_x(s) := -\frac{a(x)}{q}s^q + \frac{b(x)}{\gamma}s^\gamma.$$

Then

$$J(u) = \int\limits_\Omega j_x(|u(x)|)\,dx.$$

Since $a(x) \geqslant \sigma_a > 0$ and $b(x) \geqslant \sigma_b > 0$ for a.e. $x \in \Omega$, for such x the function $s \mapsto j_x(s)$ has a unique positive point of global minimum

$$\bar{u}_0(x) = \left(\frac{a(x)}{b(x)}\right)^{\frac{1}{(\gamma-q)}}.$$

Hence,

$$J(u) \geqslant J(\bar{u}_0) \tag{4.1}$$

for each $u \in L^\gamma(\Omega)$. Moreover, the conditions $a, b \in L^\infty(\Omega)$, $a \geqslant \sigma_a > 0$, $b \geqslant \sigma_b > 0$ imply that

$$\bar{u}_0 \in L^\infty(\Omega), \qquad \bar{u}_0 > 0 \quad \text{for a.e.} \quad x \in \Omega.$$

Let u_ε be a ground state of problem (1.1), $0 < \varepsilon < \varepsilon_e^*$. In the proof of Theorem 1.1, we constructed a positive weak solution, which was a global minimum of functional Φ_ε on $W_0^{1,p}(\Omega)$. Since this global minimum is a weak solution, the energy value of each ground state coincides with the global minimum of Φ_ε. Therefore, u_ε can also be regarded as a global minimum of Φ_ε on $W_0^{1,p}(\Omega)$. Therefore, by (4.1) we obtain

$$\Phi_\varepsilon(u_\varepsilon) \geqslant J(u_\varepsilon) \geqslant J(\bar{u}_0). \tag{4.2}$$

On the other hand, since $C_0^\infty(\Omega)$ is dense in $L^\gamma(\Omega)$, for each $\delta > 0$ one can choose a function $w_\delta \in C_0^\infty(\Omega)$, $w_\delta \geqslant 0$, such that

$$\|w_\delta - \bar{u}_0\|_\gamma < \delta.$$

Then, by the minimality of u_ε,

$$\Phi_\varepsilon(u_\varepsilon) \leqslant \Phi_\varepsilon(w_\delta) = \frac{\varepsilon}{p}\int\limits_\Omega |\nabla w_\delta|^p\,dx + J(w_\delta).$$

Therefore,

$$\limsup_{\varepsilon\to 0^+} \Phi_\varepsilon(u_\varepsilon) \leqslant J(w_\delta).$$

Since $q < \gamma$ and Ω is bounded, the functional J is continuous with respect to the strong convergence in $L^\gamma(\Omega)$. Passing to the limit as $\delta \to 0$, we obtain

$$\limsup_{\varepsilon\to 0^+} \Phi_\varepsilon(u_\varepsilon) \leqslant J(\bar u_0). \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\Phi_\varepsilon(u_\varepsilon) \to J(\bar u_0) \quad \text{as} \quad \varepsilon \to 0^+. \tag{4.4}$$

Moreover,

$$0 \leqslant J(u_\varepsilon) - J(\bar u_0) \leqslant \Phi_\varepsilon(u_\varepsilon) - J(\bar u_0),$$

and therefore,

$$J(u_\varepsilon) \to J(\bar u_0) \quad \text{as} \quad \varepsilon \to 0^+. \tag{4.5}$$

Now we are going to prove that $u_\varepsilon \to \bar u_0$ in measure. We first observe that the family $\{u_\varepsilon\}$ is bounded in $L^\gamma(\Omega)$ as $\varepsilon \to 0^+$. Indeed, since $a, b \in L^\infty(\Omega)$, $b \geqslant \sigma_b > 0$ and $q < \gamma$, there exist constants $c_1, c_2 > 0$, independent of x and $s \geqslant 0$, such that

$$j_x(s) = -\frac{a(x)}{q}s^q + \frac{b(x)}{\gamma}s^\gamma \geqslant c_1 s^\gamma - c_2.$$

This estimate and (4.5) imply the boundedness of $\{u_\varepsilon\}$ in $L^\gamma(\Omega)$ for all sufficiently small $\varepsilon > 0$.

Since $u_\varepsilon \geqslant 0$, by (4.5) we have

$$\int\limits_\Omega \big(j_x(u_\varepsilon(x)) - j_x(\bar u_0(x))\big)\,dx \to 0. \tag{4.6}$$

Since for a.e. $x \in \Omega$ the function $s \mapsto j_x(s)$ has a unique positive point of global minimum $\bar u_0(x)$, the integrand in (4.6) is nonnegative.

We let

$$\hat a := \|a\|_\infty, \qquad \hat b := \|b\|_\infty.$$

Then for a.e. $x \in \Omega$

$$0 < \rho_- := \left(\frac{\sigma_a}{\hat b}\right)^{\frac{1}{(\gamma-q)}} \leqslant \bar u_0(x) \leqslant \left(\frac{\hat a}{\sigma_b}\right)^{\frac{1}{(\gamma-q)}} =: \rho_+ < +\infty.$$

We choose $\eta > 0$. Let us show that there exists $\kappa_\eta > 0$ such that

$$j_x(s) - j_x(\bar u_0(x)) \geqslant \kappa_\eta \tag{4.7}$$

for all $s \geqslant 0$ satisfying

$$|s - \bar u_0(x)| \geqslant \eta,$$

and for a.e. $x \in \Omega$.

We consider the family of functions

$$j_{\alpha,\beta}(s) := -\frac{\alpha}{q}s^q + \frac{\beta}{\gamma}s^\gamma, \qquad (\alpha,\beta) \in [\sigma_a, \hat a] \times [\sigma_b, \hat b].$$

For each pair (α, β) the function $j_{\alpha,\beta}$ has a unique positive point of global minimum

$$\rho(\alpha,\beta) = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{(\gamma-q)}},$$

and moreover,

$$\rho(\alpha,\beta)\in[\rho_-,\rho_+].$$

We first note that

$$j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))\to+\infty\quad\text{as } s\to+\infty$$

uniformly in

$$(\alpha,\beta)\in[\sigma_a,\hat{a}]\times[\sigma_b,\hat{b}].$$

Indeed,

$$j_{\alpha,\beta}(s)\geqslant-\frac{\hat{a}}{q}s^q+\frac{\sigma_b}{\gamma}s^\gamma,$$

and the function

$$(\alpha,\beta)\mapsto j_{\alpha,\beta}(\rho(\alpha,\beta))$$

is continuous on the compact set

$$[\sigma_a,\hat{a}]\times[\sigma_b,\hat{b}].$$

Therefore, there exists $M_0>0$ such that

$$\left|j_{\alpha,\beta}(\rho(\alpha,\beta))\right|\leqslant M_0$$

for all $(\alpha,\beta)\in[\sigma_a,\hat{a}]\times[\sigma_b,\hat{b}]$. Hence,

$$j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))\geqslant\frac{\sigma_b}{\gamma}s^\gamma-\frac{\hat{a}}{q}s^q-M_0.$$

Since $\gamma>q$, the right hand side tends to $+\infty$ as $s\to+\infty$, and moreover, this lower bound is independent of (α,β). Hence, one can choose $M>0$ so that

$$j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))\geqslant1 \tag{4.8}$$

for all $s\geqslant M$ and all

$$(\alpha,\beta)\in[\sigma_a,\hat{a}]\times[\sigma_b,\hat{b}].$$

It remains to consider the domain $0\leqslant s\leqslant M$. Let

$$K_\eta:=\left\{(\alpha,\beta,s):\alpha\in[\sigma_a,\hat{a}],\ \beta\in[\sigma_b,\hat{b}],\ s\in[0,M],\ |s-\rho(\alpha,\beta)|\geqslant\eta\right\}.$$

If $K_\eta\neq\varnothing$, then this set is compact. The function

$$(\alpha,\beta,s)\mapsto j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))$$

is continuous and strictly positive on K_η, since the vanishing is possible only for $s=\rho(\alpha,\beta)$, and such points are excluded by the condition $|s-\rho(\alpha,\beta)|\geqslant\eta$. Therefore,

$$\kappa'_\eta:=\min_{K_\eta}\bigl(j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))\bigr)>0.$$

If, however, $K_\eta=\varnothing$, we let, for example, $\kappa'_\eta=1$. In both cases we can let

$$\kappa_\eta:=\min\{\kappa'_\eta,1\}>0.$$

Then it follows from (4.8) and the definition of κ'_η that

$$j_{\alpha,\beta}(s)-j_{\alpha,\beta}(\rho(\alpha,\beta))\geqslant\kappa_\eta$$

for all $s\geqslant0$ and all $(\alpha,\beta)\in[\sigma_a,\hat{a}]\times[\sigma_b,\hat{b}]$ such that

$$|s-\rho(\alpha,\beta)|\geqslant\eta.$$

Applying this statement with

$$\alpha=a(x),\qquad\beta=b(x),\qquad\rho(\alpha,\beta)=\bar{u}_0(x),$$

we obtain (4.7).

Therefore,

$$\kappa_\eta \operatorname{mes}\{x \in \Omega : \ |u_\varepsilon(x) - \bar{u}_0(x)| \geqslant \eta\} \leqslant \int\limits_\Omega \big(j_x(u_\varepsilon(x)) - j_x(\bar{u}_0(x))\big)\, dx.$$

By (4.6) the right hand side tends to zero. Hence, for each $\eta > 0$

$$\operatorname{mes}\{x \in \Omega : \ |u_\varepsilon(x) - \bar{u}_0(x)| \geqslant \eta\} \to 0,$$

that is,

$$u_\varepsilon \to \bar{u}_0 \quad \text{in measure in} \quad \Omega. \tag{4.9}$$

Now we are going to show the strong convergence in $L^r(\Omega)$ for $1 \leqslant r < \gamma$. Since the family $\{u_\varepsilon\}$ is bounded in $L^\gamma(\Omega)$ and $\bar{u}_0 \in L^\gamma(\Omega)$, the family $\{u_\varepsilon - \bar{u}_0\}$ is also bounded in $L^\gamma(\Omega)$. Therefore, for each $1 \leqslant r < \gamma$ the family $\{|u_\varepsilon - \bar{u}_0|^r\}$ is bounded in $L^{\gamma/r}(\Omega)$, where $\gamma/r > 1$. Hence, it is uniformly integrable. Taking into account (4.9), from the Lebesgue — Vitali convergence theorem (see, e.g., [9, Thm. 4.5.4]) we obtain

$$\int\limits_\Omega |u_\varepsilon - \bar{u}_0|^r\, dx \to 0.$$

Thus,

$$u_\varepsilon \to \bar{u}_0 \quad \text{strongly in} \quad L^r(\Omega)$$

for all $1 \leqslant r < \gamma$.

Moreover,

$$u_\varepsilon \rightharpoonup \bar{u}_0 \quad \text{weakly in} \quad L^\gamma(\Omega).$$

Indeed, the boundedness of u_ε in $L^\gamma(\Omega)$ and the reflexivity of this space imply that each sequence u_{ε_m}, $\varepsilon_m \to 0^+$, contains a subsequence weakly convergent in $L^\gamma(\Omega)$. Let

$$u_{\varepsilon_m} \rightharpoonup w \quad \text{weakly in} \quad L^\gamma(\Omega)$$

along such a subsequence. On the other hand, the proven strong convergence in $L^r(\Omega)$, $1 \leqslant r < \gamma$, in particular gives

$$u_{\varepsilon_m} \to \bar{u}_0 \quad \text{strongly in} \quad L^1(\Omega).$$

Therefore,

$$\int\limits_\Omega u_{\varepsilon_m} \varphi\, dx \to \int\limits_\Omega \bar{u}_0 \varphi\, dx$$

for each $\varphi \in L^\infty(\Omega)$. The weak convergence in $L^\gamma(\Omega)$ also implies that

$$\int\limits_\Omega u_{\varepsilon_m} \varphi\, dx \to \int\limits_\Omega w \varphi\, dx.$$

Hence $w = \bar{u}_0$ almost everywhere in Ω. Therefore, each weak limit point of the family u_ε in $L^\gamma(\Omega)$ coincides with $\bar{u}_0$, and hence the entire family converges weakly to $\bar{u}_0$ in $L^\gamma(\Omega)$. In other words,

$$u_\varepsilon - \bar{u}_0 \rightharpoonup 0 \quad \text{weakly in} \quad L^\gamma(\Omega).$$

Finally, by the definition of $\bar{u}_0$ we have

$$a(x)\bar{u}_0^{q-1}(x) - b(x)\bar{u}_0^{\gamma-1}(x) = 0 \quad \text{for a.e.} \quad x \in \Omega.$$

Thus, $\bar{u}_0$ satisfies the limit equation of problem (1.1) for $\varepsilon = 0$:

$$a(x)\bar{u}_0^{q-1} = b(x)\bar{u}_0^{\gamma-1} \quad \text{a.e. in} \quad \Omega.$$

The proof is complete.

BIBLIOGRAPHY

1. A.M. Il'in. *Matching of asymptotic expansions of solutions of boundary value problems*. Nauka, Moscow; *English translation*: Amer. Math. Soc., Providence, RI (1992).
2. Ya.Sh. Il'yasov. *On the curve of critical exponents for nonlinear elliptic problems in the case of a zero mass* // Comput. Math. Math. Phys. **57**:3, 497–514 (2017). https://doi.org/10.1134/S096554251703006X
3. A. Ambrosetti, H. Brezis, G. Cerami. *Combined effects of concave and convex nonlinearities in some elliptic problems* // J. Funct. Anal. **122**:2, 519–543 (1994). https://doi.org/10.1006/jfan.1994.1078
4. A. Ambrosetti, A. Malchiodi. *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^n$. Birkhäuser, Basel (2006).
5. A. Ambrosetti, A. Malchiodi, W.-M. Ni. *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I* // Commun. Math. Phys. **235**:3, 427–466 (2003). https://doi.org/10.1007/s00220-003-0811-y
6. A. Ambrosetti, P.H. Rabinowitz. *Dual variational methods in critical point theory and applications* // J. Funct. Anal. **14**:4, 349–381 (1973). https://doi.org/10.1016/0022-1236(73)90051-7
7. M. Badiale, J. García Azorero, I. Peral. *Perturbation results for an anisotropic Schrödinger equation via a variational method* // NoDEA, Nonlinear Differ. Equ. Appl. **7**:2, 201–230 (2000). https://doi.org/10.1007/s000300050005
8. T. Bartsch, M. Willem. *On an elliptic equation with concave and convex nonlinearities* // Proc. Amer. Math. Soc. **123**:11, 3555–3561 (1995). https://doi.org/10.2307/2161107
9. V.I. Bogachev. *Measure Theory. Vol. I*. Springer, Berlin (2007). https://mathoverflow.net/q/439397
10. V. Bobkov, P. Drábek, J. Hernández. *Existence and multiplicity results for a class of semilinear elliptic equations* // Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods **200**, 112017, 24 p. (2020). https://doi.org/10.1016/j.na.2020.112017
11. V. Bobkov, Y. Il'yasov. *Asymptotic behaviour of branches for ground states of elliptic systems* // Electron. J. Differ. Equ. **2013**, 212 (2013). https://emis.de/ft/1087
12. R. Carles, Y. Il'yasov. *On ground states for the 2D Schrödinger equation with combined nonlinearities and harmonic potential* // Stud. Appl. Math. **150**:1, 92–118 (2023). https://doi.org/10.1111/sapm.12534
13. M.L. Carvalho, Y. Il'yasov, C.A. Santos. *Existence of S-shaped type bifurcation curve with dual cusp catastrophe via variational methods* // J. Differ. Equations **334**, 256–279 (2022). https://doi.org/10.1016/j.jde.2022.06.021
14. E.N. Dancer, S. Yan. *Multipeak solutions for a singularly perturbed Neumann problem* // Pac. J. Math. **189**:2, 241–262 (1999). https://doi.org/10.2140/pjm.1999.189.241
15. E. DiBenedetto. $C^{1,\alpha}$ *local regularity of weak solutions of degenerate elliptic equations* // Nonlinear Anal., Theory Methods Appl. **7**:8, 827–850 (1983). https://doi.org/10.1016/0362-546X(83)90061-5
16. P. Drábek, A. Kufner, F. Nicolosi. *Quasilinear Elliptic Equations with Degenerations and Singularities*. Walter de Gruyter, Berlin (1997).
17. P. Drábek, J. Milota. *Methods of Nonlinear Analysis: Applications to Differential Equations*. Springer, Basel (2013). https://doi.org/10.1007/978-3-0348-0387-8
18. J. García Azorero, I. Peral Alonso. *Some results about the existence of a second positive solution in a quasilinear critical problem* // Indiana Univ. Math. J. **43**:3, 941–957 (1994). https://doi.org/10.1512/iumj.1994.43.43041
19. F.A. Howes. *Singularly perturbed semilinear elliptic boundary value problems* // Commun. Partial Differ. Equations **4**:1, 1–39 (1979). https://doi.org/10.1080/03605307908820090
20. Y. Il'yasov. *On nonlocal existence results for elliptic equations with convex–concave nonlinearities* // Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods **61**:1–2, 211–236 (2005). https://doi.org/10.1016/j.na.2004.10.022
21. Y.S. Il'yasov. *On extreme values of Nehari manifold method via nonlinear Rayleigh quotient* // Topol. Methods Nonlinear Anal. **49**:2, 683–714 (2017). https://doi.org/10.12775/TMNA.2017.005

22. Y. Il'yasov, K. Silva. *On branches of positive solutions for p–Laplacian problems at the extreme value of the Nehari manifold method* // Proc. Am. Math. Soc. **146**:7, 2925–2935 (2018). https://doi.org/10.1090/proc/13972
23. Y.S. Il'yasov. *Rayleigh quotients of the level set manifolds related to the nonlinear PDE* // Minimax Theory Appl. **7**:2, 277–302 (2022). https://geodesic.mathdoc.fr/item/MTA_2022_7_2_a4/
24. E.M. de Jager, F. Jiang. *The Theory of Singular Perturbations*. North–Holland, Amsterdam (1996).
25. M.K. Kadalbajoo, K.C. Patidar. *Singularly perturbed problems in partial differential equations: a survey* // Appl. Math. Comput. **134**:2–3, 371–429 (2003). https://doi.org/10.1016/S0096-3003(01)00291-0
26. C.S. Lin, W.-M. Ni, I. Takagi. *Large amplitude stationary solutions to a chemotaxis system* // J. Differ. Equations **72**:1, 1–27 (1988). https://doi.org/10.1016/0022-0396(88)90147-7
27. J.D. Murray. *Mathematical Biology. I: An Introduction*. Springer, New York (2007). https://doi.org/10.1007/b98868
28. A.H. Nayfeh. *Perturbation Methods*. John Wiley & Sons, New York (1973).
29. W.-M. Ni, J. Wei. *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems* // Commun. Pure Appl. Math. **48**:7, 731–768 (1995). https://doi.org/10.1002/cpa.3160480704
30. H. Ramos Quoirin, G. Siciliano, K. Silva. *Critical points with prescribed energy for a class of functionals depending on a parameter: existence, multiplicity and bifurcation results* // Nonlinearity **37**:6, 065010 (2024). https://doi.org/10.1088/1361-6544/ad39dd
31. H.-G. Roos, M. Stynes, L. Tobiska. *Robust Numerical Methods for Singularly Perturbed Differential Equations. Convection–diffusion–reaction and flow problems*. Springer, Berlin (2008). https://doi.org/10.1007/978-3-540-34467-4
32. P. Tolksdorf. *Regularity for a more general class of quasilinear elliptic equations* // J. Differ. Equations **51**:1, 126–150 (1984). https://doi.org/10.1016/0022-0396(84)90105-0
33. A.B. Vasil'eva, V.F. Butuzov, L.V. Kalachev. *The Boundary Function Method for Singular Perturbation Problems*. SIAM, Philadelphia (1995).
34. J.L. Vázquez. *A strong maximum principle for some quasilinear elliptic equations* // Appl. Math. Optimization **12**:1, 191–202 (1984). https://doi.org/10.1007/BF01449041
35. A. Volpert, V. Volpert, V. Volpert. *Traveling Wave Solutions of Parabolic Systems*. Amer. Math. Soc., Providence, RI (1994). http://www.ams.org/online_bks/mmono140/

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