ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 4 (2026). P. 43-51.

doi:10.13108/2026-18-4-43

ONE APPROACH TO ANALYSIS OF NON–HOMOGENEOUS POISSON PROCESS WITH PERIODIC INTENSITY

A.V. ZORINE

Abstract. We consider probabilistic properties of the jump times of a non-homogeneous Poisson process. Under the assumption that the instantaneous intensity is a periodic function separated from zero, the "phases" (fractional parts with respect to the period length) of the jump times form a general Markov chain on an interval, and the density of the invariant probability distribution is proportional to the instantaneous intensity function.

Possible applications of the discovered property of the jump times to statistical inference about a non-homogeneous Poisson process are discussed. Using the Central Limit Theorem for Markov chains, we prove the convergence of the finite–dimensional distributions of the empirical process to the finite–dimensional distributions of a Gaussian process of the same type as in the case of an independent repeated sample.

Keywords: non-homogeneous Poisson process, periodic intensity, general Markov chain, empirical process, goodness–of–fit test, central limit theorem.

Mathematics Subject Classification: 60G55, 60G17, 62G05, 62G10

1. Introduction

In this paper we consider properties of one class of point processes on the half–line $\mathbb{R}_+ = \{t; t \geqslant 0\}$. In many real applications it is assumed that we are dealing with the instants of time $\{\tau_i; i = 0, 1, \ldots\}$ of occurrence of a sequence of certain events. We obtain an equivalent model if we pass to the study of the intervals $(\tau_i - \tau_{i-1})$ between events, $i = 1, 2, \ldots$, or of the counting function

$$N(t) = \sum_{i=0}^{\infty} I(\tau_i < t)$$

for the number of events on the interval $[0, t)$. The classical monograph by Cox and Lewis [7] provides one of the first comprehensive surveys of statistical models for a point process on the half–line. These include models of the Poisson process and the renewal process, and regression type models for the intervals between events. Important variations of the Poisson process include the non–stationary Poisson process with a non–random, time–dependent instantaneous intensity, mentioned in this monograph, and the process in which the instantaneous intensity is itself a random process (the so–called Cox processes). Besides questions of estimating first– and second–order numerical characteristics, in [7] the authors described procedures for testing important statistical hypotheses about whether an observed segment of a point process belongs to the family of Poisson processes with constant intensity, or to the family of renewal processes.

A.V. Zorine, On one approach to analysis of non-homogeneous Poisson process with periodic intensity.

© Zorine A.V. 2026.

Submitted July 25, 2025.

With regard to non-stationary Poisson processes (they are also called non-homogeneous Poisson processes), the following groups of problems naturally arise: the development of tests for hypotheses on constant intensity against an alternative in the form of a time function from a given class, and also the problem of estimating the instantaneous intensity and the mathematical expectation of the number of events for the reason that we shall recall below. Testing the hypothesis that the intensity is independent of time against the alternative of a linear dependence was already considered in monograph [7]. Later, the alternative hypothesis of a quadratic dependence of the intensity on time was studied in [1].

Passing to the problem of estimating the instantaneous intensity of a non-stationary Poisson process, it is necessary to remember the following. A non-stationary Poisson process can be defined as a random process $\{N(t); t \geqslant 0\}$, $N(0) = 0$, with independent increments, where the increment $N(t) - N(s)$ has a Poisson distribution with parameter $\Lambda(t) - \Lambda(s)$, where $\Lambda(t) = \mathbb{E}(N(t))$ is a nondecreasing function, $\Lambda(0) = 0$. The function $\Lambda(t)$ is often called the *leading function* or the *compensator* of the Poisson process. Under the additional assumption that

$$\Lambda(t) = \int_0^t \lambda(s)\, ds$$

for some nonnegative function $\lambda(s)$, it is easy to provide a simple physical interpretation for this new function $\lambda(s)$: the probability of a single jump of the Poisson process on the interval $[t, t+h)$ equals $\lambda(t)h+o(h)$, and the probability of no jump equals $1-\lambda(t)h+o(h)$. The quantity $\lambda(t)$ is called the instantaneous intensity at the instant $t \geqslant 0$. Non-stationary Poisson processes have various applications in engineering modelling, in physics, biology, sociology, and many others. Various approaches to the non-parametric estimation of the instantaneous intensity function $\lambda(t)$ were proposed in [12], [8] on the basis of kernel functions, in [9] by means of Reproducing Kernel Hilbert Spaces, in [4] by methods of asymptotic analysis.

Recent monograph [10] richly presented results relating both to parametric estimation of the intensity of a non-stationary Poisson process and to methods of estimating the mean function $\Lambda(t)$, as well as some goodness–of–fit tests. Moreover, the exposition of the general theory for a non-stationary Poisson process is illustrated by examples with periodic intensities, $\lambda(t+\tau) = \lambda(t)$ for all $t \geqslant 0$ for some $\tau > 0$. The main input data in the book are considered to be either one realization $\{N(t); 0 \leqslant t \leqslant \Theta_n\}$ with growing $\Theta_n \to \infty$, or n independent realizations $\{N_j(t); 0 \leqslant t \leqslant \tau\}$, $j = 1, 2, \ldots, n$. For the case of periodic intensities one chooses $\Theta_n = n\tau$. Apparently, while acknowledging the richness of obtained specific practical results and the diversity of the considered statistical problems, we can make a remark about when this approach may lead to insufficient use of all available data, at least from "physical considerations". Namely, if the period of the intensity function is comparable with the mathematical expectation of the interval between events, or even smaller than it, then a large portion of the realizations turns out to be "empty". Also, cutting a non-stationary Poisson process into periods, even if statistically independent, may lead to the loss of important information about the function $\lambda(t)$ because of breaks into unrelated parts of a single integral interval between events that fall into different periods.

In the present work we point out one unexpected property of a non-stationary Poisson process, which allows statistical analysis even of such "rare" events. Here the main data are not independent realizations of the process $\{N(t); t \geqslant 0\}$, but the instants τ_i, $i = 1, 2, \ldots, n$, themselves.

2. One property of point process and its possible use. Summary of new results

Without loss of generality, we assume that the intensity $\lambda(t)$ is periodic with period $\tau = 1$ (this can always be achieved by a change of time), i.e., $\lambda(t+1) = \lambda(t)$ for all $t \geqslant 0$. Note that it is not required that the number 1 be the smallest one with this property. We also assume that the instantaneous intensity function is separated from zero, that is, $\lambda(t) \geqslant \delta$ for some $\delta > 0$.

Let $T_0 = 0$ and let T_i, $i = 1, 2, \ldots$ be the jump times of the process $\{N(t); t \geqslant 0\}$. We shall also need the following two sequences of random variables: the sequence of fractional parts $X_i = \{T_i\}$, $i = 0, 1, \ldots$, and the sequence of integer parts of the intervals between jumps $M_i = [T_i - T_{i-1}]$, $i = 1, 2, \ldots$

Theorem 2.1. *The sequence $\{X_i; i = 0, 1, \ldots\}$ is a general Markov chain on the interval $[0, 1]$ with transition density of the form*

$$p(u, v) = \begin{cases} \dfrac{\lambda(v) e^{-(\Lambda(1+v)-\Lambda(u))}}{1 - e^{-\Lambda(1)}}, & 0 \leqslant v \leqslant u \leqslant 1, \\ \dfrac{\lambda(v) e^{-(\Lambda(v)-\Lambda(u))}}{1 - e^{-\Lambda(1)}}, & 0 \leqslant u < v \leqslant 1. \end{cases} \tag{2.1}$$

The corresponding invariant probability distribution is given by a density $\pi(u) = \dfrac{\lambda(u)}{\Lambda(1)}$.

Theorem 2.1 relates the instantaneous intensity $\lambda(t)$ on a period of the original nonhomogeneous Poisson process to the stationary measure of a certain Markov chain. Moreover, the normalizing coefficient $\Lambda(1)$ is also determined by the values of the intensity on this period. This observation allows us to propose procedures of statistical inference concerning this, usually unknown, instantaneous intensity. Let us list some possible statistical applications of the indicated property of a non-homogeneous Poisson process.

1. *Estimation of the unknown intensity $\lambda(t)$, $0 \leqslant t < 1$.* To solve this problem it suffices, by the available realization $(t_1, t_2, \ldots, t_n)$ of the random vector $(T_1, T_2, \ldots, T_n)$ of a pre-chosen length n, to estimate the stationary distribution density $\hat{\pi}(t)$, $0 \leqslant t < 1$, of the Markov chain with transition probability given by function (2.1) and, moreover, to estimate the normalizing constant $\Lambda(1)$. And if as estimates for $\hat{\pi}(t)$ it is natural to consider the empirical distribution density or a kernel density estimate, then an estimate for $\Lambda(1)$ can be obtained on the base of the following statement.

Lemma 2.1. *The random variables M_i, $i = 1, 2, \ldots$ are independent and distributed according to the geometric law:*

$$\mathbb{P}(M_i = m) = e^{-m\Lambda(1)}(1 - e^{-\Lambda(1)}), \qquad m = 0, 1, \ldots. \tag{2.2}$$

It is not difficult to verify that the method of moments estimator reads

$$\widehat{\Lambda(1)} = \ln(1 + \overline{m}) - \ln \overline{m},$$

where

$$\overline{m} = \frac{1}{n} \sum_{i=1}^{n} [t_i - t_{i-1}].$$

Then the instantaneous intensity can be estimated as $\hat{\lambda}(t) = \widehat{\Lambda(1)}\hat{\pi}(t)$, $0 \leqslant t < 1$.

2. *Testing goodness–of–fit with a given intensity function* $\lambda(t)$. The distribution function corresponding to the stationary density $\frac{\lambda(t)}{\Lambda(1)}$ is $\frac{\Lambda(t)}{\Lambda(1)}$, $0 \leqslant t \leqslant 1$. Exactly with this function we shall compare the sample distribution function constructed from the realization $(x_1, x_2, \ldots, x_n)$ of the random vector $(X_1, X_2, \ldots, X_n)$.

We introduce the empirical process

$$w_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} I(X_n < t) - \frac{\Lambda(t)}{\Lambda(1)}\right), \qquad 0 \leqslant t \leqslant 1.$$

The empirical process $\{w_n(t); 0 \leqslant t \leqslant 1\}$ is fundamental in constructing such popular goodness-of-fit tests as Pearson χ^2 test, the Kolmogorov test, the Cramér — von Mises ω^2 test, and the Anderson — Darling test (see [2], [5]). In the classical theory, where the sample values X_1, X_2, …, X_n are independent, weak convergence of the joint distributions for the increments of the empirical process to a Gaussian distribution with zero mean vector and a covariance matrix of known form is proved. It should be mentioned that also for the general Markov chain $\{X_i; i = 0, 1, \ldots\}$ introduced above the result on the limiting distribution for the empirical process remains valid. Let us state this result.

We consider the partition

$$[0,1] = [z_0, z_1) \cup [z_1, z_2) \cup \ldots \cup [z_{r-1}, z_r], \qquad 0 = z_0 < z_1 < \ldots < z_r = 1,$$
$$\Delta_j = \frac{(\Lambda(z_j) - \Lambda(z_{j-1}))}{\Lambda(1)}.$$

Here the quantity Δ_j is the probability assigned to the interval $[z_{j-1}, z_j)$ by the stationary distribution.

Theorem 2.2. *As* $n \to \infty$, *the distribution of the random vector*

$$(w_n(z_1) - w_n(z_0), w_n(z_2) - w_n(z_1), \ldots, w_n(z_r) - w_n(z_{r-1}))$$

converges weakly to the r*-dimensional normal distribution with zero mean and covariance matrix* $\Sigma = (\sigma_{j,l})$, *where*

$$\sigma_{j,j} = \Delta_j(1 - \Delta_j), \qquad \sigma_{j,l} = -\Delta_j\Delta_l, \quad j \neq l.$$

Thus, the limiting distribution for the increments

$$(w_n(z_1) - w_n(z_0), w_n(z_2) - w_n(z_1), \ldots, w_n(z_r) - w_n(z_{r-1}))$$

as $n \to \infty$ is the same as in the case of an independent repeated sample with distribution function $\frac{\Lambda(t)}{\Lambda(1)}$, $0 \leqslant t \leqslant 1$ (see, e.g., [2]). As it was said above, this fact is remarkable, since it is not difficult to construct examples of general Markov chains, also with values on the interval $[0, 1]$, for which the limiting distribution of the vector of increments of the empirical process will possess a different covariance matrix.

We denote

$$\nu_j = \frac{\Lambda(1)}{n}\sum_{i=1}^{n} I(z_{j-1} \leqslant X_i < z_j), \qquad j = 1, 2, \ldots, r. \tag{2.3}$$

Theorem 2.3. *The statistic*

$$\frac{n}{\Lambda(1)}\sum_{j=1}^{r}\frac{(\nu_j - (\Lambda(z_j) - \Lambda(z_{j-1})))^2}{\Lambda(z_j) - \Lambda(z_{j-1})} \tag{2.4}$$

converges in distribution to χ^2_{r-1} *as* $n \to \infty$.

This theorem is obviously implied, up to notation, from Theorem 2.2, so we leave its proof to the reader.

3. *Testing a composite hypothesis of homogeneity of a given Poisson process.* Indeed, let $\lambda(t) = \lambda = \text{const}$, then the stationary distribution must have constant density on the interval $[0, 1]$, and hence this is the standard uniform distribution. Thus, the problem has reduced to testing a simple null hypothesis about the form of the stationary distribution, which was discussed in the previous item.

It is necessary to note the main difference with the methods available in the literature: instead of observing a part of the counting process $\{N(t); 0 \leqslant t \leqslant T\}$ for some $T > 0$, a fixed number n of jump times $\{T_i; i = 1, 2, \dots, n\}$ is observed. Here, the choice of a fixed T gives a sample of random size $n = N(T)$ for the jump times, while fixing n gives a random observation time $T = T_n$.

3. Proofs of Theorems

In this section we present proofs of Theorems 2.1, 2.2 and Lemma 2.1, and also formulate and prove the necessary auxiliary propositions. The notation of Section 2 is fully preserved.

Proof of Theorem 2.1. The Markov property is implied by the fact that the distribution of the quantity $T_{i+1} - T_i$ under the condition $T_i = t$ is independent of the quantities T_{i-1}, T_{i-2}, …, T_1; recall that it has a distribution function of the form

$$\mathbb{P}(T_{i+1} - T_i < s \mid T_i = t) = 1 - \exp\Big\{-\int_t^{s+t} \lambda(u)\,du\Big\}, \qquad s \geqslant 0.$$

For $u < v$ the relations $\{T_i\} = u$ and $v \leqslant \{T_{i+1}\} < v + dv$ mean that the difference $T_{i+1} - T_i$ can take values only in the intervals

$$[v - u, v + dv - u), \quad [1 + v - u, 1 + v + dv - u), \quad [2 + v - u, 2 + v + dv - u), \quad \dots.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(X_{i+1} \in [v, v + dv) \mid X_i = u\right) \\
&\qquad = \sum_{m=0}^{\infty} \left(\exp\left(-\int_u^{u+m+(v-u)} \lambda(s)\,ds\right) - \exp\left(-\int_u^{u+m+(v-u)+dv} \lambda(s)\,ds\right)\right) \\
&\qquad = \sum_{m=0}^{\infty} \exp(-m\Lambda(1)) \exp\left(-\int_u^{v} \lambda(s)\,ds\right)\left(1 - \exp\left(-\int_v^{v+dv} \lambda(s)\,ds\right)\right) \\
&\qquad = (1 - e^{-\Lambda(1)})^{-1} \exp\left(-\int_u^{v} \lambda(s)\,ds\right)\left(1 - \exp\left(-\int_v^{v+dv} \lambda(s)\,ds\right)\right).
\end{aligned}$$

For $u > v$ the case $m = 0$ is excluded. The remaining calculations are analogous. This implies formulas (2.1) for the transition probability density.

By direct substitution it is easy to verify that the stationary density $\pi(u) = \dfrac{\lambda(u)}{\Lambda(1)}$, $0 \leqslant u \leqslant 1$, satisfies the integral equation

$$\pi(v) = \int_0^1 \pi(u) p(u,v)\, du, \qquad 0 \leqslant v \leqslant 1.$$

The proof is complete. □

To prove the limit theorems we shall have to apply the Central Limit Theorem to a Markov chain. The version of the Central Limit Theorem used in the present work assumes that the Markov chain is Harris recurrent [6]. Let us establish this property.

Theorem 3.1. *Under the assumption $\lambda(t) \geqslant \delta > 0$ for some δ and all $t \geqslant 0$, the Markov chain $\{X_i; i = 0, 1, \ldots\}$ is Harris recurrent.*

Proof. We use the following criterion of Harris recurrence: if every bounded harmonic function is a constant, then the given chain is either dissipative or Harris recurrent [6, Thm. 3.8 (ii)]. We need to exclude dissipativity [6, Def. 3.3].

Let $h(u)$, $0 \leqslant u \leqslant 1$, be a bounded harmonic function. For our Markov chain this means that it satisfies the equation

$$h(u) = \int_0^1 p(u,v) h(v)\, dv.$$

Suppose that this function is not identically constant. It is continuous, since the right–hand side reads

$$\int_0^u h(v)\lambda(v) \frac{e^{-(\Lambda(v)-\Lambda(u))}}{1-e^{-\Lambda(1)}}\, dv + \int_u^1 h(v)\lambda(v) \frac{e^{-(\Lambda(1+v)-\Lambda(u))}}{1-e^{-\Lambda(1)}}\, dv$$
$$= \frac{e^{\Lambda(u)}}{1-e^{-\Lambda(1)}} \left(\int_0^u h(v)\lambda(v) e^{-\Lambda(v)}\, dv + \int_u^1 h(v)\lambda(v) e^{-\Lambda(1+v)}\, dv \right)$$

and this is the product of two continuous functions. Let

$$\min_{0 \leqslant u \leqslant 1} h(u) = h_*, \qquad H_* = \{u \colon h(u) = h_*\},$$

then for each $u \in H_*$ we have the inequality

$$h_* = h(u) = \left(\int_{H_*} + \int_{[0,1]\setminus H_*} \right) p(u,v) h(v)\, dv = h_* + \int_{[0,1]\setminus H_*} p(u,v)(h(v) - h_*)\, dv > h_*.$$

Here we have taken into consideration that the density $p(u,v)$ is nonzero everywhere. This gives the contradiction of $h_* < h_*$.

It remains to exclude dissipativity. In order to do this, we study the properties of the potential

$$G(u, A) = \sum_{i=0}^{\infty} \mathbb{P}(X_i \in A \mid X_0 = u), \qquad A \in \mathcal{B}([0,1]), \quad 0 \leqslant u \leqslant 1,$$

where $\mathcal{B}([0,1])$ is the σ-algebra of Borel subsets in the interval $[0,1]$. Our Markov chain would be dissipative if it were possible to find a strictly positive measurable function $g(u)$, $0 \leqslant u \leqslant 1$,

such that

$$\int_0^1 g(v)G(u,dv) < \infty \quad \text{for all } u \in [0,1].$$

Let $p_i(u,v)$ be the transition probability density over i steps,

$$p_1(u,v) = p(u,v), \qquad p_{i+1}(u,v) = \int_0^1 p_i(u,t)p(t,v)\,dt.$$

Then, by induction,

$$p_{i+1}(u,v) \geqslant \int_0^1 p_i(u,t)\delta\,dt = \delta, \qquad i = 0,1,\ldots.$$

Hence, for each set $A \in \mathcal{B}([0,1])$ of positive Lebesgue measure the corresponding potential satisfies $G(u,A) = \infty$ since

$$\mathbb{P}(X_i \in A \mid X_0 = u) \geqslant \delta \int_A du = \text{const}$$

and the series diverges.

Suppose that there exists a measurable function $g(u) > 0$ such that $\int_0^1 G(u,dv)g(v) < \infty$. Then there exists a positive ε such that $g(u) \geqslant \varepsilon$ for some u and the Lebesgue measure satisfies $\text{mes}\{u\colon g(u) \geqslant \varepsilon\} > 0$. But then

$$\int_0^1 G(u,dv)g(v) \geqslant \int_{u\colon g(u)\geqslant\varepsilon} G(u,dv)g(v) \geqslant \varepsilon G(u,\{u\colon g(u)\geqslant\varepsilon\}) = \infty.$$

The obtained contradiction proves the theorem. □

The proof of Theorem 2.2 consists in applying the multi–dimensional version of the Central Limit Theorem for a Markov random sequence (not necessarily stationary). Let us present here the needed notation and the formulation of the result, since the multi–dimensional version of the Central Limit Theorem for a Markov chain has not been encountered by us in the literature, but it is easily obtained by the well–known Cramér — Wold device (see, e.g., [3, Sect. 6, Ch. 8]) from the one–dimensional Central Limit Theorem [11].

Theorem 3.2. *Let $\{\Phi_n; n = 0,1,\ldots\}$ be a Harris recurrent Markov chain with transition kernel $\tilde{P}(\cdot,\cdot)$ and invariant probability measure $\tilde{\pi}(\cdot)$, and let the functions $f_1(\cdot)$, $f_2(\cdot)$, …, $f_r(\cdot)$ from E to $\mathbb{R}$ be such that the solutions $g_j(\cdot)$ of the corresponding Poisson equation*

$$g_j(x) - \int_E g_j(y)\tilde{P}(x,dy) = f_j(x) - \int_E f_j(y)\tilde{\pi}(dy) \tag{3.1}$$

exist and the matrix $\Sigma = (\sigma_{j,k})_{j,k=\overline{1,r}}$ with elements

$$\sigma_{j,j} = \int_E \left(g_j^2(x) - \left(\int_E g_j(y)\tilde{P}(x,dy) \right)^2 \right) \tilde{\pi}(dx),$$

$$\sigma_{j,k} = \int_E \left(g_j(x)g_k(x) - \left(\int_E g_j(y)\tilde{P}(x,dy) \right) \left(\int_E g_k(y)\tilde{P}(x,dy) \right) \right) \tilde{\pi}(dx), \quad j \neq k,$$

is positive definite. Then

$$\sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{i=0}^{n-1} f_1(\Phi_i) - \int_E f_1(y)\tilde{\pi}(dy) \\ \frac{1}{n}\sum_{i=0}^{n-1} f_2(\Phi_i) - \int_E f_2(y)\tilde{\pi}(dy) \\ \vdots \\ \frac{1}{n}\sum_{i=0}^{n-1} f_r(\Phi_i) - \int_E f_r(y)\tilde{\pi}(dy) \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma).$$

Proof of Theorem 2.2. Let us apply the multi–dimensional Central Limit Theorem to the set of functions

$$f_j(u) = I(z_{j-1} \leqslant u < z_j), \quad j = 1, 2, \ldots, r.$$

We note that

$$w_n(z_j) - w_n(z_{j-1}) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{j} f_j(X_i) - \int_0^1 f_j(u)\pi(u)\,du\Big).$$

In our case,

$$\Delta_j = \int_0^1 f_j(u)\pi(u)\,du.$$

Consider the corresponding Poisson equation (3.1), which in our case reads

$$\begin{aligned} g_j(u) =& I(z_{j-1} \leqslant u < z_j) - \Delta_j \\ &+ (1 - e^{-\Lambda(1)})^{-1}\Big(\int_0^u \lambda(v) g_j(v) e^{-(\Lambda(1+v)-\Lambda(u))}\,dv + \int_u^1 \lambda(v) g_j(v) e^{-(\Lambda(v)-\Lambda(u))}\,dv\Big). \end{aligned}$$

Since the integrals on the right–hand side are continuous functions of u, the function $g_j(u)$ at the points z_{j-1} and z_j experiences a jump of magnitude 1 and -1, respectively. Differentiating with respect to u (and eliminating the sum of integrals by means of the original equation), we obtain

$$g_j'(u) = \lambda(u)(\Delta_j - I(z_{j-1} \leqslant u < z_j)).$$

For $j > 1$, taking into consideration the jump at the points z_{j-1} and z_j, we have

$$g_j(u) = \begin{cases} \Delta_j \Lambda(u), & 0 \leqslant u < z_{j-1}, \\ \Lambda(u)(\Delta_j - 1) + 1 + \Lambda(z_{j-1}), & z_{j-1} \leqslant u < z_j, \\ (\Lambda(u) - \Lambda(1))\Delta_j, & z_j \leqslant u \leqslant 1. \end{cases}$$

The second and third cases will also be used for $j = 1$. Now, having the explicit form of the functions $g_j(u)$ and noting, for example, that

$$\int_0^1 \Big(\int_0^1 p(u,v) g_j(v)\,dv\Big)^2 \pi(u)\,du = \int_0^1 (g_j(u) - I(z_{j-1} \leqslant u < z_j) + \Delta_j)^2 \pi(u)\,du,$$

after routine algebraic simplifications we complete the proof. □

BIBLIOGRAPHY

1. N.K. Bakirov, M.V. Snachev. *The test: does the input flow intensity grow in queuing system?* // Ufimskij Matem. Zh. **2**:3, 17–30 (2010). (in Russian). https://www.mathnet.ru/eng/ufa59
2. A.A. Borovkov. *Mathematical Statistics.* Lan, St.–Petersburg (2010); *English translation:* Gordon & Breach, Amsterdam (1998).
3. A.A. Borovkov. *Probability Theory.* Librokom, Moscow (2009); *English translation:* Springer, London (2013).
4. I.A. Ibragimov. *An estimation problem for the intensity density of Poisson processes* // J. Math. Sci., New York **251**:1, 88–95 (2020). https://doi.org/10.1007/s10958-020-05068-6
5. G.V. Martynov. *Omega–Square Critertions.* Nauka, Moscow (1978).
6. E. Nummelin. *General Irreducible Markov Chains and Non–Negative Operators.* Cambridge University Press. Cambridge (1984). https://doi.org/10.1017/CBO9780511526237
7. D.R. Cox, P.A.W. Lewis. *The Statistical Analysis of Series of Events.* John Wiley, New York (1966).
8. P. Diggle. *A kernel method for smoothing point process data* // J. R. Stat. Soc., Ser. C **34**:2, 138–147 (1985). https://doi.org/10.2307/2347366
9. S. Flaxman, Y.W. Teh, D. Sejdinovic. *Poisson intensity estimation with reproducing kernels* // Electron. J. Stat. **11**:2, 5081–5104 (2017). https://doi.org/10.1214/17-EJS1339SI
10. Yu. Kutoyants. *Introduction to the Statistics of Poisson Processes and Applications.* Springer, Cham (2023). https://doi.org/10.1007/978-3-031-37054-0
11. S. Meyn, R.L. Tweedie, P.W. Glynn. *Markov Chains and Stochastic Stability.* Cambridge University Press, New York (2009).
12. H. Ramlau–Hansen. *Smoothing counting process intensities by means of kernel functions* // Ann. Statist. **11**:2, 453–466 (1983). https://doi.org/10.1214/aos/1176346152

Zorine Andrei Vladimirovich,
Lobachevsky State University of Nizhny Novgorod,
Gagarin av. 23,
603022, Nizhny Novgorod, Russia
E-mail: `andrei.zorine@itmm.unn.ru`