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APPROXIMATION BY ABEL — POISSON AND CONJUGATE ABEL — POISSON MEANS IN SEMINORMED SPACES

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Abstract. In this paper we establish the degree of approximation of a periodic function by the Abel — Poisson means of its Fourier series in a seminormed space. In addition, the corresponding problem for the conjugate function, approximated by the conjugate Abel — Poisson means of the conjugate Fourier series, is considered. Several corollaries of the main results are derived as special cases.

Keywords: Degree of approximation, Fourier series, conjugate function, seminormed space, Abel — Poisson kernel.

Mathematics Subject Classification: 42A10, 41A30, 42A50, 41A65, 46E30

1. Introduction

The Hölder class of functions was first studied by Prössdorf [32] and later explored and generalized by Leindler [21], [22], Chandra [1], Mohapatra and Chandra [28], Singh et al. [33]в Ъ“[35], Das et al. [2], [3], Mittal and Rhoades [27], Krasniqi [10]в Ъ“[12] and [18], [19], Krasniqi and Szal [14], Lenski et al. [23], Krasniqi et al. [15]в Ъ“[20], Nayak et al. [29], [30], Singh and Sonker [36], Değer [4], Pradhan et al. [31], and Kim [9]. In most cases, researchers studied how a 2π-periodic function from these spaces deviates from an approximating polynomial in the corresponding metrics. For the approximation of conjugate function $\widetilde{f}$ in Hölder spaces, relevant studies include those by Mishra and Khatri [8], [26], Landon et al. [25], and Krasniqi [13]. Among others, notable contributions to this area come from Sunouchi and Watari [37], Xu [38], Guven [5], [6], and Jafarov [7], who studied approximation in hexagonal domains and Morrey spaces. However, only a few studies have focused on approximation using Abel — Poisson means of Fourier series or their conjugates. Recent results on approximation in seminormed spaces using specific Fourier means can be found in [20], [23], and [24]. Although numerous related results are available in [11]–[18], they are not included here to keep the paper focused and concise.

For an integrable 2π-periodic function $f(x)$, by

$$s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

we denote the n-th partial sums of Fourier series of f at the point x,

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \tag{1.1}$$

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where the Fourier coefficients a_k and b_k are defined by

$$a_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \quad k \in \mathbb{N}\cup\{0\}, \qquad b_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \quad k \in \mathbb{N}.$$

The sums

$$\widetilde{s}_n(f;x) = \sum_{k=1}^{n}(b_k\cos kx - a_k\sin kx)$$

are the n-partial sums of the series

$$\sum_{k=1}^{\infty}(b_k\cos kx - a_k\sin kx), \tag{1.2}$$

which indeed is the conjugate series of Fourier series (1.1).

If f is integrable functions in the sense of Lebesgue, then it is known that

$$\widetilde{f}(x) = -\frac{1}{2\pi}\int_{0}^{\pi}\psi_x(t)\cot\frac{t}{2}\,dt = \lim_{\varepsilon\to 0}\widetilde{f}(x;\varepsilon)$$

is well–defined s for almost all x [39], where

$$\widetilde{f}(x;\varepsilon) = -\frac{1}{2\pi}\int_{\varepsilon}^{\pi}\psi_x(t)\cot\frac{t}{2}\,dt$$

with

$$\psi(x;t) := f(x+t) - f(x-t).$$

Let L^s $(1 \leqslant s < \infty)$ $[C]$ be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with s-th power [continuous] over $Q := [-\pi,\pi]$ and let $X^s = L^s$ for $1 \leqslant s < \infty$ or $X^s = C$ for $s = \infty$.

We introduce the norms in X^s as

$$\|f\|_{X^s} = \|f(\cdot)\|_{X^s} := \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_{Q}|f(x)|^s\,dx\right)^{\frac{1}{s}} & \text{for} \quad 1 \leqslant s < \infty, \\ \sup\limits_{x\in Q}|f(x)| & \text{for } s = \infty; \end{cases}$$

By Riesz theorem [39], if $f \in L^s$ for $1 < s < \infty$, then $\widetilde{f} \in L^s$.

As is well–known, the conjugate series of a Fourier series is not necessarily itself a Fourier series. A classical example illustrating this fact is the series $\sum\limits_{k=2}^{\infty}\frac{\sin(kx)}{\ln k}$, which serves as the conjugate series for $\sum\limits_{k=2}^{\infty}\frac{\cos(kx)}{\ln k}$. However, the latter series does not constitute a Fourier series in the classical sense [39].

For two positive quantities w_1 and w_2 we write $w_1 = \mathcal{O}(w_2)$ whenever $w_1 \leqslant Kw_2$, where K denotes a positive constant.

Let $\xi(h)$ be a function of the modulus of continuity type, that is, the function $\xi(h)$ is a positive non-decreasing continuous function on $[0,2\pi]$ with properties:

(i) $\xi(0) = 0$,

(ii) $\xi(h_1+h_2) \leqslant \xi(h_1)+\xi(h_2)$ for $0 \leqslant h_1 \leqslant h_2 \leqslant h_1+h_2 \leqslant 2\pi$.

The Banach space

$$H_s^{(\xi)} = \left\{ f \in L^s : \|f(x+h) - f(x)\|_s = \mathcal{O}(\xi(h)),\ h > 0,\ s \geqslant 1 \right\},$$

equipped with the norm

$$\|f\|_s^{(\xi)} = \|f\|_s + \sup_{h \neq 0} \frac{\|f(x+h) - f(x)\|_s}{\xi(h)}$$

was previously introduced, for example, in [30].

Very recently, we utilized the Abel — Poisson means $f_r(x)$ of the Fourier series of the function f and the conjugate Abel — Poisson means $\widetilde{f}_r(x)$ of the conjugate series of Fourier series. They are defined as follows [39]

$$f_r(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) r^k$$

and

$$\widetilde{f}_r(x) = \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx) r^k.$$

In [17] we proved the following two theorems.

Theorem 1.1. *Let $f \in H(\alpha, s)$, $s \geqslant 1$, $0 \leqslant \beta < \alpha \leqslant 1$, and $0 < r < 1$. Then*

$$\|f_r - f\|_{\beta,s} = \mathcal{O}(1) \begin{cases} (1-r)^{\alpha} & \text{for} \quad 0 < \alpha < 1, \\ (1-r)\left(\log \frac{\pi e}{1-r}\right) & \text{for} \quad \alpha = 1, \end{cases}$$
$$+ \mathcal{O}(1) \begin{cases} (1-r)^{\alpha-\beta} & \text{for} \quad 0 < \alpha < 1, \\ (1-r)^{1-\beta}\left(1 + (1-r)\log^{1-\beta} \frac{\pi}{1-r}\right) & \text{for} \quad \alpha = 1. \end{cases}$$

Theorem 1.2. *Let $f \in H(\alpha, s)$, $s \geqslant 1$, $0 < \beta < \alpha \leqslant 1$, and $0 < r < 1$. Then*

$$\|\widetilde{f}_r - \widetilde{f}\|_{\beta,s} = \mathcal{O}_{\alpha}\left((1-r)^{\alpha}\right) + \mathcal{O}_{r,\alpha,\beta}(1)(1-r)^{\alpha-\beta+\frac{\beta}{\alpha}}.$$

In this research study we are going to consider the seminorms P satisfying, for all $f, g \in (X^p, P)$, the following conditions.

1. For each $h \in \mathbb{R}$

$$P(f(\cdot + h)) = P(f(\cdot)). \tag{1.3}$$

2. If $|f(x)| \leqslant |g(x)|$ for every $x \in [-\pi, \pi]$, then

$$P(f) \leqslant P(g). \tag{1.4}$$

Let ω be a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a non-decreasing continuous function such that

$$\omega(0) = 0,\ \omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2)$$

for all $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1 + \delta_2 \leqslant 2\pi$.

For $f \in (X^p, P)$, the seminormed space $H_P^{(\omega)}$ is defined as [20]

$$H_P^{(\omega)} = \{f \in (X^p, P) :\ M(f;\omega)_P < \infty\}$$

with

$$P(f)^{(\omega)} := P(f) + M(f;\omega)_P,$$

where

$$M(f;\omega)_P := \sup_{t \neq 0} \frac{P(f(\cdot + t) - f(\cdot))}{\omega(|t|)}.$$

Hereinafter, the modulus of continuity $\omega f(\delta)_P$ of f, is defined as

$$\omega f(\delta)_P = \sup_{|t|\leqslant\delta} P\left(f(\cdot + t) - f(\cdot)\right).$$

We note that a function w of modulus of continuity type satisfies the inequality $\frac{w(\delta)}{\delta} \leqslant 2\frac{w(\eta)}{\eta}$ for $0 < \eta \leqslant \delta$. However, we shall focus on functions v and w of modulus of continuity type where the ratio w/v is almost non–decreasing, meaning that there exists a positive constant C such that $\frac{w(\delta)}{v(\delta)} \leqslant C\frac{w(\eta)}{v(\eta)}$ for $0 < \delta \leqslant \eta$.

Motivated by previous studies on approximation, we explore this topic further in seminormed spaces $H_P^{(\omega)}$ using the Abel — Poisson means.

At this point, a natural question arises: What is the degree of approximation of a function from seminormed spaces $H_P^{(\omega)}$ by Abel — Poisson means of its Fourier series and their conjugate series, measured in $P(\cdot)^{(\bullet)}$?

From this point onward, we focus on answering the above question specifically for 2π-periodic functions f (and their conjugate functions $\widetilde{f}$), which constitutes the main objective of this paper. We assert, based on the available literature, that the results presented in this paper are novel and have not been published previously. The proofs of our results rely on several key statements, which are provided in the next section.

2. Some auxiliary lemmas

The following lemmas play a critical role in proving our results.

Lemma 2.1 ([23]). *Let $f \in (L^p, P)$ with $1 \leqslant p < \infty$ and $K \in L^1$. If the seminorm P satisfies conditions (1.3) and (1.4), then*

$$P\left(\int_{-\pi}^{\pi} f(\cdot + t)K(t)\,dt\right) \leqslant \int_{-\pi}^{\pi} P(f)\,|K(t)|\,dt.$$

Lemma 2.2 ([20]). *Let $f \in (X^p, P)$, where the seminorm P satisfies condition (1.3). If v and w be functions of modulus of continuity type such that $\frac{w}{v}$ is an almost non–decreasing function, then for $f \in \mathrm{Lip}(w, P) := \{f \in (X^p, P) : \omega f(\delta)_P = \mathcal{O}(w(\delta))\}$ we have:*

(i) *For $t \geqslant 0$,*

$$P(\varphi_\cdot(t)) = \mathcal{O}(1)\omega f(t)_P = O(w(t)),$$

(ii) *For $t, y > 0$,*

$$P(\varphi_{\cdot+y}(t) - \varphi_\cdot(t)) = \mathcal{O}\left(v(y)\frac{w(t)}{v(t)}\right).$$

Arguing similarly to the proof of Lemma 2.2, we establish the following result.

Lemma 2.3. *Let $f \in (X^p, P)$, where the seminorm P satisfies condition (1.3). If v and w be functions of modulus of continuity type such that $\frac{w}{v}$ is an almost non–decreasing function, then for $f \in Lip(w, P) := \{f \in (X^p, P) : \omega f(\delta)_P = \mathcal{O}(w(\delta))\}$:*

(i) *For $t \geqslant 0$,*

$$P(\psi_\cdot(t)) = \mathcal{O}(1)\omega f(t)_P = O(w(t)),$$

(ii) *For $t, y > 0$,*

$$P(\psi_{\cdot+y}(t) - \psi_\cdot(t)) = \mathcal{O}\left(v(y)\frac{w(t)}{v(t)}\right).$$

Proof. (i) It follows from direct computation that

$$\begin{aligned} P\left(\psi_{\cdot}\left(t\right)\right) =& P\left(f\left(\cdot+t\right)-f\left(\cdot-t\right)\right)=P\left(f\left(\cdot+t\right)-f\left(\cdot\right)-f\left(\cdot-t\right)+f\left(\cdot\right)\right) \\ \leqslant& P\left(f\left(\cdot+t\right)-f\left(\cdot\right)\right)+P\left(f\left(\cdot-t\right)-f\left(\cdot\right)\right)\leqslant 2\omega f(t)_P=O\left(w(t)\right). \end{aligned}$$

(ii) By applying property (1.3) of the seminorm P and case (i), we deduce

$$P\left(\psi_{\cdot+y}\left(t\right)-\psi_{\cdot}\left(t\right)\right)\leqslant P\left(\psi_{\cdot+y}\left(t\right)\right)+P\left(\psi_{\cdot}\left(t\right)\right)=P\left(\psi_{\cdot}\left(t\right)\right)+P\left(\psi_{\cdot}\left(t\right)\right)\leqslant 4\omega f(t)_P=O\left(w(t)\right)$$

and in the same way,

$$P\left(\psi_{\cdot+y}\left(t\right)-\psi_{\cdot}\left(t\right)\right)=O\left(w(y)\right).$$

Thus, for $t,y>0$, we leverage the monotonicity of v and w, as well as the almost monotonicity of the ratio $\frac{w}{v}$, to derive:

$$P\left(\psi_{\cdot+y}\left(t\right)-\psi_{\cdot}\left(t\right)\right)=O\left(\frac{w(t)}{v\left(t\right)}v\left(t\right)\right)=O\left(\frac{w(t)}{v\left(t\right)}v\left(y\right)\right)\quad\text{for}\quad t\leqslant y,$$

and

$$P\left(\psi_{\cdot+y}\left(t\right)-\psi_{\cdot}\left(t\right)\right)=O\left(\frac{w(y)}{v\left(y\right)}v\left(y\right)\right)=O\left(\frac{w(t)}{v\left(t\right)}v\left(y\right)\right)\quad\text{for}\quad y\leqslant t.$$

This completes the proof. □

Undoubtedly, we will use the well–known Abel — Poisson kernel $P(r,t)$ and its conjugate kernel $Q(r,t)$, $0\leqslant r<1$, which are defined by [39]

$$P(r,t):=\frac{1}{2}+\sum_{v=1}^{\infty}r^v\cos(vt),\qquad Q(r,t):=\sum_{v=1}^{\infty}r^v\sin(vt).$$

The following lemma can be proved easily.

Lemma 2.4. *Let r be a fixed number from $(0,1)$ and $0<t\leqslant\pi$. Then, the following identities are valid:*

$$P(r,t)=(1-r)\sum_{v=0}^{\infty}r^v\frac{\sin\left(v+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)},\tag{2.1}$$

$$P(r,t)=(1-r)^2\sum_{v=0}^{\infty}r^v\left(\frac{\sin\frac{(v+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2.\tag{2.2}$$

Proof. First, we define $\cos 0:=\frac{1}{2}$. We employ summation by parts and use the identity

$$\sum_{j=0}^{k}\cos jt=\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)},$$

which allows to to express

$$\begin{aligned} P(r,t)=&\lim_{k\to\infty}\sum_{v=0}^{k}r^v\cos(vt)=\lim_{k\to\infty}\left(\sum_{v=0}^{k-1}(r^v-r^{v+1})\sum_{j=0}^{v}\cos jt+r^k\sum_{j=0}^{k}\cos jt\right) \\ =&\sum_{v=0}^{\infty}(r^v-r^{v+1})\frac{\sin\left(v+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}=(1-r)\sum_{v=0}^{\infty}r^v\frac{\sin\left(v+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}, \end{aligned}$$

which proves (2.1). We then apply summation by parts and use the identity

$$\sum_{j=0}^{k}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}=\left(\frac{\sin\frac{(k+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2$$

to derive

$$P(r,t) = \lim_{k\to\infty}(1-r)\left(\sum_{v=0}^{k-1}(r^v - r^{v+1})\sum_{j=0}^{v}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} + r^k\sum_{j=0}^{k}\frac{\sin\left(j+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right)$$
$$=(1-r)\sum_{v=0}^{\infty}(r^v - r^{v+1})\left(\frac{\sin\frac{(v+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2$$
$$=(1-r)^2\sum_{v=0}^{\infty}r^v\left(\frac{\sin\frac{(v+1)t}{2}}{\sin\left(\frac{t}{2}\right)}\right)^2,$$

and this completes the proof. □

In the forthcoming section, we present the principal contributions of the paper.

3. Main results

The initial result is presented below.

Theorem 3.1. *Let v and w be functions of the modulus of continuity type, where w/v is an almost non-decreasing function. Assume that $f \in H_P^{(w)}$, with the seminorm P satisfying conditions (1.3) and (1.4). Then, for a fixed $r \in (0,1)$,*

$$P\left(f_r(\cdot) - f(\cdot)\right)^{(v)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)\int_{1-r}^{\pi}\frac{w(t)}{t^2 v(t)}\,dt\right). \tag{3.1}$$

Proof. The Abel — Poisson mean $f_r(x)$ associated with series (1.1) admits the following representation [39]:

$$f_r(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)P(r;t)\,dt$$

where $P(r;t)$ is the Abel — Poisson kernel.

According to the property

$$\frac{2}{\pi}\int_{0}^{\pi} P(r,t)\,dt = 1,$$

it follows that

$$A_r f(x) := f_r(x) - f(x) = \frac{1}{\pi}\int_{0}^{\pi}\varphi_x(t)P(r,t)\,dt,$$

where

$$\varphi_x(t) := f(x+t) - 2f(x) + f(x-t).$$

We aim to derive an upper bound for the quantity

$$P\left(A_r f(\cdot)\right)^{(v)} = P\left(A_r f(\cdot)\right) + \sup_{y\neq 0}\frac{P\left(A_r f(\cdot+y) - A_r f(\cdot)\right)}{v(y)}.$$

Given $f \in H_P^{(\omega)}$, we apply Lemma 2.1 to the identity

$$A_r f(x+y) - A_r f(x) = \frac{1}{\pi}\int_{0}^{\pi}(\varphi_{x+y}(t) - \varphi_x(t))P(r,t)\,dt,$$

to obtain

$$\begin{aligned} P\left(A_r f(\cdot+y)-A_r f(\cdot)\right) \leqslant & \frac{1}{\pi}\int_{-\pi}^{\pi} P\left(\varphi_{\cdot+y}(t)-\varphi_{\cdot}(t)\right) P(r,t)\,dt \\ = & \frac{1}{\pi}\underbrace{\int_{0}^{1-r} P\left(\varphi_{\cdot+y}(t)-\varphi_{\cdot}(t)\right) P(r,t)\,dt}_{=:A_{11}} \\ & + \frac{1}{\pi}\underbrace{\int_{1-r}^{\pi} P\left(\varphi_{\cdot+y}(t)-\varphi_{\cdot}(t)\right) P(r,t)\,dt}_{=:A_{12}}, \end{aligned} \tag{3.2}$$

since $P(r,t)$ is positive for all $t \in (0,\pi)$.

By applying standard trigonometric inequalities $\pi \sin\theta \geqslant 2\theta$ for $\theta \in [0,\pi/2]$, and $|\sin\theta| \leqslant |\theta|$ for all θ, by (2.2) we readily obtain

$$P(r,t) \leqslant (1-r)^2 \sum_{v=0}^{\infty} r^v \left(\frac{\frac{(v+1)t}{2}}{\left(\frac{t}{\pi}\right)}\right)^2 \leqslant \left(\frac{\pi(1-r)}{2}\right)^2 \sum_{v=0}^{\infty}(v+1)^2 r^v. \tag{3.3}$$

Substituting the identities

$$\begin{aligned} &\sum_{v=0}^{\infty} r^v = \frac{1}{1-r} \quad \text{for} \quad |r|<1, \\ &\sum_{v=0}^{\infty} v r^v = \frac{r}{(1-r)^2} \quad \text{for} \quad |r|<1, \\ &\sum_{v=0}^{\infty} v^2 r^v = \frac{r(1+r)}{(1-r)^3} \quad \text{for} \quad |r|<1, \end{aligned} \tag{3.4}$$

into (3.3), we obtain

$$\begin{aligned} P(r,t) \leqslant & \left(\frac{\pi(1-r)}{2}\right)^2 \left(\frac{r(1+r)}{(1-r)^3}+\frac{2r}{(1-r)^2}+\frac{1}{1-r}\right), \\ = & \frac{\pi^2}{4}\left(\frac{r(1+r)}{1-r}+2r+(1-r)\right) = \frac{\pi^2(1+r)}{4(1-r)}. \end{aligned} \tag{3.5}$$

Now, by applying (2.2) together with the estimate in (3.5), we obtain

$$A_{11} = \mathcal{O}(v(y))\frac{1+r}{1-r}\int_{0}^{1-r}\frac{w(t)}{v(t)}\,dt = \mathcal{O}\left(v(y)\right)\frac{w(1-r)}{v(1-r)}, \tag{3.6}$$

where the final estimate follows from the assumption that $w(t)/v(t)$ is a non–decreasing function.

We seek an alternative upper bound for the kernel $P(r,t)$, as stated in (2.2). By employing the inequalities $\pi \sin\theta \geqslant 2\theta$ for $\theta \in [0,\pi/2]$, and $|\sin\theta| \leqslant 1$ for all θ, we derive (use also (3.4))

$$P(r,t) \leqslant (1-r)^2 \sum_{v=0}^{\infty} r^v \left(\frac{\pi}{t}\right)^2 = (1-r)\frac{\pi^2}{t^2}. \tag{3.7}$$

Hence, by Statement (ii) in Lemma 2.2, we find

$$A_{12} = \mathcal{O}(v(y))(1-r)\int\limits_{1-r}^{\pi} \frac{w(t)}{t^2 v(t)}\, dt. \tag{3.8}$$

Combining relations from (3.2), (3.6), and (3.8), we obtain

$$\frac{P\left(A_r f(\cdot + y) - A_r f(\cdot)\right)}{v(y)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)\int\limits_{1-r}^{\pi} \frac{w(t)}{t^2 v(t)}\, dt\right). \tag{3.9}$$

Furthermore, we aim to establish an upper bound for the quantity $P\left(A_r f(\cdot)\right)$. To this end, we first apply Lemma 2.1, which yields

$$P(A_r f(x)) \leqslant \frac{1}{\pi}\int\limits_{0}^{\pi} P(\varphi_x(t))P(r,t)\ dt,$$

and subsequently invoke Lemma 2.2 (i) to obtain

$$\begin{aligned} P(A_r f(x)) =& \mathcal{O}(1)\int\limits_{0}^{\pi} \omega(t)P(r,t)\, dt \\ =& \mathcal{O}(1)\Bigg(\underbrace{\int\limits_{0}^{1-r} \omega(t)P(r,t)\, dt}_{=:A_{21}} + \underbrace{\int\limits_{1-r}^{\pi} \omega(t)P(r,t)\, dt}_{=:A_{22}}\Bigg). \end{aligned} \tag{3.10}$$

Since $\omega(t)$ is non-decreasing, the estimate in relation (3.5) implies

$$A_{21} = \mathcal{O}(\omega(1-r))\int\limits_{0}^{1-r} \frac{1+r}{1-r}\, dt = \mathcal{O}\left(v(\pi)\right)\frac{\omega(1-r)}{v(1-r)}. \tag{3.11}$$

To handle A_{22}, we proceed as follows: the estimate given in relation (3.7) implies that

$$A_{22} = \mathcal{O}(v(\pi))(1-r)\int\limits_{1-r}^{\pi} \frac{\omega(t)}{t^2 v(t)}\, dt. \tag{3.12}$$

By collecting relations from (3.10), (3.11), and (3.12), we obtain

$$P(A_r f(x)) = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)\int\limits_{1-r}^{\pi} \frac{w(t)}{t^2 v(t)}\, dt\right). \tag{3.13}$$

Finally, relations in (3.9) and (3.13) imply

$$\begin{aligned} P\left(A_r f(\cdot)\right)^{(v)} &= P\left(A_r f(\cdot)\right) + \sup_{t\neq 0} \frac{P\left(A_r f(\cdot + y) - A_r f(\cdot)\right)}{v(t)} \\ &= \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)\int\limits_{1-r}^{\pi} \frac{w(t)}{t^2 v(t)}\, dt\right). \end{aligned}$$

The proof is complet. □

We proceed by making some useful observations. Note that the first term in (3.1) is bounded from above, up to a positive constant factor, by

$$(1-r)\int\limits_{1-r}^{\pi}\frac{w(t)}{t^2v(t)}\,dt.$$

Indeed, since the function $\frac{w(t)}{v(t)}$ is non–decreasing and $r\in(0,1)$, it follows that

$$\begin{aligned}\int\limits_{1-r}^{\pi}\frac{w(t)}{t^2v(t)}\,dt &\geqslant \frac{w(1-r)}{v(1-r)}\int\limits_{1-r}^{\pi}t^{-2}\,dt=\frac{w(1-r)}{v(1-r)}\left(\frac{1}{1-r}-\frac{1}{\pi}\right)\\ &=\frac{w(1-r)}{(1-r)v(1-r)}\left(1-\frac{1-r}{\pi}\right)\geqslant C\frac{w(1-r)}{(1-r)v(1-r)},\end{aligned}$$

where $C:=1-\pi^{-1}$ (corresponding to the case $r=0$).

This leads us to the following result.

Corollary 3.1. *Assume that the assumptions of Theorem 3.1 are satisfied. Then*

$$P\left(f_r(\,\cdot\,)-f(\,\cdot\,)\right)^{(v)}=\mathcal{O}\left((1-r)\int\limits_{1-r}^{\pi}\frac{w(t)}{t^2v(t)}\,dt\right).$$

If, in addition, $\frac{w(t)}{tv(t)}$ is a non–increasing function, then

$$\int\limits_{1-r}^{\pi}\frac{w(t)\,dt}{t^2v(t)}\leqslant\frac{w(1-r)}{(1-r)v(1-r)}\ln\left(\frac{\pi}{1-r}\right).$$

Thus, we obtain the following.

Corollary 3.2. *Assume that all the conditions in Theorem 3.1 are satisfied. If, in addition,* $\frac{w(t)}{tv(t)}$ *is a non-increasing function, then*

$$P\left(f_r(\,\cdot\,)-f(\,\cdot\,)\right)^{(v)}=\mathcal{O}\left(\frac{w(1-r)}{v(1-r)}\ln\left(\frac{\pi}{1-r}\right)\right).$$

We consider the specific choices $w(t)=t^{\alpha}$ and $v(t)=t^{\beta}$, where the parameters α and β satisfy $0\leqslant\beta<\alpha\leqslant1$. Using these conditions in Corollary 3.2, we deduce that the function

$$\frac{w(t)}{v(t)}=t^{\alpha-\beta}$$

is non–decreasing, whereas

$$\frac{w(t)}{tv(t)}=\frac{1}{t^{1-\alpha+\beta}}$$

is non–increasing. Consequently, we establish the following corollary.

Corollary 3.3. *Assume that all the conditions in Theorem 3.1 are satisfied. If* $0\leqslant\beta<\alpha\leqslant1$, *then*

$$P\left(f_r(\,\cdot\,)-f(\,\cdot\,)\right)^{(v)}=\mathcal{O}\left((1-r)^{\alpha-\beta}\ln\left(\frac{\pi}{1-r}\right)\right).$$

In the particular case where $\beta=0$ in Corollary 3.3, we obtain the following statement.

Corollary 3.4. *Assume that all the conditions in Theorem 3.1 are satisfied. If* $0 < \alpha \leqslant 1$, *then*

$$P\left(f_r(\,\cdot\,) - f(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left((1-r)^{\alpha} \ln\left(\frac{\pi}{1-r}\right)\right).$$

Next result deals with determining the degree of approximation by the conjugate Abel – Poisson means of the conjugate function $\widetilde{f}$, whenever the function f itself belongs to $H_P^{(w)}$. At this point, we establish the following result.

Theorem 3.2. *Let* v *and* w *be functions of the modulus of continuity type, where* w/v *is an almost non–decreasing function. Assume that* $f \in H_P^{(w)}$, *with the seminorm* P *satisfying conditions (1.3) and (1.4). Then, for a fixed* $r \in (0,1)$,

$$P\left(\widetilde{f}_r(\,\cdot\,) - \widetilde{f}(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)^2 \int\limits_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)}\right) \tag{3.14}$$

provided that

$$\int\limits_{0}^{1-r} \frac{w(t)\,dt}{tv(t)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)}\right).$$

Proof. Although the proof is similar (with some important differences) to the proof of Theorem 3.1, we will outline its main steps. In fact, we know that the conjugate Abel – Poisson mean $\widetilde{f}_r(x)$ of the series in (1.2) can be expressed as [39]

$$\widetilde{f}_r(x) = -\frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x+t) Q(r;t)\,dt,$$

where

$$Q(r;t) := \sum_{k=1}^{\infty} r^k \sin(kt) = \frac{r \sin t}{1 - 2r\cos t + r^2}, \quad 0 < t < \pi.$$

Since

$$\begin{aligned}
Q(r;t) &= \frac{2r \sin\frac{t}{2}\cos\frac{t}{2}}{1 - 2r\cos t + r^2} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \frac{2r\sin^2\frac{t}{2}}{1 - 2r\cos t + r^2} \\
&= \frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}} \frac{2r(1-\cos t)}{1 - 2r\cos t + r^2} = \frac{1}{2\tan\frac{t}{2}} \frac{2r - 2r\cos t}{1 - 2r\cos t + r^2} \\
&= \frac{1}{2\tan\frac{t}{2}} \left(\frac{1 - 2r\cos t + r^2 - 1 + 2r - r^2}{1 - 2r\cos t + r^2}\right) \\
&= \frac{1}{2\tan\frac{t}{2}} \left(1 - \frac{(1-r)^2}{1 - 2r\cos t + r^2}\right) \\
&= \frac{1}{2\tan\frac{t}{2}} - \frac{1}{2\tan\frac{t}{2}} \frac{(1-r)^2}{1 - 2r\cos t + r^2},
\end{aligned}$$

then a simple computation reveals that

$$\widetilde{A}_r f(x) := \widetilde{f}_r(x) - \widetilde{f}(x) = \frac{(1-r)^2}{2\pi} \int\limits_{0}^{\pi} \frac{\psi_x(t)}{(1 - 2r\cos t + r^2)\tan\frac{t}{2}}\,dt$$

considering that $Q(r;t)$ is odd with respect to t.

According to the definition, we are required to provide an upper estimate for the quantity

$$P\left(\widetilde{A}_r f(\cdot)\right)^{(v)} = P\left(\widetilde{A}_r f(\cdot)\right) + \sup_{y\neq 0} \frac{P\left(\widetilde{A}_r f(\cdot+y) - \widetilde{A}_r f(\cdot)\right)}{v(y)}.$$

We now proceed to estimate the second term on the right–hand side of the above equality. The use of Lemma 2.1 to

$$\widetilde{A}_r f(x+y) - \widetilde{A}_r f(x) = \frac{(1-r)^2}{2\pi} \int_0^\pi \frac{\psi_{x+y}(t) - \psi_x(t)}{(1-2r\cos t + r^2)\tan\frac{t}{2}}\, dt,$$

leads to

$$\begin{aligned} P\left(\widetilde{A}_r f(\cdot+y) - \widetilde{A}_r f(\cdot)\right) &\leqslant \frac{(1-r)^2}{2\pi} \int_0^\pi \frac{P\left(\psi_{\cdot+y}(t) - \psi_\cdot(t)\right)}{(1-2r\cos t + r^2)\tan\frac{t}{2}}\, dt \\ &= \frac{1}{\pi} \underbrace{(1-r)^2 \int_0^{1-r} \frac{P\left(\psi_{\cdot+y}(t) - \psi_\cdot(t)\right)}{(1-2r\cos t + r^2)2\tan\frac{t}{2}}\, dt}_{=:B_{11}} \\ &\quad + \frac{1}{\pi} \underbrace{(1-r)^2 \int_{1-r}^\pi \frac{P\left(\psi_{\cdot+y}(t) - \psi_\cdot(t)\right)}{(1-2r\cos t + r^2)2\tan\frac{t}{2}}\, dt}_{=:B_{12}}. \end{aligned} \tag{3.15}$$

Statement (ii) in Lemma 2.3 implies

$$\begin{aligned} B_{11} =& \mathcal{O}\left(v\left(y\right)\right)(1-r)^2 \int_0^{1-r} \frac{w(t)\,dt}{v(t)((1-r)^2 + 4r\sin^2\frac{t}{2})t} \\ =& \mathcal{O}\left(v\left(y\right)\right)(1-r)^2 \int_0^{1-r} \frac{w(t)\,dt}{v(t)(1-r)^2 t} = \mathcal{O}\left(v\left(y\right)\frac{w(1-r)}{v(1-r)}\right), \end{aligned} \tag{3.16}$$

taking into account the inequality $\tan u \leqslant u$ for $u \geqslant 0$.

Again, applying Statement (ii) of Lemma 2.3 and using the inequalities $\pi\sin\theta \geqslant 2\theta$ for $\theta \in [0, \pi/2]$ and $\tan\theta \geqslant \theta$ for $\theta \geqslant 0$, we arrive at

$$\begin{aligned} B_{12} =& \mathcal{O}\left(v\left(y\right)\right)(1-r)^2 \int_{1-r}^{\pi} \frac{w(t)\,dt}{v(t)((1-r)^2 + 4r\sin^2\frac{t}{2})t} \\ =& \mathcal{O}\left(v\left(y\right)\right)(1-r)^2 \int_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)}. \end{aligned} \tag{3.17}$$

By combining relations from (3.15), (3.16), and (3.17), we obtain

$$\frac{P\left(\widetilde{A}_r f(\cdot+y) - \widetilde{A}_r f(\cdot)\right)}{v(y)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)^2 \int_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)}\right). \tag{3.18}$$

Following a similar argument as above, we also deduce that

$$P\left(\widetilde{A}_r f(\,\cdot\,)\right) = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)} + (1-r)^2 \int\limits_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)}\right). \tag{3.19}$$

It is clear that relations (3.18) and (3.19) imply relation (3.14). This completes the proof. □

We now present several observations and derive some corollaries. Observe that the first term in (3.14) is bounded above, up to a positive constant factor, by

$$(1-r)^2 \int\limits_{1-r}^{\pi} \frac{w(t)}{t^3 v(t)}\,dt.$$

Indeed, since the function $\frac{w(t)}{v(t)}$ is non–decreasing and $r \in (0,1)$, it follows that

$$\begin{aligned}\int\limits_{1-r}^{\pi} \frac{w(t)}{t^3 v(t)}\,dt &\geqslant \frac{w(1-r)}{v(1-r)} \int\limits_{1-r}^{\pi} t^{-3}\,dt = \frac{w(1-r)}{v(1-r)}\left(\frac{1}{2(1-r)^2} - \frac{1}{2\pi^2}\right)\\ &= \frac{w(1-r)}{2(1-r)^2 v(1-r)}\left(1 - \frac{(1-r)^2}{\pi^2}\right) \geqslant C\frac{w(1-r)}{(1-r)^2 v(1-r)},\end{aligned}$$

where $C := 1 - \pi^{-2}$ (corresponding to the case $r = 0$).

Obviously, the following holds true.

Corollary 3.5. *Assume that all the conditions of Theorem 3.2 are satisfied. Then,*

$$P\left(\widetilde{f}_r(\,\cdot\,) - \widetilde{f}(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left((1-r)^2 \int\limits_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)}\right)$$

provided that

$$\int\limits_{0}^{1-r} \frac{w(t)\,dt}{tv(t)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)}\right).$$

If, moreover, $\frac{w(t)}{tv(t)}$ is a non–increasing function, then

$$\int\limits_{1-r}^{\pi} \frac{w(t)\,dt}{t^3 v(t)} = \mathcal{O}\left(\frac{w(1-r)}{(1-r)^2 v(1-r)}\right).$$

Hence, we arrive at the following statement.

Corollary 3.6. *Assume that all the conditions in Theorem 3.2 are satisfied. If, in addition,* $\frac{w(t)}{tv(t)}$ *is a non–increasing function, then*

$$P\left(\widetilde{f}_r(\,\cdot\,) - \widetilde{f}(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)}\right)$$

provided that

$$\int\limits_{0}^{1-r} \frac{w(t)\,dt}{tv(t)} = \mathcal{O}\left(\frac{w(1-r)}{v(1-r)}\right). \tag{3.20}$$

We now examine the specific choices $w(t) = t^\alpha$ and $v(t) = t^\beta$, where the parameters α and β satisfy $0 \leqslant \beta < \alpha \leqslant 1$. By applying these conditions, we see that condition (3.20) is satisfied immediately. Based on these observations, we derive the following corollary.

Corollary 3.7. *Assume that all the conditions in Theorem 3.2 are satisfied. If* $0 \leqslant \beta < \alpha \leqslant 1$, *then*

$$P\left(\widetilde{f}_r(\,\cdot\,) - \widetilde{f}(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left((1-r)^{\alpha-\beta}\right).$$

When $\beta = 0$, Corollary 3.7 yields the following result.

Corollary 3.8. *Assume that all the conditions in Theorem 3.2 are satisfied. If* $0 < \alpha \leqslant 1$, *then*

$$P\left(\widetilde{f}_r(\,\cdot\,) - \widetilde{f}(\,\cdot\,)\right)^{(v)} = \mathcal{O}\left((1-r)^{\alpha}\right).$$

Remark 3.1. *Note that the degree of approximation in the corollaries of Theorem 3.2 is of Jackson order, whereas that in the corollaries of Theorem 3.1 is not, due to the presence of the factor* $\ln\left(\frac{\pi}{1-r}\right)$. *This difference can be attributed to the distinct mathematical properties of the kernels* $P(r,t)$ *and* $Q(r,t)$.

4. Conclusion

The problem of determining the degree of approximation in a seminormed space by Abel — Poisson means of the Fourier series of a periodic function, as well as by the conjugate Abel — Poisson means of its conjugate function, has been investigated. This study paves the way for future research on the degree of approximation of such functions using generalized Abel — Poisson operators.

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