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ESTIMATE FOR DEVIATION OF LOGARITHM OF MAXIMAL TERM IN DIRICHLET SERIES UNDER HADAMARD COMPOSITION

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Abstract. We study the problem of estimating the equivalence degree of the logarithms of the maximal terms in the Hadamard composition (the modified series) $\sum\limits_n a_n b_n e^{\lambda_n z}$ of the Dirichlet series $\sum\limits_n a_n e^{\lambda_n z}$ and $\sum\limits_n b_n e^{\lambda_n z}$ with positive exponents, the domains of convergence of which are a half–plane. A similar problem for entire Dirichlet series was first studied by A.M. Gaisin in 2003; he obtained a criterion for the stability of the maximal term $\mu(\sigma) = \max\limits_{n\geqslant 1}\{|a_n|e^{\lambda_n\sigma}\}$. This result turned out to be useful in studying the asymptotic properties of a Dirichlet series on arbitrary curves going to infinity, namely in the proof of the well–known Pólya conjecture.

As in the case of entire Dirichlet series, and in the case of series converging only in a half–plane, in problems of this type Leontiev's formulas for the coefficients play a key role. The functions of the corresponding biorthogonal system contain a factor, which is the reciprocal of the derivative of the characteristic function at the points λ_n, $n \geqslant 1$. This circumstance naturally leads to the formulation of the considered problem on a quantitative characteristic of the stability of the maximal term. This question is relevant from the point of view of its applications to the problem of the minimum of modulus of an entire transcendental function and to other related problems of analysis and complex dynamics.

In terms of the deviation estimate, we obtain a condition for the logarithm of the maximal term of a Dirichlet series on an asymptotic set to be equivalent to the logarithm of the maximal term of the modified series.

Keywords: Dirichlet series, convergence half–plane, maximal term, Hadamard composition, estimate for deviation, asymptotic set.

Mathematics Subject Classification: 30D10

1. Introduction

The stability of the maximal term of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad 0 < \lambda_n \uparrow \infty, \tag{1.1}$$

absolutely converging in the entire plane, was first studied in [4]. Moreover, the stability property of the maximal term played an important role in the proof of the Pólya problem (for more detail [4]). Later the results of [4] were extended to the case of a half–plane (see [6]–[8]). In [6] there was proved a on the stability of the maximal term of series (1.1), absolutely convergent

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in the half–plane $\Pi_0 = \{s : \operatorname{Re} s < 0\}$; work [7] was devoted to applications of this theorem to the study of the behaviour of Dirichlet series (1.1) on a curve arbitrarily approaching the boundary of the half–plane Π_0, the line of convergence. For clarity it is worth explaining the essence of the problems considered in [7] in a particular case, namely for the lacunary power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^{p_n}, \qquad 0 < p_n \uparrow \infty, \qquad p_n \in \mathbb{N}, \tag{1.2}$$

the convergence domain of which is the unit disk $D(0,1) = \{z : |z| < 1\}$. Let γ be a curve starting in $D(0,1)$ and ending on the boundary of $D(0,1)$ or asymptotically approaching it, for example, along a spiral. We consider a special modified series

$$\sum_{n=1}^{\infty} a_n Q'(p_n) z^{p_n}, \tag{1.3}$$

where

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{{p_n}^2}\right).$$

Under conditions on $\{p_n\}$ ensuring the equivalence of the logarithms of the maximal terms of series (1.2), (1.3) (i.e., under the stability of the maximal term of series (1.2)), in [7] the following fact was shown: there exists a sequence $\{\xi_n\}$, $\xi_n \in \gamma$, $|\xi_n| \to 1$, such that

$$\ln M_f(|\xi_n|) = (1 + o(1)) \ln |f(\xi_n)|,$$

where

$$M_f(r) = \max_{|z|=r} |f(z)|, \qquad 0 < r < 1.$$

Later works [1], [2], [9] were also devoted to applications of the results of [8], where there were proved theorems on stronger regularity of growth of the sum of Dirichlet series (1.1), in particular, of series (1.2).

As it can be seen, in [6] the stability of the maximal term of Dirichlet series (1.1) (or of series (1.2)) is considered in connection with specific problems investigated in [1], [2], [7]–[9]. For this reason it was sufficient to consider only the sequences $\Lambda = \{\lambda_n\}$, where λ_n are the zeroes of an entire function of a finite order. However, the problem of stability of the maximal term is also an independent problem. Therefore the conditions on the exponents λ_n can be substantially weakened.

The aim of the present article is to consider a sufficiently wide class of sequences $\Lambda = \{\lambda_n\}$ and to obtain a quantitative measure of the stability of maximal term of series (1.1) in terms of an estimate for the quantity

$$\left|1 - \frac{\ln \mu_b^*(\sigma)}{\ln \mu(\sigma)}\right|,$$

i.e., the deviation of the ratio $[\ln \mu^*(\sigma)]/[\ln \mu(\sigma)]$ from the unity, where $\mu(\sigma)$ is the maximal term of series (1.1), and $\mu_b^*(\sigma)$ is the maximal term of modified series $\sum\limits_{n=1}^{\infty} a_n b_n e^{\lambda_n s}$.

2. Definitions. Main result

Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence obeying the condition

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\lambda_n} = 0. \tag{2.1}$$

By $D_c(\Lambda)$ we denote the class of all functions F, which can be represented by Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \tag{2.2}$$

converging only in the half–plane $\Pi_c = \{s : \operatorname{Re} s < c\}$, $-\infty < c \leqslant \infty$. It follows from condition (2.1) that series (2.2) converges in the half–plane Π_c absolutely, and its sum F is an analytic function in Π_c [11].

Along with the series (2.2), we also introduce into consideration the series

$$F^*(s) = \sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s}, \qquad s = \sigma + it, \tag{2.3}$$

where $B = \{b_n\}$ is a sequence of complex numbers b_n ($b_n \neq 0$ for $n \geqslant N$), satisfying the condition

$$\lim_{n\to\infty} \frac{\ln |b_n|}{\lambda_n} = 0. \tag{2.4}$$

Then series (2.3) also converges absolutely in the half–plane Π_c, and F^* is an analytic function in this half–plane. Condition (2.4) also allows us to consider the Dirichlet series $\sum\limits_{n=N}^{\infty} a_n b_n^{-1} e^{\lambda_n s}$, absolutely convergent in the half–plane Π_c.

If F is a function defined in the half–plane Π_0 by series (2.2), and

$$G(s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s}, \tag{2.5}$$

then series (2.3) is the Hadamard composition of series (2.2) and (2.5), i.e.

$$(F * G)(s) = \sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s} = F^*(s).$$

It is clear that if $F \in D_0(\Lambda)$, then $F^* \in D_0(\Lambda)$.

Let $\mu(\sigma)$ and $\mu^*(\sigma)$ be the maximal terms of series (2.2) and (2.3), respectively. By L we denote the class of all continuous, unboundedly increasing functions on $[0, \infty)$. We consider the classes of functions [8]

$$W = \{w \in L : \int_1^{\infty} \frac{w(x)}{x^2} dx < \infty\},$$
$$\underline{W}_{\varphi} = \{w \in W : \varliminf_{t\to\infty} \varphi(t) J(t; w) = 0\},$$
$$W_{\varphi} = \{w \in W : \lim_{t\to\infty} \varphi(t) J(t; w) = 0\},$$

where $\varphi \in L$, and

$$J(t; w) = \int_t^{\infty} \frac{w(x)}{x^2} dx.$$

Let M be the class of functions Φ from L such that $x\Phi(x) < \Phi(kx)$ for $x \geqslant x_0$, where k is some constant. It is clear that all functions from M grow faster than any power x^n, $n = 1, 2, \ldots$.

To each function Φ from M we assign its inverse function φ. Then we obtain a new class of functions, which we denote by M^{-1}. Thus, the classes $M = \{\Phi\}$ and $M^{-1} = \{\varphi\}$ consist of mutually inverse functions. It is easy to show that if $\varphi \in M^{-1}$, then the function $\omega(x) = \sqrt{x}$ belongs to the class $\underline{W}_{\varphi}$.

Let $e \subset [-1,0)$ be a set measurable with respect to the Lebesgue measure. The upper De and lower de densities of the set e are the quantities [3]

$$De = \overline{\lim_{\sigma\to 0-}} \frac{\operatorname{mes}(e\cap[\sigma,0))}{|\sigma|}, \qquad de = \underline{\lim_{\sigma\to 0-}} \frac{\operatorname{mes}(e\cap[\sigma,0))}{|\sigma|}.$$

If $0 \leqslant q < 1$, $De \leqslant q$, then the set e is called the C_q–set, and for $de \leqslant q$ it is the c_q–set. Let

$$\underline{D}_0(\Phi) = \left\{ F \in D_0(\Lambda) : \sup_{\tau>0} \underline{\lim_{\sigma\to\ 0-}} \frac{\ln\mu(\sigma)}{|\sigma|\Phi(\frac{\tau}{|\sigma|})} > 0 \right\},$$

where $\mu(\sigma)$ is the maximal term of series (2.2).

We shall say that the maximal term $\mu(\sigma)$ of series (2.2) is $B(d)$- $(B(D))$-stable [6] if, as $\sigma \to 0-$ outside some set $e \subset [-1,0)$ of zero lower density de (zero upper density De), the asymptotic identity holds

$$\ln\mu(\sigma) = (1+o(1))\ln\mu^*(\sigma).$$

The set $A = [-1,0)\setminus e$ is called an asymptotic set.

We shall say that the sequence $\{b_n\}$, $b_n \neq 0$ for $n \geqslant N$ is W_φ-normal if there exists a function $\theta \in W_\varphi$ such that

$$-\ln|b_n| \leqslant \theta(\lambda_n), \qquad n \geqslant N.$$

Let $n(t) = \sum\limits_{\lambda_n \leqslant t} 1$ be the counting function of sequence Λ, and let $n_l(t)$ be the least concave majorant of the function $\ln n(t)$. By virtue of condition (2.1), $n_l(t) = o(t)$ as $t \to \infty$.

In [8] the following theorem was proved.

Theorem 2.1. *Let Φ be some fixed function from the class M, and φ the function inverse to Φ. Let $n_l \in W_\varphi$, and $B = \{b_n\}$ a sequence satisfying the condition*

$$|b_n| + \frac{1}{|b_n|} \leqslant e^{w(\lambda_n)}, \qquad n \geqslant N, \tag{2.6}$$

where $w \in W$ and such that

$$\lim_{n\to\infty} \frac{\varphi(\lambda_n)\ln|b_n|}{\lambda_n} = 0. \tag{2.7}$$

Then estimates (2.6) with some function $w \in \underline{W}_\varphi$ is a sufficient condition ensuring that for each function $F \in \underline{D}_0(\Phi)$ the maximal term of its series (2.2) is $B(d)$-stable; for W_φ-normal sequence $\{b_n\}$ this condition is also necessary.

In [6] for the sequence Λ a too restrictive condition

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln\lambda_n} < \infty$$

was imposed. This means that $\ln n = \ln n(\lambda_n) \leqslant a\ln\lambda_n$, $n \geqslant 1$. Hence, $n_l(t) \leqslant a\ln t$ and therefore $n_l \in W_\varphi$. The converse is obviously not true.

We also note that matching condition (2.7) in Theorem 2.1 is essential [6].

Let $p > 0$ be any number, $\gamma \geqslant 0$, $M_{p\gamma}$ the class of functions $\Phi \in L$, possessing the property: for all $A > 1$, $B > 1$ there exists $K > 1$ such that for $x \geqslant x_1$

$$AB^{(1+\gamma)p}x^{(1+\gamma)p}\Phi(x) \leqslant \Phi(Kx).$$

We note that for $p = 1$, $\gamma = 0$ the class $M_{p\gamma}$ coincides with the class M given above.

Let $p > 0$,

$$\underline{D}_0^p(\Phi) = \left\{ F \in D_0(\Lambda) : \sup_{\tau>0} \varliminf_{\sigma \to\ 0-} \frac{\ln \mu(\sigma)}{|\sigma| \Phi\left(\frac{\tau}{|\sigma|^{\frac{1}{p}}}\right)} > 0 \right\}.$$

It is clear that $\underline{D}_0^1(\Phi) = \underline{D}_0(\Phi)$.

We introduce the following class of majorants:

$$\underline{W}_\varphi^p = \{w \in W : \varliminf_{t\to\infty} \varphi^p(t) J(t;w) = T_p^- < \infty\},$$
$$\overline{W}_\varphi^p = \{w \in W : \varlimsup_{t\to\infty} \varphi^p(t) J(t;w) = T_p^+ < \infty\}.$$

A sequence $\{b_n\}$ ($b_n \neq 0$, $n \geqslant N$) is called W_φ^p–normal if there exists a function $\theta \in W_\varphi^p$ such that

$$-\ln |b_n| \leqslant \theta(\lambda_n), \qquad n \geqslant N.$$

Our main result is the following theorem.

Theorem 2.2. *Let $p > 0$, $\beta > 0$, $\gamma > 0$, $\nu = 1 + \beta + \gamma$, $\alpha \in (0, 1]$, $\Phi \in M_{p\gamma}$, $n_l(t)$ be the function introduced above. Suppose that $n_l \in W_\varphi^{p\nu}$, and moreover $n_l(t)\varphi^{p\nu}(t) = o(t)$ as $t \to \infty$.*

Let $\{b_n\}$ be a sequence ($b_n \neq 0$ for $n \geqslant N$), satisfying the condition

$$|\ln |b_n|| \leqslant w(\lambda_n), \quad n \geqslant N, \tag{2.8}$$

where $w \in W$, and moreover

$$\lim_{n\to\infty} \frac{\varphi^{p\nu}(\lambda_n) \ln |b_n|}{\lambda_n} = 0. \tag{2.9}$$

Suppose that $|\sigma|^\gamma \ln \mu(\sigma) \uparrow$, $|\sigma|^\gamma \ln \mu^*(\sigma) \uparrow$ as $\sigma \to 0-$. If for some majorant $w \in \underline{W}_\varphi^{p\nu}$ estimates (2.8) hold, then for any function $F \in \underline{D}_0^p(\Phi)$ as $\sigma \to 0-$ outside some c_q–set $e_{\alpha\beta\gamma}^p \subset [-1, 0)$, $q = q(\alpha) = O(\alpha)$, $\alpha \to 0$, the estimate holds

$$\left| 1 - \frac{\ln \mu^*(\sigma)}{\ln \mu(\sigma)} \right| \leqslant \text{Const}\, |\sigma|^\gamma. \tag{2.10}$$

Question: Whether condition (2.8) with a majorant $w \in \underline{W}_\varphi^{p\nu}$ is also necessary for estimate (2.10) to hold outside some c_q–set $e_{\alpha\beta\gamma}^p \subset [-1, 0)$ for any function $F \in \underline{D}_0^{p\nu}$?

For each W_φ^p–normal sequence $\{b_n\}$ for which condition (2.8) with a majorant w from $\underline{W}_\varphi^{p\nu}$ is not satisfied, there exists an example of a function $F \in \underline{D}_0^{p\nu}(\Phi)$, $\nu > 1$, for which inequality (2.10) fail; this example is constructed in exactly the same way as the corresponding example from the class $\underline{D}_0(\Phi)$ [8]. However, the question of the existence of an analogous example for the class $\underline{D}_0^p(\Phi)$ still remains open.

3. Preliminaries

3.1. Convex Newton polygon. To prove Theorem 2.2, we shall need some properties of the maximal term of Dirichlet series. The geometric description of the maximal term of a power series or Dirichlet series defining an entire function in terms of the convex Newton polygon is well–known (see, e.g., [8]). An analogous interpretation of the maximal term of a power series converging only in the unit disk is given also in a series of works (see, e.g., [12]).

Let us construct the convex Newton polygon for Dirichlet series (2.2) absolutely convergent only in the half–plane Π_0. Assuming $\sup\limits_n |a_n| = \infty$ (one may also assume that $a_1 \neq 0$), we mark

on the plane the points $P_n = (\lambda_n, g_n)$, where $g_n = -\ln|a_n|$; if $a_n = 0$, we set $g_n = \infty$. Since $F \in D_0(\Lambda)$, we have

$$\overline{\lim_{n\to\infty}} \frac{\ln|a_n|}{\lambda_n} = 0. \tag{3.1}$$

Taking this into consideration, by $Q(F)$ we denote the convex hull of the points P_n, $n \geqslant 1$. Let $\gamma(x) = \inf\{y : (x,y) \in Q(F)\}$. The line described by the equation $y = \gamma(x)$, $x \geqslant \lambda_1$, is called the diagram or convex Newton polygon [8]. It follows from (3.1) that the Newton diagram, which we denote by $L(F)$, is a downward convex polygonal line.

Let $F \in D_0(\Lambda)$,

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad \sup_n |a_n| = \infty,$$

and

$$\widehat{F}(s) = \sum_{n=1}^{\infty} T_n e^{\lambda_n s}, \qquad T_n = e^{-\gamma(\lambda_n)}, \quad n \geqslant 1. \tag{3.2}$$

The function $\widehat{F}$ is called the Newton majorant of the function $F \in D_0(\Lambda)$.

Let $\gamma(\lambda_n) = G_n$, $n \geqslant 1$. Then $(\lambda_n, G_n) \in L(F)$. For an infinite set of values λ_n, in particular, for the abscissas λ_{n_i}, $i \geqslant 1$, $n_1 = 1$, of all vertices of the polygon $L(F)$ we have $G_n = -\ln|a_n|$ (n_i are called the principal indices). We note that the point $P_n = (\lambda_n, -\ln|a_n|)$ lies either on the polygon $L(F)$ (the point P_{n_i} necessarily lies on the polygon), or above it. The slope of the segment joining the vertices P_{n_i} and $P_{n_{i+1}}$ of the polygon $L(F)$ is equal to

$$R_i = \frac{G_{n_{i+1}} - G_{n_i}}{\lambda_{n_{i+1}} - \lambda_{n_i}}, \qquad i \geqslant 1, \quad \lambda_1 = 1.$$

It is clear that $R_i \uparrow 0$ as $i \to \infty$. Hence, for $R_{i-1} \leqslant \sigma < R_i$ the central index $\nu(\sigma) = n_i = \text{const}$, and $\ln\mu(\sigma) = \ln|a_{n_i}| + \lambda_{n_i}\sigma$ [8], [11, Ch. II, §6, item 2]. In particular this implies that $\mu(\sigma) = \widehat{\mu}(\sigma)$, $\nu(\sigma) = \widehat{\nu}(\sigma)$, where $\widehat{\mu}(\sigma)$ and $\widehat{\nu}(\sigma)$ are the maximal term and the central index of series (3.2). It is also known that the function $\ln\mu(\sigma)$, as a convex function, is continuous, and for $\sup_n |a_n| = \infty$ it increases unboundedly on the interval $[-1, 0)$ [8].

Let us determine under which conditions the functions $a(\sigma) = |\sigma|^\gamma \ln\mu(\sigma)$ and $a^*(\sigma) = |\sigma|^\gamma \ln\mu^*(\sigma)$, $\gamma \geqslant 0$, are increasing on some half-interval $[\sigma_0, 0) \subset [-1, 0)$ (the case $\gamma = 0$ is obvious, since $\ln\mu(\sigma)$ is a convex increasing function).

Let $\nu(\sigma)$ be the central index of series (2.2), and $k(\sigma)$ the central index of series (2.3). For $R_{i-1} \leqslant \sigma < R_i$ we have $\nu(\sigma) = n_i$, $\ln\mu(\sigma) = -G_{n_i} + \lambda_{n_i}\sigma$, where $G_{n_i} = -\ln|a_{n_i}|$ ($G_{n_i} \downarrow -\infty$ as $i \to \infty$). Then $a(\sigma) = -G_{n_i}|\sigma|^\gamma - \lambda_{n_i}|\sigma|^{1+\gamma}$. This yields that for

$$R_i \leqslant \frac{G_{n_i}}{\lambda_{n_i}} \frac{\gamma}{1+\gamma}$$

the function $a(\sigma)$ is increasing on $[R_{i-1}, R_i)$, $i \geqslant 2$. The condition for $a(\sigma)$ to increase on the half–interval $[R_1, 0)$ can be written as follows:

$$\frac{G_{n_{i+1}}}{G_{n_i}} \geqslant \frac{\gamma}{1+\gamma}\left(\frac{\lambda_{n_{i+1}}}{\lambda_{n_i}} + \frac{1}{\gamma}\right) \stackrel{\text{def}}{=} c_i,$$

for $i \geqslant 1$, or, equivalently,

$$\frac{\ln|a_{n_{i+1}}|}{\ln|a_{n_i}|} \geqslant c_i, \quad i = 1, 2, \ldots.$$

The corresponding condition for the function $a^*(\sigma)$ to increase on the half–interval $[R_1^*, 0)$ ($\{R_i^*\}$, $R_i^* \uparrow 0$, $i \to \infty$ being the analogous sequence for the series (2.3)) is

$$\frac{\ln|a_{k_{i+1}} b_{k_{i+1}}|}{\ln|a_{k_i} b_{k_i}|} \geqslant \frac{\gamma}{1+\gamma}\left(\frac{\lambda_{k_{i+1}}}{\lambda_{k_i}} + \frac{1}{\gamma}\right) \stackrel{\text{def}}{=} c_i^*$$

for $i \geqslant 1$, where $k_i = k(\sigma)$ for $R_{i-1}^* \leqslant \sigma < R_i^*$.

3.2. Borel — Nevanlinna type lemma. Let L be the class of all continuous, unboundedly increasing on $[0, \infty)$ functions, and W the convergence class introduced above. For a function w from W we introduce the notation

$$J(t; w) = \int_t^\infty \frac{w(x)}{x^2} dx.$$

By H we denote the subclass of L consisting of functions Φ obeying [3]

$$\varphi(2t) \leqslant c\varphi(t), \quad 0 < c < \infty; \qquad \varphi(t) t^{-1} \ln t = o(1), \quad t \to \infty,$$

where the function φ is the inverse of Φ.

The following Borel — Nevanlinna theorem is well–known and is widely used for studying the asymptotic properties of functions defined by Dirichlet series [5], [10].

Theorem 3.1 (Borel — Nevanlinna theorem). *Let $u(r)$ be a continuous function on $[r_0, \infty)$, which is nondecreasing and tending to $+\infty$ as $r \to \infty$. Let $\varphi(u)$ be a continuous, positive function defined on $[u_0, +\infty)$ ($u_0 = u(r_0)$), nonincreasing, tending to 0 as $u \to \infty$, and moreover*

$$\int_{u_0}^\infty \varphi(u)\, du < \infty.$$

Then for each $r > r_0$, except possibly for a set of finite measure[1]*, the inequality holds*

$$u(r + \varphi(u(r))) < u(r) + 1.$$

There are different versions of this theorem [5]. Here we shall need statements of this type for the half–interval $[-1, 0)$.

In [3] the following version of the Borel — Nevanlinna theorem was proved.

Lemma 3.1. *Let $u(t)$ be a continuous, nondecreasing on $[-1, 0)$ function, $u(t) \to \infty$ as $t \to 0-$, and moreover for some function Φ from the class H the condition holds*

$$\overline{\lim_{t \to 0-}}\, e^{u(t)} \Phi^{-1}\left(\frac{1}{|t|}\right) > 0.$$

If for some function w from W the asymptotic relation holds

$$\varphi(t) J(t; w) = o(1), \quad t \to \infty,$$

then as $t \to 0-$ outside some set $e \subset [-1, 0)$, $de = 0$, the identity holds

$$u(t + \delta(t)) = u(t) + o(1), \quad \delta(t) = \frac{w(v(t))}{v(t)},$$

where $v = v(t)$ is the solution of the equation

$$w(v) = e^{u(t)}. \tag{3.3}$$

In the work [6] a more general variant of Lemma 3.1 was proved (see below), in which we refine the main asymptotic identity, namely, we obtain a specific estimate for the quantity $o(1)$.

[1] In theorems of Borel — Nevanlinna type the exceptional sets $E \subset \mathbb{R}_+$ or $e \subset [-1, 0)$ in the case of a half-interval are the union of intervals $[a_n, a_n']$, pairwise having no common interior points [5]).

Lemma 3.2. *Let $u(t)$ be a continuous, nondecreasing on $[-1,0)$ function, $u(t)\to\infty$ as $t\to 0-$. Let $w\in W$, and let $v=v(t)$ be the solution of equation (3.3).*

If

$$\frac{w(v(t))}{|t|v(t)}=o(1),\quad t\to 0-,$$

and for some sequence $\{\tau_j\}$, $\tau_j\uparrow 0$, the condition holds

$$\lim_{\tau_j\to 0-}\frac{1}{|\tau_j|}J(v_j;w)=0,\qquad v_j=v(\tau_j),\tag{3.4}$$

then as $t\to 0-$ outside some set $e\subset[-1,0)$,

$$\operatorname{mes}(e\cap[\tau_j,0))=o(|\tau_j|),\quad \tau_j\to 0-,$$

the asymptotic identity holds

$$u\left(t+\frac{w(v(t))}{v(t)}\right)=u(t)+o(1).$$

The proof of Theorem 2.2 is based on a qualitatively new version of Lemma 3.2 of Borel — Nevanlinna type.

Lemma 3.3. *Let $u(t)$ be a continuous, nondecreasing on $[-1,0)$ function, $u(t)\to\infty$ as $t\to 0-$, $w\in W$, and let $v=v(t)$ be the solution of the equation*

$$w(v)=e^{u(t)}.\tag{3.5}$$

Fix $\beta\geqslant 0$, $\gamma\geqslant 0$. Let $\nu=1+\beta+\gamma$, $\alpha\in(0,1]$.

If

$$\frac{w(v(t))}{|t|^{1+\gamma}v(t)}=o(1),\quad t\to 0-,$$

and for some sequence $\{\tau_j\}$, $\tau_j\uparrow 0$,

$$\lim_{\tau_j\to 0-}\frac{1}{|\tau_j|^{\nu}}J(v_j;w)=\widehat{T}_{\nu}^{-}<\infty,\qquad v_j=v(\tau_j),\qquad \nu=1+\beta+\gamma,\tag{3.6}$$

then as $t\to 0-$ outside some c_q–set $e_{\alpha\beta\gamma}\subset[-1,0)$, $q=q(\alpha)=O(\alpha)$, $\alpha\to 0$,

$$\overline{\lim_{\tau_j\to 0-}}\frac{\operatorname{mes}(e_{\alpha\beta\gamma}\cap[\tau_j,0])}{|\tau_j|}\leqslant q<1$$

for $0<\alpha\leqslant\alpha_0<1$, the estimate holds[1]

$$u\left(t+d|t|^{-\gamma}\frac{w(v(t))}{v(t)}\right)<u(t)+\frac{|t|^{\beta}}{\alpha}.$$

We note that condition (3.4) from Lemma 3.2 in Lemma 3.3 is replaced by $\widehat{T}_{\nu}^{-}<\infty$.

Proof. We have $w\in W$, $w(v(t))=o(|t|^{1+\gamma}v(t))$, $t\to 0-$, and condition (3.6) holds. Let us show that outside some c_q–set $e_{\alpha\beta\gamma}\subset[-1,0)$ the inequality holds

$$u(t+d|t|^{-\gamma}\delta(t))<u(t)+\frac{|t|^{\beta}}{\alpha},\tag{3.7}$$

where $\delta(t)=\dfrac{w(v(t))}{v(t)}$, $0<d<\infty$.

[1]Since $w(v(t))=o(|t|^{1+\gamma}v(t))$ as $t\to 0-$, the expression in parentheses on the left–hand side of this inequality is negative for $t_0\leqslant t<0$.

Let $e_{\alpha\beta\gamma}$ be a closed set contained in $[-1,0)$ on which

$$u(t+d|t|^{-\gamma}\delta(t)) \geqslant u(t)+\frac{|t|^{\beta}}{\alpha}. \tag{3.8}$$

Let $e(t)=e_{\alpha\beta\gamma}\cap[t,0)$. If $e(t)=\emptyset$ for some $t\in[-1,0)$, then statement is proved. Otherwise we let

$$t_1=\inf\{t: t\in e_{\alpha\beta\gamma}\},\qquad t_1'=\inf\left\{t: u(t)=u(t_1)+\frac{|t_1|^{\beta}}{\alpha}\right\}.$$

Then $0<t_1'-t_1\leqslant d|t_1|^{-\gamma}\delta(t_1)$.

Let

$$t_2=\inf\{t: t\in e(t_1')\},\qquad t_2'=\inf\left\{t:\ u(t)=u(t_2)+\frac{|t_2|^{\beta}}{\alpha}\right\}.$$

Then, in view of (3.8), we obtain that

$$0<t_2'-t_2\leqslant d|t_2|^{-\gamma}\delta(t_2),\qquad u(t_2)-u(t_1)\geqslant\frac{|t_1|^{\beta}}{\alpha}.$$

Arguing by induction, we find sequences $\{t_n\}$, $\{t_n'\}$ such that

$$0<t_{n+1}'-t_{n+1}\leqslant d|t_{n+1}|^{-\gamma}\delta(t_{n+1}),\qquad u(t_{n+1})-u(t_n)\geqslant\frac{|t_n|^{\beta}}{\alpha}, \tag{3.9}$$

and moreover

$$e_{\alpha\beta\gamma}\subset\bigcup_{n=1}^{\infty}[t_n,t_n'].$$

Let $t\in(t_n,t_{n+1}]$. Then, taking into consideration (3.9), we find

$$\operatorname{mes}(e_{\alpha\beta\gamma}\cap[t,0))=\begin{cases}t_n'-t+\sum\limits_{i=n+1}^{\infty}d|t_i|^{-\gamma}\delta_i, & \text{if}\quad t\in(t_n,t_n'],\\ \sum\limits_{i=n+1}^{\infty}d|t_i|^{-\gamma}\delta_i, & \text{if}\quad t\in(t_n',t_{n+1}],\end{cases} \tag{3.10}$$

where $\delta_i=\dfrac{w(v_i)}{v_i}$, $v_i=v(t_i)$, $i\geqslant 1$.

If $2v_i\leqslant v_{i+1}$, then

$$\delta_i\leqslant w(v_i)2\int_{v_i}^{v_{i+1}}\frac{dx}{x^2}\leqslant 2\int_{v_i}^{v_{i+1}}\frac{w(x)}{x^2}dx\leqslant 2\left[\frac{w(v_{i+1})}{v_{i+1}}-\frac{w(v_i)}{v_i}\right]+4\int_{v_i}^{v_{i+1}}\frac{w(x)}{x^2}dx.$$

Otherwise, due to (3.5), (3.9) we obtain

$$\begin{aligned}\delta_i&\leqslant\alpha|t_i|^{-\beta}\frac{w(v_i)}{v_i}(u(t_{i+1})-u(t_i))\leqslant\alpha|t_i|^{-\beta}\frac{w(v_i)}{v_i}(\ln w(v_{i+1})-\ln w(v_i))\\ &\leqslant 2\alpha|t_i|^{-\beta}\int_{v_i}^{v_{i+1}}\frac{w(x)}{x}d\ln w(x)=2\alpha|t_i|^{-\beta}\left(\frac{w(v_{i+1})}{v_{i+1}}-\frac{w(v_i)}{v_i}+\int_{v_i}^{v_{i+1}}\frac{w(x)}{x^2}dx\right).\end{aligned}$$

Hence, the estimate holds

$$\delta_i\leqslant 2d_i\left(\frac{w(v_{i+1})}{v_{i+1}}-\frac{w(v_i)}{v_i}+2\int_{v_i}^{v_{i+1}}\frac{w(x)}{x^2}dx\right),\quad i\geqslant 1, \tag{3.11}$$

where $d_i=\max(1,\alpha|t_i|^{-\beta})$.

Let us now estimate the quantity $\operatorname{mes}(e_{\alpha\beta\gamma}\cap[\tau_j,0))$. If $t_n<\tau_j\leqslant t_{n+1}$, then it follows from (3.10), (3.11) that

$$\begin{aligned}&\frac{1}{|\tau_j|}\operatorname{mes}(e_{\alpha\beta\gamma}\cap[\tau_j,0))\leqslant\frac{t_n'-\tau_j}{|\tau_j|}+\frac{4dd_j|\tau_j|^{-\gamma}}{|\tau_j|}J(v_j;w),\quad&&\text{if }\tau_j\in(t_n,t_n'];\\&\frac{1}{|\tau_j|}\operatorname{mes}(e_{\alpha\beta\gamma}\cap[\tau_j,0))\leqslant\frac{4dd_j|\tau_j|^{-\gamma}}{|\tau_j|}J(v_j;w),&&\text{if }\tau_j\in(t_n',t_{n+1}].\end{aligned}\tag{3.12}$$

Recall that here $v_j=v(\tau_j)$, $v=v(t)$, is the solution of equation (3.5). Since

$$0<t_n'-t_n<d|t_n|^{-\gamma}\frac{w(v(t_n))}{v(t_n)},$$

as $t_n\to0-$ we have

$$0<1-\left|\frac{t_n'}{t_n}\right|<d\frac{w(v(t_n))}{|t_n|^{1+\gamma}v(t_n)}\to0.$$

Therefore, $t_n'=(1+o(1))t_n$ as $t_n\to0-$. Hence, $t_n'-\tau_j=o(|\tau_j|)$ as $\tau_j\to0-$. Hence, in view of condition (3.6), by (3.12) we obtain that for $0<\alpha\leqslant\alpha_0<1$ outside the set $e_{\alpha\beta\gamma}\subset[-1,0)$,

$$de_{\alpha\beta\gamma}\leqslant\varlimsup_{\tau_j\to0-}\frac{\operatorname{mes}(e_{\alpha\beta\gamma}\cap[\tau_j,0))}{|\tau_j|}\leqslant4\alpha d\lim_{\tau_j\to0-}\frac{1}{|\tau_j|^{\nu}}J(v_j;w)=4\alpha d\widehat{T}_{\nu}^{-}\leqslant q<1,$$

the required estimate holds

$$u(t+d|t|^{-\gamma}\delta(t))<u(t)+\frac{|t|^{\beta}}{\alpha},\quad\beta\geqslant0,\quad\gamma\geqslant0.$$

The proof is complete.

□

4. Proof of Theorem 2.2

Proof of Theorem 2.2. Let $\nu=1+\beta+\gamma$ and let condition (2.9) and

$$|b_n|+\frac{1}{|b_n|}\leqslant e^{w(\lambda_n)},\quad n\geqslant N,\tag{4.1}$$

hold, where $w=w(x)$ is some function from $\underline{W}_{\varphi}^{p\nu}$. We can assume that $w(x)\varphi^{p\nu}(x)=o(x)$ as $x\to\infty$ (this follows from (2.9), if the majorant in (4.1) is chosen in a suitable way). Let $w^*(x)=n_l(x)+w(x)$. It is clear that $w^*\in\underline{W}_{\varphi}^{p\nu}$, $w^*(x)\varphi^{p\nu}(x)=o(x)$ as $x\to\infty$. Let $v=v(\sigma)$ be the solution of the equation

$$w^*(v)=|\sigma|^{\gamma}\ln\mu(\sigma),\quad\gamma>0.\tag{4.2}$$

It is clear that $v(\sigma)\uparrow\infty$ as $\sigma\uparrow0$. Since $w^*\in\underline{W}_{\varphi}^{p\nu}$, there exists a sequence $\{\tau_j\}$, $\tau_j\uparrow0$, such that

$$\lim_{v_j\to\infty}\varphi^{p\nu}(v_j)J(v_j;w^*)=T_{p\nu}^{-}<\infty,\tag{4.3}$$

where $v_j=v(\tau_j)\to\infty$ as $\tau_j\to0-$,

$$J(v_j;w^*)=\int\limits_{v_j}^{\infty}\frac{w^*(x)}{x^2}dx.$$

Further, equation (4.2) can be written in the form

$$w^*(v)=e^{u(\sigma)},\qquad u(\sigma)=\ln|\sigma|^{\gamma}+\ln\ln\mu(\sigma).\tag{4.4}$$

By the assumptions of the theorem, $u(\sigma) \uparrow \infty$ as $\sigma \uparrow 0$. Further, since $F \in \underline{D}_0^p(\Phi)$, then, moreover, for some $\tau > 0$

$$\varliminf_{\sigma \to\ 0-} \frac{e^{u(\sigma)}}{|\sigma|\Phi\left(\frac{\tau}{|\sigma|^{\frac{1}{p}}}\right)} > 0, \quad p > 0.$$

Hence, in view of (4.4) and the fact that $\Phi \in M_{p\gamma}$, for some sufficiently small $\varepsilon_o > 0$, $\varepsilon_1 > 0$ and $\sigma' < \sigma < 0$, we have

$$w^*(v(\sigma)) = |\sigma|^\gamma \ln \mu(\sigma) > \varepsilon_o |\sigma|^{1+\gamma} \Phi\left(\frac{\tau}{|\sigma|^{\frac{1}{p}}}\right) > \varepsilon_o \Phi\left(\frac{\varepsilon_1}{|\sigma|^{\frac{1}{p}}}\right).$$

Since $w^*(x) = o(x)$ as $x \to \infty$, by these estimates we obtain

$$\frac{1}{|\sigma|} < \varepsilon_1^{-p} \varphi^p(v), \tag{4.5}$$

where $v = v(\sigma)$, $\sigma' < \sigma'' < \sigma < 0$. The last estimate holds, in particular, for $\sigma = \tau_j$, $j \geqslant j_0$. Thus, by (4.2)–(4.4) we see that for the function w^* conditions (3.5), (3.6) of Lemma 3.3 are satisfied. Moreover, it follows from (4.5) that

$$0 < \frac{w^*(v(\sigma))}{|\sigma|^{1+\gamma} v(\sigma)} < \varepsilon_1^{-p\nu} \frac{\varphi^{p\nu}(v(\sigma)) w^*(v(\sigma))}{v(\sigma)} \to 0$$

as $\sigma \to 0-$. Therefore, applying Lemma 3.3 for $h = \dfrac{3|\sigma|^{-\gamma} w^*(v)}{v}$, $v = v(\sigma)$, for $0 < \alpha \leqslant \alpha_1 < 1$ and $\sigma \to 0-$ outside some set $e_1 \subset [-1, 0)$, $e_1 = e_1^p(\alpha, \beta, \gamma)$,

$$\varlimsup_{\tau_j \to 0-} \frac{\operatorname{mes}(e_1 \cap [\tau_j, 0))}{|\tau_j|} \leqslant q_1 < \frac{1}{4},$$

we obtain the asymptotic estimate

$$\ln |\sigma + h|^\gamma + \ln \ln \mu(\sigma + h) < \ln |\sigma|^\gamma + \ln \ln \mu(\sigma) + \frac{|\sigma|^\beta}{\alpha},$$

where $\beta > 0$, $\gamma > 0$. Since $w^*(v(\sigma)) = o(|\sigma|^{1+\gamma}) v(\sigma)$ as $\sigma \to 0-$, we have

$$-\gamma \ln |\sigma + h| + \gamma \ln |\sigma| = -\gamma \ln \left|1 - \frac{h}{|\sigma|}\right| = 3\gamma \frac{w^*(v)}{|\sigma|^{1+\gamma} v} = r_\gamma(\sigma),$$

$r_\gamma(\sigma) \to 0$ as $\sigma \to 0-$, $v = v(\sigma)$. Hence, as $\sigma \to 0-$ outside e_1

$$\mu(\sigma + h) < \mu(\sigma)^{1+o(1)}, \qquad h = 3|\sigma|^{-\gamma} \frac{w^*(v)}{v}, \qquad v = v(\sigma). \tag{4.6}$$

Let

$$R_v = \sum_{\lambda_n > v} |a_n| e^{\lambda_n \sigma}.$$

Further, we have $\ln n = \ln n(\lambda_n) \leqslant n_l(\lambda_n)$. Since the function $n_l(t)$ is concave, and moreover,

$$n_l(\lambda_n) \leqslant \frac{w^*(v)}{v} \lambda_n \tag{4.7}$$

for $\lambda_n \geqslant v$, for $\sigma \in [\sigma_0, 0) \setminus e_1$ we have

$$R_v \leqslant \mu(\sigma + h) \sum_{\lambda_n > v} e^{-h\lambda_n} \leqslant C_0 \mu(\sigma + h) \exp[\max_{t \geqslant v} \psi(t)], \qquad h = 3|\sigma|^{-\gamma} \frac{w^*(v)}{v}, \qquad v = v(\sigma),$$

where

$$\psi(t) = 2n_l(t) - ht, \qquad C_0 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus, in view of estimate (4.7), as $\sigma \to 0-$ we have

$$\max_{t \geqslant v}(\psi(t)) \leqslant 2\frac{w^*(v)}{v}t - ht \leqslant -3|\sigma|^{-\gamma}w^*(v)(1+o(1)) < -2|\sigma|^{-\gamma}w^*(v), \quad v = v(\sigma).$$

Then, by (4.2), (4.6), as $\sigma \to 0-$ outside e_1 we obtain that for $\sigma \in [\sigma_1, 0) \setminus e_1$

$$R_v \leqslant C_0[\mu(\sigma)]^{1+o(1)} \exp[-2|\sigma|^{-\gamma}w^*(v)] = [\mu(\sigma)]^{-1+o(1)} < 1. \tag{4.8}$$

Hence, for $\sigma \in [\sigma_1, 0) \setminus e_1$ we find $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$, where $\nu = \nu(\sigma)$ is the central index of series (2.2). Then, as $\sigma \to 0-$ outside e_1, by (4.1), (4.2), we have

$$\mu(\sigma) = |a_\nu|e^{\lambda_\nu \sigma} = |a_\nu b_\nu|e^{\lambda_\nu \sigma}|b_\nu|^{-1} \leqslant \mu^*(\sigma)e^{w^*(v)} = \mu^*(\sigma)\mu(\sigma)^{|\sigma|^\gamma}, \quad 0 < \gamma.$$

Thus, as $\sigma \to 0-$, outside $e_1 \subset [-1, 0)$ we obtain the estimate

$$(1 - |\sigma|^\gamma)\ln \mu(\sigma) \leqslant \ln \mu^*(\sigma). \tag{4.9}$$

On the other hand, since $|b_n| \leqslant e^{w^*(\lambda_n)}$, $n \geqslant 1$, we have

$$\mu^*(\sigma) = |a_k b_k|e^{\lambda_k \sigma} \leqslant \mu(\sigma)e^{w^*(\lambda_k)}, \tag{4.10}$$

where $k = k(\sigma)$ is the central index of series (2.3).

Let $\chi = \chi(\sigma)$ be the solution of the equation

$$w^*(\chi) = 2|\sigma|^\gamma \ln \mu^*(\sigma), \tag{4.11}$$

and

$$R^*_\chi = \sum_{\lambda_n > \chi} |a_n||b_n|e^{\lambda_n \sigma}.$$

We are going to obtain an estimate of type (4.8) for R^*_χ.

Let $\{\tau_j\}$ be the above introduced sequence. By (4.9) for $\sigma \in [\sigma_2, 0) \setminus e_1$ we find

$$\ln \mu(\sigma) < 2 \ln \mu^*(\sigma). \tag{4.12}$$

If for some subsequence $\{\tau_{j_n}\}$ of the sequence $\{\tau_j\}$ the estimates $\ln \mu(\tau_{j_n}) < 2 \ln \mu^*(\tau_{j_n})$, hold, then, due to (4.2), (4.11), we obtain

$$w^*(v(\tau_{j_n})) = |\tau_{j_n}|^\gamma \ln \mu(\tau_{j_n}) < 2|\tau_{j_n}|^\gamma \ln \mu^*(\tau_{j_n}) = w^*(\chi(\tau_{j_n})), \quad n \geqslant 1.$$

Hence, $v(\tau_{j_n}) < \chi(\tau_{j_n})$, $n \geqslant 1$. Further, since $F \in \underline{D}^p_0(\Phi)$, $\Phi \in M_{p\gamma}$, for $\sigma_3 < \sigma < 0$ we obtain

$$\frac{1}{|\sigma|} \leqslant A_p \varphi^p(v), \quad v = v(\sigma).$$

In particular, this yields

$$\frac{1}{|\tau_{j_n}|} < A_p \varphi^p(v(\tau_{j_n})), \quad n \geqslant n_1. \tag{4.13}$$

But $v(\tau_{j_n}) < \chi(\tau_{j_n})$, $n \geqslant 1$. Thus, by (4.3), (4.13) we obtain

$$\overline{\lim_{\tau_{j_n} \to 0-}} \frac{1}{|\tau_{j_n}|^\nu} J(\chi(\tau_{j_n}); w^*) \leqslant A_p^\nu \lim_{\tau_{j_n} \to 0-} \varphi^{p\nu}(v_{j_n}) J(v_{j_n}; w^*) = A_p^\nu T^-_{p\nu} < \infty.$$

Taking this into consideration, we apply Lemma 3.3 to the function $u(\sigma) = \ln(2|\sigma|^\gamma \ln \mu^*(\sigma))$. Then as $\sigma \to 0-$ outside some set $e_2 \subset [-1, 0)$, $e_2 = e_2^p(\alpha, \beta, \gamma)$, $0 < \alpha \leqslant \alpha_2 < 1$,

$$\overline{\lim_{\tau_{j_n} \to 0-}} \frac{\operatorname{mes}(e_2 \cap [\tau_{j_n}, 0))}{|\tau_{j_n}|} \leqslant q_2 < \frac{1}{4}, \tag{4.14}$$

we obtain the estimate

$$\mu^*(\sigma+h) < [\mu^*(\sigma)]^{1+o(1)}, \quad h = 6|\sigma|^\gamma \frac{w^*(\chi)}{\chi}, \quad \chi = \chi(\sigma). \tag{4.15}$$

From this, in the same way as estimate (4.8) was obtained, we find that as $\sigma \to 0-$ outside e_2, $e_2 = e_2^p(\alpha,\beta,\gamma)$,

$$R_\chi^* < [\mu^*(\sigma)]^{-1+o(1)} < 1. \tag{4.16}$$

Now suppose that for each τ_j, $j \geqslant j_1$, the inequality $2\ln\mu^*(\tau_j) \leqslant \ln\mu(\tau_j)$ holds. Then consider the set $A_j \subset [-1,0)$,

$$A_j = \{x : x \geqslant \tau_j, \quad |\tau_j|^\gamma \ln\mu(\tau_j) < |x|^\gamma \ln\mu^*(x)\}, \quad j \geqslant j_1.$$

Since $\tau_j \notin A_j$ for $j \geqslant j_1$, the continuity of the function $|x|^\gamma \ln\mu^*(x)$ implies

$$|\tau_j|^\gamma \ln\mu(\tau_j) = 2|x_j|^\gamma \ln\mu^*(x_j), \qquad x_j = \inf\{x : x \in A_j\}, \quad x_j \geqslant \tau_j.$$

Therefore, by (4.2), (4.11) we get $w^*(v(\tau_j)) = w^*(\chi(x_j))$, i.e. $v(\tau_j) = \chi(x_j)$, $j \geqslant j_1$. It is clear that $\tau_j \in e_1$, for sufficiently large j. Indeed, on the one hand $2\ln\mu^*(\tau_j) \leqslant \ln\mu(\tau_j)$, $j \geqslant j_1$. On the other hand, for $\sigma \in [\sigma_2, 0) \setminus e_1$ inequality (4.12) holds. If for some sequence $j_n \to \infty$ the points τ_{j_n} did not belong to e_1, then we would have estimate (4.12) for $\sigma = \tau_{j_n}$, which is impossible. So, $\tau_j \in e_1$ for $j \geqslant j_2$. Let us now verify that $x_j \in e_1$ for $j \geqslant j_3$. Indeed, for each $x_j \in [\sigma_2, 0) \setminus e_1$ estimate (4.12) holds, which is equivalent to the double relation

$$|x_j|^\gamma \ln\mu(x_j) < 2|x_j|^\gamma \ln\mu^*(x_j) = |\tau_j|^\gamma \ln\mu(\tau_j).$$

Since $|x|^\gamma \ln\mu(x) \uparrow$ as $x \uparrow 0$, this shows that $\tau_j > x_j$ ($|\tau_j| < |x_j|$). But by the definition of A_j we have the inequality $x_j \geqslant \tau_j$, and this means that $x_j \in e_1$ for $j \geqslant j_3$.

Further, since $|\tau_j| > |x_j|$, and τ_j, x_j belong to e_1, then

$$\overline{\lim_{\tau_j \to 0-}} \frac{\operatorname{mes}(e_1 \cap [\tau_j, 0))}{|\tau_j|} \leqslant q_1 < \frac{1}{4}, \qquad \overline{\lim_{\tau_j \to 0-}} \frac{x_j - \tau_j}{|\tau_j|} \leqslant q_1 < \frac{1}{4}. \tag{4.17}$$

This yields

$$|x_j| \geqslant (1-q_1)(1+o(1))|\tau_j|, \quad j \to \infty.$$

This, by the identity $\chi(x_j) = v(\tau_j)$, $j \geqslant j_1$, (4.13) and (4.3),

$$\frac{1}{|x_j|^\nu} J(\chi(x_j); w^*) \leqslant \frac{1}{(1-q_1)(1+o(1))|\tau_j|^\nu} J(v(\tau_j); w^*) = O(1), \quad j \to \infty.$$

Thus, the functions $w^* = w^*(t)$ and $u(\sigma) = \ln(2|\sigma|^\gamma \ln\mu^*(\sigma))$ satisfy conditions (3.5), (3.6) of Lemma 3.3. Moreover, $w^*(x)\varphi^{p\nu}(x) = o(x)$, $x \to \infty$. By (4.5) we then obtain $w^*(v(t_j) = o(|t|^{1+\gamma} v(t))$ as $t \to 0-$. Thus all assumptions of Lemma 3.3 are satisfied. According to this lemma, estimate (4.15), and hence (4.16), hold as $\sigma \to 0-$ outside some set $e_3 \subset [-1,0)$, $e_3 = e_3^p(\alpha,\beta,\gamma)$, $0 < \alpha \leqslant \alpha_3 < 1$,

$$\overline{\lim_{x_j \to 0-}} \frac{\operatorname{mes}(e_3 \cap [x_j, 0))}{|x_j|} < \frac{1}{8}.$$

In view of (4.17) we then obtain

$$\overline{\lim_{\tau_j \to 0-}} \frac{\operatorname{mes}(e_3 \cap [\tau_j, 0))}{|\tau_j|} \leqslant q_3 < \frac{3}{8}. \tag{4.18}$$

Thus, according to (4.14), (4.18), as $\sigma \to 0-$ outside some set $e_4 = e_2 \cup e_3$,

$$\overline{\lim_{\tau_{j_n} \to 0-}} \frac{\operatorname{mes}(e_4 \cap [\tau_{j_n}, 0))}{|\tau_{j_n}|} \leqslant q_4 < \frac{5}{8},$$

estimate (4.16) is valid. This means that $\lambda_{k(\sigma)} < \chi(\sigma)$, if $\sigma \in [\sigma_3, 0) \setminus e_4$. Therefore, by (4.10) we obtain that for such σ

$$\mu^*(\sigma) \leqslant \mu(\sigma) e^{w^*(\chi(\sigma))} = \mu(\sigma)[\mu^*(\sigma)]^{2|\sigma|^\gamma},$$

that is,

$$(1 - 2|\sigma|^\gamma) \ln \mu^*(\sigma) \leqslant \ln \mu(\sigma). \tag{4.19}$$

By estimates (4.9), (4.19) we finally obtain that as $\sigma \to 0-$ outside the set $e = e_1 \cup e_4$, $de \leqslant \dfrac{7}{8}$, the desired estimate holds

$$\left| 1 - \frac{\ln \mu^*(\sigma)}{\ln \mu(\sigma)} \right| \leqslant 5|\sigma|^\gamma.$$

The proof is complete.

□

Let the domain of convergence of the power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{4.20}$$

be the unit disk $\{z : |z| < 1\}$. We make the change of variable $z = e^s$ and consider the function $F(s) = f(e^s)$. It is clear that $F \in D_0(\Lambda)$, where $\Lambda = \{0, 1, 2, \dots, n, \dots\}$. As $\operatorname{Re} s = \sigma \to 0-$ we have

$$|z| = r \to 1-, \quad |\sigma| = |\ln r| = 1 - r + o(1 - r).$$

Therefore, if $F \in \underline{D}_0^p(\Phi)$, then $f \in \underline{K}^p(\Phi)$, where

$$\underline{K}^p(\Phi) = \left\{ f : \sup_{\tau > 0} \varliminf_{r \to\ 1-} \frac{\ln \mu_f(r)}{(1 - r)\Phi\left(\frac{\tau}{(1-r)^{\frac{1}{p}}}\right)} > 0 \right\}, \quad p > 0,$$

and $\mu_f(r)$ is the maximal term of series (4.20). Further, if e_1 is the image of the set $e \subset [-1, 0)$ under the mapping $r = e^\sigma$, then $de < 1$ means that $d_1 e_1 < 1$, where

$$d_1 e_1 = \varliminf_{r \to\ 1-} \frac{\operatorname{mes}(e_1 \cap [r, 1))}{1 - r}.$$

Theorem 2.2 implies the following statement. assertion follows.

Theorem 4.1. *Let $p > 0$, $\beta > 0$, $\gamma > 0$, $\nu = 1 + \beta + \gamma$, $\Phi \in M_{p\gamma}$, φ be the function inverse to Φ, and let $\{b_n\}$ be a sequence satisfying conditions (2.8) and (2.9). Suppose that $n_l \in W_\varphi^{p\nu}$, and moreover $n_l(t)\varphi^{p\nu}(t) = o(t)$ as $t \to \infty$. Let $(\ln r)^\gamma \ln \mu_f(r)$ and $(\ln r)^\gamma \ln \mu_f^*(r)$ be functions increasing as $r \uparrow 1$.*

If the sequence $\{b_n\}$ satisfies estimates (2.8) with some majorant $w \in \underline{W}_\varphi^{p\nu}$, then for each function $f \in \underline{K}^p(\Phi)$ as $r \to 1-$ outside some set $e_1 \subset [0, 1)$ of lower density $d_1 e_1 < 1$ the estimate holds

$$\left| 1 - \frac{\ln \mu_f^*(r)}{\ln \mu_f(r)} \right| \leqslant \operatorname{Const}(1 - r)^\gamma.$$

Here $\mu_f^*(r)$ is the maximal term of the modified power series

$$f^*(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

BIBLIOGRAPHY

1. T.I. Belous, A.M. Gaisin, R.A. Gaisin. *An estimate for the sum of a Dirichlet series on an arc of bounded slope* // Russ. Math. **68**:1, 1–10 (2024). https://doi.org/10.3103/S1066369X24700014
2. T.I. Belous, A.M. Gaĭsin, R.A. Gaĭsin. *Specification of asymptotic Pólya type estimate for Dirichlet series converging in half–plane* // Ufa Math. J. **16**:4, 12–20 (2024). https://doi.org/10.13108/2024-16-4-12
3. A.M. Gajsin. *Behavior of the logarithm of the modulus of the sum of a Dirichlet series converging in a half–plane* // Russ. Acad. Sci., Izv., Math. **45**:1, 175–186 (1995). https://doi.org/10.1070/IM1995v045n01ABEH001625
4. A.M. Gajsin. *Estimates for the growth and decrease of an entire function of infinite order on curves* Sb. Math. **194**:8, 1167–1194 (2003). https://doi.org/10.1070/SM2003v194n08ABEH000761
5. A.M. Gaisin. *Borel — Nevanlinna type theorem. Application.* Bashkir State University Publ., Ufa (2010). (in Russian).
6. A.M. Gajsin, T.I. Belous. *B-stability of the maximal term of the Hadamard composition of two Dirichlet series* // Sib. Math. J. **43**:6, 1027–1038. https://emis.de/ft/37568
7. A.M. Gajsin, T.I. Belous. *Estimation over curves of the functions given by Dirichlet series on a half–plane* // Sib. Math. J. **44**:1, 22–36 (2003). https://emis.de/ft/37504
8. A.M. Gaisin, T.I. Belous. *The maximum term of a Dirichlet series converging in the halfplane: a theorem on stability* // Ufa Math. J. **14**:3, 22–32 (2022). https://doi.org/10.13108/2022-14-3-22
9. A.M. Gaisin, R.A. Gaisin, T.I. Belous. *Regularity of the growth of Dirichlet series with respect to a strongly incomplete exponential system* // Sib. Math. J. **64**:4, 854–863 (2023). https://doi.org/10.1134/S0037446623040079
10. A.A. Goldberg, I.V. Ostrovskii. *Value distribution of meromorphic functions.* Nauka, Moscow (1970); *English translation:* Amer. Math. Soc., Providence, RI (2008).
11. A.F. Leontiev. *Exponential series.* Nauka, Moscow (1976).
12. G.G. Tsegelik. *Properties of the Newton majorant and diagram of a function analytic in the disk* // Ukr. Math. J. **29**:4, 431–433 (1977). https://doi.org/10.1007/BF01085816

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