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OPERATOR ESTIMATES FOR WEAKLY NON–LINEAR SIEVE PROBLEMS WITH MIXED BOUNDARY CONDITIONS: HOMOGENIZED DIRICHLET CONDITION

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Abstract. We consider a boundary value problem for a weakly nonlinear elliptic equation in a multi–dimensional domain, which contains a finely perforated surface. The perforation is arbitrary and non–periodic. We arbitrarily partition each side of surface into two subsets, on one subsect we impose the Dirichlet condition, the other is subject to a weakly nonlinear Robin condition. As a particular case, the mentioned partition includes the one consisting of small pieces, which leads to the frequent alternation of boundary conditions. We formulate quite weak geometric conditions of the perforation of surface and the partitions of its sides, under which the homogenization leads to the Dirichlet condition on both sides of the surface. Our main results are W_2^1– and L_2–operator estimates.

Keywords: sieve problem, alternation of boundary conditions, homogenization, operator estimate.

Mathematics Subject Classification: 35B27, 35B40

1. Introduction

Operator estimates for homogenization problems is a rather new field, which is developed quite intensively nowadays. In contrast to classical results in the homogenization theory, which established the convergence of solutions to perturbed problems to ones of homogenized problems strongly or weakly in appropriate Lebesgue and Sobolev spaces for fixed right hand sides in equations or boundary conditions, the operators estimates ensure the uniform in right hand sides convergence. Such operator estimates are much stronger and in fact, this is the best possible convergence result.

One of wide direction in the homogenization theory aims on studying boundary homogenization problems, which involve perturbation concentrated on or near some surfaces of codimension one or higher. The homogenization usually preserves the equation, but brings some effective boundary condition on the surface, along which the perturbation is distributed. The type and form of the boundary condition depends essentially on the structure of the perturbation. As possible examples we can mention problems with alternating boundary conditions, with a fine perforation along a given manifold, with fast oscillating boundaries. The operators estimates for such type of perturbations were developed during last 20 years in series of works, see, for instance, [4], [10], [11], [12], [13], [14], [15], [5], [6], [3], [7], [8]. The main advantage is that we succeeded to study the cases of essentially non–periodic perturbations under rather weak assumptions formulated in geometric terms.

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One more class of problems in the boundary homogenization theory are so–called sieve problems. The main feature of such problem is the presence of finely perforated manifold inside the domain, and on this surface one of classical conditions is imposed. There are quite a lot of classical works, in which such problems were considered and the weak or strong convergence was established, see, for instance, [2], [16], [17], [18], [23], [24], [1], [22]. The operator estimates for the Neumann sieve problems were recently established in [20], [21].

In the present paper we consider a boundary value problem in a multi–dimensional domain with a finely perforated surface for a weakly nonlinear elliptic equation. The perforation is arbitrary and non–periodic. We arbitrarily partition each side of surface into two subsets, on one subsect we impose the Dirichlet condition, the other is subject to a weakly nonlinear Robin condition. As a particular case, the mentioned partition includes the one consisting of small pieces, which leads to the frequent alternation of boundary conditions. We formulate quite weak geometric conditions of the perforation of surface and the partitions of its sides, under which the homogenization leads to the Dirichlet condition on both sides of the surface. Our main results are W_2^1– and L_2–operator estimates.

2. Problem and main results

Let $x = (x_1, \dots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 2$, and Ω be some bounded or unbounded domain in $\mathbb{R}^n$ with a C^1–boundary. By S we denote an oriented manifold of codimension one with no edge, which is located inside Ω and is separated from $\partial\Omega$ by a positive distance. By $S_\pm$ we denote the sides of S and the symbol ν stands for the unit normal vector associated with the sides $S_\pm$. By τ we denote the distance to a point measured along the vector ν_+, and

$$\Pi^r := \{x \in \Omega : \ \operatorname{dist}(x, S) < r\}, \qquad \Pi^r_\pm := \Pi^r \cap \{x : \ \pm\tau > 0\}.$$

We suppose that the surface is regular, namely, we make the following assumption.

(A1). There exists $\tau_0 > 0$ and some local variables $s = (s_1, \dots, s_{n-1})$ on S such that in $\Pi^{\tau_0} \subset \Omega$ the variables (s, τ) are well–defined, and all partial derivatives of x in (s, τ) and of (s, τ) in x up to the second order are uniformly bounded.

In the domain Ω we define a matrix function $\mathrm{A} = \mathrm{A}(x)$ of size $n \times n$ and an n–dimensional vector function $\mathrm{B} = \mathrm{B}(x)$ as well as two complex–valued functions $V = V(x, u)$ on $\Omega \times \mathbb{C}$ and $a_\pm = a_\pm(x, u)$ on $S_\pm \times \mathbb{C}$. We suppose that $\mathrm{A} \in W^1_\infty(\Omega)$, $\mathrm{B} \in L_\infty(\Omega)$, the functions V and $a_\pm$ are measurable, and

$$\begin{aligned} &\operatorname{Re} \mathrm{A}(x)z \cdot z \geqslant c_1|z|^2, \qquad |V(x, u_1) - V(x, u_2)| \leqslant c_2|u_1 - u_2|, \\ &|a_\pm(x, u_1) - a_\pm(x, u_2)| \leqslant c_2|u_1 - u_2|, \qquad V(x, 0) = 0, \qquad a_\pm(x, 0) = 0 \end{aligned} \tag{2.1}$$

where $\cdot$ stands for the scalar product in $\mathbb{C}^n$, and c_1, c_2 are some fixed positive constants independent of $x \in \Omega$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, u_1, $u_2 \in \mathbb{C}$.

By ε we denote a small positive parameter. On the surface S we make an arbitrary number of holes of arbitrary shapes, which can depend on ε, and the remaining part is denoted by a subset S^ε. The sides of S^ε are denoted by $S^\varepsilon_\pm$, and we suppose that for each ε the edge of S^ε is C^1–smooth manifold in $\mathbb{R}^n$ of codimension two. We let $\Omega^\varepsilon := \Omega \setminus S^\varepsilon$. On both sides $S^\varepsilon_\pm$ of S^ε we arbitrarily choose subsets $\gamma^\varepsilon_\pm$, which can also depend on ε.

The main object of our study is the following non–linear boundary value problem

$$\begin{aligned} &(\hat{\mathcal{H}} - \lambda)u^\varepsilon = f \quad \text{in} \quad \Omega^\varepsilon, \\ &u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \cup \gamma^\varepsilon_+ \cup \gamma^\varepsilon_-, \qquad \frac{\partial u^\varepsilon}{\partial \overline{\nu}_\pm} - a_\pm(\,\cdot\,, u^\varepsilon) = 0 \quad \text{on} \quad S^\varepsilon_\pm \setminus \gamma^\varepsilon_\pm. \end{aligned} \tag{2.2}$$

where $\lambda \in \mathbb{C}$ is a scalar parameter, $f \in L_2(\Omega)$ is an arbitrary function, and

$$\mathcal{H}^\varepsilon u := -\operatorname{div} \mathrm{A}\nabla u + \mathrm{B}^t \nabla u + V(\,\cdot\,, u), \qquad \frac{\partial}{\partial \overline{\nu}_\pm} = \nu_\pm \cdot \mathrm{A}\nabla - \frac{1}{2}\nu \cdot \overline{\mathrm{B}}$$

The main aim of this work is to study solvability of this problem and behavior of its solution as $\varepsilon \to +0$ under some appropriate conditions on S^ε and $\gamma^\varepsilon_\pm$.

We treat this problem in the generalized sense. Namely, by $\mathring{W}^1_2(\Omega, \Upsilon)$ we denote the Sobolev space of functions vanishing on a given surface $\Upsilon \subset \Omega$. In $L_2(\Omega_\varepsilon)$ we introduce the non–linear form

$$\begin{aligned}\mathfrak{h}^\varepsilon(u,v) =& (\mathrm{A}\nabla u, \nabla v)_{L_2(\Omega)} + \frac{1}{2}(\mathrm{B}^t\nabla u, v)_{L_2(\Omega)} - \frac{1}{2}\left(u, \operatorname{div}\overline{\mathrm{B}}v\right)_{L_2(\Omega)} \\ &+ \left(V(\,\cdot\,,u), v\right)_{L_2(\Omega)} + \left(a_+(\,\cdot\,,u), v\right)_{L_2(S^\varepsilon_+\setminus\gamma^\varepsilon_+)} + \left(a_-(\,\cdot\,,u), v\right)_{L_2(S^\varepsilon_-\setminus\gamma^\varepsilon_-)}\end{aligned}$$

on the domain $\mathfrak{D}(\mathfrak{h}^\varepsilon) := \mathring{W}^1_2(\Omega_\varepsilon, \partial\Omega \cup \gamma^\varepsilon_+ \cup \gamma^\varepsilon_-)$. A solution to problem (2.2) is a function in $\mathfrak{D}(\mathfrak{h}^\varepsilon)$, which satisfies the integral identity

$$\mathfrak{h}^\varepsilon(u,v) - \lambda(u,v)_{L_2(\Omega)} = (f,v)_{L_2(\Omega)} \quad \text{for all} \quad v \in \mathfrak{D}(\mathfrak{h}^\varepsilon). \tag{2.3}$$

We proceed to the conditions on S^ε and $\gamma^\varepsilon_\pm$. Let $\gamma \subset \mathbb{R}^{n-1} \times \{0\}$ be a non–empty bounded $(n-1)$–dimensional domain with C^1–boundary. Then the problem

$$\Delta_\xi X_\gamma = 0, \quad \xi_n > 0, \qquad X_\gamma = 0, \quad \xi \in \gamma, \qquad \frac{\partial X_\gamma}{\partial \xi_n} = 0, \quad \xi \in (\mathbb{R}^{n-1} \times \{0\}) \setminus \overline{\gamma}, \tag{2.4}$$

has a unique solution (see, for instance, [3, Lm. 6.1]), which has a differentiable asymptotics

$$X_\gamma(x) = 1 + K_\gamma |\xi|^{-n+2} + O(|\xi|^{-n+1}), \qquad \xi \to \infty, \tag{2.5}$$

where K_γ is some constant.

Let $\eta = \eta(\varepsilon)$ be a small positive parameter and a positive function, which obeys the conditions

$$0 < \eta(\varepsilon) < 1, \qquad \varepsilon\eta^{-n+2}(\varepsilon) \to 0, \quad \varepsilon \to +0. \tag{2.6}$$

By $B_r(y)$ we denote the open ball in $\mathbb{R}^n$ of a radius r centered at a point y.

(A2). There exist countably many points $M^\varepsilon_{k,\pm} \subset \gamma^\varepsilon_\pm$, $k \in \mathbb{N}$, and a fixed numbers R_1, R_2, $R_3 > 0$ independent of ε and k such that

$$\Pi^{R_2\varepsilon}_\pm \cap \bigcup_{k\in\mathbb{N}} B_{R_1\varepsilon}(M^\varepsilon_{k,\pm}) = \Pi^{R_2\varepsilon}_\pm, \qquad |M^\varepsilon_{k,\pm} - M^\varepsilon_{j,\pm}| \geqslant R_3\varepsilon, \quad k \neq j. \tag{2.7}$$

(A3). For each point $M^\varepsilon_{k,\pm}$ there exists a subset $\gamma^\varepsilon_{k,\pm} \subset \gamma^\varepsilon_\pm$ such that $M^\varepsilon_{k,\pm} \in \gamma^\varepsilon_{k,\pm} \subset B_{\varepsilon\eta}(M^\varepsilon_{k,\pm})$ and for the set

$$\tilde{\gamma}^\varepsilon_{k,\pm} := \{\xi' \in \mathbb{R}^{n-1} : \ \xi' = (s(x) - s(M^\varepsilon_{k,\pm}))\varepsilon^{-1}\eta^{-1}(\varepsilon), \ x \in \gamma^\varepsilon_{k,\pm}\},$$

treated as located on $\mathbb{R}^{n-1}\times\{0\}$, in asymptotics (2.5) for the corresponding solution $X_{\tilde{\gamma}^\varepsilon_{k,\pm}}$ the error term and its first derivatives are uniform in k and ε, while the constant $K_{\tilde{\gamma}^\varepsilon_{k,\pm}}$ satisfies the estimate

$$0 < c_3 \leqslant |K_{\tilde{\gamma}^\varepsilon_{k,\pm}}| \leqslant c_4, \tag{2.8}$$

with some fixed positive constants c_i, $i = 3, 4$, independent of ε and k.

Under the made assumptions, the homogenized problem for problem (2.2) is that with the Dirichlet condition on S :

$$(\hat{\mathcal{H}} - \lambda)u^0 = f \quad \text{in} \quad \Omega \setminus S, \qquad u = 0 \quad \text{on} \quad \partial\Omega \cup S. \tag{2.9}$$

This problem is again treated in the generalized sense. The corresponding form $\mathfrak{h}^0$ in $L_2(\Omega)$ on the domain $\mathfrak{D}(\mathfrak{h}^0) := \mathring{W}_2^1(\Omega, \partial\Omega \cup S)$ and the integral identity read

$$\mathfrak{h}^0(u,v) = (\mathrm{A}\nabla u, \nabla v)_{L_2(\Omega)} + \frac{1}{2}(\mathrm{B}^t\nabla u, v)_{L_2(\Omega)} - \frac{1}{2}\left(u, \operatorname{div}\overline{B}v\right)_{L_2(\Omega)} + \left(V(\,\cdot\,,u), v\right)_{L_2(\Omega)}, \quad (2.10)$$

$$\mathfrak{h}^0(u^0,v) - \lambda(u^0,v) = (f,v)_{L_2(\Omega)}. \quad (2.11)$$

Now we are in position to formulate our main result.

Theorem 2.1. *Under Assumption (A1), there exists λ_0 independent of ε such that both problems (2.2) and (2.9) are uniquely solvable for $\operatorname{Re}\lambda \leqslant \lambda_0$ for all sufficiently small ε. Under additional assumptions (A2), (A3) and condition (2.6) the operator estimate holds*

$$\|u^\varepsilon - u^0\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}\eta^{-\frac{n}{2}+1}(\varepsilon)\|f\|_{L_2(\Omega)}, \quad (2.12)$$

where C is a constant independent of ε and f. If, in addition,

$$\mathrm{B} \in W_\infty^1(\Omega), \quad (2.13)$$

then the operator estimate holds

$$\|u_\varepsilon - u_0\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon\eta^{-n+2}(\varepsilon)\|f\|_{L_2(\Omega)} \quad (2.14)$$

with a constant C independent of ε and f.

Let us briefly discuss the main features of our problem and main result. Our domain Ω^ε contains the perforated surface S^ε, and the perforation can be chosen quite arbitrarily and essentially non–periodic. The surface S^ε is considered as two–sided and the boundary conditions are imposed on both sides independently. Each side of the surface S^ε is partitioned into two subsets, on one of them the Dirichlet condition is imposed, the other is subject to the weakly non–linear Robin condition, see (2.2). These are the main features of the perturbation we consider. In earlier works on homogenization of sieve problems the boundary condition was of the same type on both sides of the surface, the alternation of boundary conditions was not considered and in most part of works the perforation was periodic or locally–periodic. This is why our setting is new and in such generality it was not considered before. One more feature is that we consider the weakly nonlinear problem, the nonlinearity is involved both in the equation and boundary conditions, while the most part of earlier problems were for linear operators.

It should also be said that we consider the case, when the homogenization leads to the Dirichlet condition on the surface S. Such case is due to Assumptions (A2), (A3) and convergence (2.6). The most important here are Assumption (A2) and convergence (2.6), exactly these conditions ensure the Dirichlet condition. Once we omit at least one of these conditions, the homogenized problem changes. Assumption (A3) means that the sets $\gamma_\pm^\varepsilon$ are regular enough.

Our main results are two operator estimates (2.12) and (2.14). They state the uniform in the right hand f convergence of solution of the perturbed problem to that of the homogenized problem. In (2.12) we estimate the difference $u^\varepsilon - u^0$ in $W_2^1(\Omega)$–norm, while in (2.14) this is done in $L_2(\Omega)$–norm. Since the second norm is weaker, the convergence rate in (2.14) is twice higher than in (2.12). Such estimates are best possible in the sense that they are uniform in $L_2(\Omega)$–norm of the right hand side f; in most classical results in the homogenization theory the convergence usually established for each fixed f, and the constants in the estimates for the convergence rates just depend somehow on f. For the linear problems estimates (2.12), (2.14) imply the norm resolvent convergence with the corresponding estimates for the convergence rates. Despite our problem is very general and the perturbation is essential non–periodic, we succeeded to prove such best possible estimates and this is the main advantage of our main result.

3. Solvability and convergence

The proof of solvability follows quite standard scheme based on Browder — Minti theory. In our case such proof almost literally reproduces the proofs of Lemma 5.1 in [4] and of Lemma 2.1 in [7] and this is why we do not dwell on it. We also mention that within this proof, the estimates can be established

$$\begin{aligned}&\operatorname{Re}(\mathfrak{h}^\varepsilon(u,u-v)-\mathfrak{h}^\varepsilon(v,u-v)-\lambda\|u-v\|_{L_2(\Omega)}\geqslant C\|u-v\|^2_{W_2^1(\Omega)},\quad u,v\in\mathfrak{D}(\mathfrak{h}^\varepsilon),\\&\|u^0\|_{W_2^2(\Pi^{\tau_0}\setminus S)}\leqslant C\|f\|_{L_2(\Omega)},\end{aligned}\tag{3.1}$$

where C is some fixed positive constant independent of u and ε. The first estimate also holds for $\varepsilon=0$; then $u,v\in\mathfrak{D}(\mathfrak{h}^0)$.

The proof of operator estimates also follows quite standard scheme originally proposed in [5], and later it was successfully employed in many other works, see, for instance, [4], [6], [7], [8]. Namely, let $f\in L_2(\Omega)$ be an arbitrary function, $\operatorname{Re}\lambda\leqslant\lambda_0$, and u^ε and u^0 are the corresponding solutions to problems (2.2), (2.9), and $\chi=\chi(t)$ be an infinitely differentiable cut–off function, which equals to one as $|t|>\frac{2}{3}$ and vanishes as $|t|<\frac{1}{3}$. We then let

$$\chi^\varepsilon(x):=\chi\left(\frac{\tau}{R_2\varepsilon}\right)\quad\text{in}\quad\Pi^{R_2\varepsilon},\qquad\chi^\varepsilon(x):=1\quad\text{outside}\quad\Pi^{R_2\varepsilon}.$$

This function is infinitely differentiable and vanishes identically in $\Pi^{\frac{R_2\varepsilon}{3}}$. We then let $v^\varepsilon:=u^\varepsilon-\chi^\varepsilon u^0$ and write integral identities (2.3), (2.9) with $v=v^\varepsilon$ and $v=v^\varepsilon\chi^\varepsilon$, respectively. Then we take the difference of these identities using the obvious identity

$$\mathfrak{h}^0(u^0,v^\varepsilon\chi^\varepsilon)=\mathfrak{h}^0(u^0\chi^\varepsilon,v^\varepsilon)+F^\varepsilon(u^0,v^\varepsilon),\tag{3.2}$$

$$\begin{aligned}F^\varepsilon(u,v):=&(\mathrm{A}\nabla u,v\nabla\chi^\varepsilon)_{L_2(\Pi^{R_2\varepsilon})}-(u\mathrm{A}\nabla\chi^\varepsilon,\nabla v)_{L_2(\Pi^{R_2\varepsilon})}\\&-\frac{1}{2}(\mathrm{B}^t u\nabla\chi^\varepsilon,v)_{L_2(\Pi^{R_2\varepsilon})}+(\chi^\varepsilon V(\,\cdot\,,u)-V(\,\cdot\,,\chi^\varepsilon u),v)_{L_2(\Pi^{R_2\varepsilon})}.\end{aligned}\tag{3.3}$$

This gives

$$\mathfrak{h}^\varepsilon(u^\varepsilon,v^\varepsilon)-\mathfrak{h}^\varepsilon(u^0\chi^\varepsilon,v^\varepsilon)-\lambda\|v^\varepsilon\|^2_{L_2(\Omega)}=(f,(1-\chi^\varepsilon)v^\varepsilon)_{L_2(\Pi^{R_2\varepsilon})}+F^\varepsilon(u^0,v^\varepsilon).\tag{3.4}$$

The next step is to estimate the left hand side of this identity. We first of all mention the estimates [7, Lm. 2.2]

$$\|u\|_{L_2(\Pi^{R_2\varepsilon})}\leqslant C\varepsilon^{\frac{1}{2}}\|u\|_{W_2^1(\Omega)},\quad u\in W_2^1(\Omega_\varepsilon),\qquad\|u\|_{L_2(\Pi^{R_2\varepsilon})}\leqslant C\varepsilon^{\frac{3}{2}}\|u\|_{W_2^2(\Omega)},\tag{3.5}$$

where C are some constants independent of ε and u. On the base of these estimates and the second estimate in (3.1) all terms in $F^\varepsilon(u^0,v^\varepsilon)$ except for the first one are estimated by $C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v^\varepsilon\|_{W_2^1(\Omega)}$; hereinafter C are some constants independent of ε, f, u^0, u^ε, v^ε. The most nontrivial estimate here is that for the first term in F^ε; this term can be immediately estimated by $C\varepsilon^{-\frac{1}{2}}\|f\|_{L_2(\Omega)}\|v^\varepsilon\|_{L_2(\Pi^{R_2\varepsilon})}$. The required estimate for this term is provided by the next lemma.

Lemma 3.1. *Under Assumptions (A2), (A3) for all $v\in W_2^1(\Omega^\varepsilon,\partial\Omega\cup\gamma_+^\varepsilon\cup\gamma_-)$ the estimate holds*

$$\|v\|_{L_2(\Pi^{R_2\varepsilon})}\leqslant C\varepsilon\eta^{-\frac{n}{2}+1}(\varepsilon)\|v\|_{W_2^1(\Omega)},$$

where C is a constant independent of ε and v.

We shall prove this crucial lemma in what follows and this is in fact the main content of our work. Once this lemma holds, the right hand side of (3.4) is estimated by $C\varepsilon^{\frac{1}{2}}\eta^{-\frac{n}{2}+1}(\varepsilon)\|f\|_{L_2(\Omega)}\|v^\varepsilon\|_{W_2^1(\Omega)}$. This immediately gives

$$\|v^\varepsilon\|_{W_2^1(\Omega^\varepsilon)}\leqslant C\varepsilon^{\frac{1}{2}}\eta^{-\frac{n}{2}+1}(\varepsilon)\|f\|_{L_2(\Omega)}.\tag{3.6}$$

Then it follows from (3.1), (3.5) that

$$\|(1-\chi^\varepsilon)u^0\|_{L_2(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{3}{2}}\|f\|_{L_2(\Omega)}, \qquad \|\nabla(1-\chi^\varepsilon)u^0\|_{L_2(\Omega^\varepsilon)} \leqslant C\varepsilon^{\frac{1}{2}}\|f\|_{L_2(\Omega)}, \tag{3.7}$$

and this gives estimate (2.12).

The proof of estimate (2.14) follows the scheme used in [7], [8], [10] and based on duality approach [25], [26]. We consider the boundary value problem

$$\left(-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\overline{A_{ij}}\frac{\partial}{\partial x_i} - \sum_{j=1}^n \frac{\partial}{\partial x_j}\overline{A_j} - \overline{\lambda}\right) w = v^\varepsilon \quad \text{in} \quad \Omega\setminus S, \qquad w = 0 \quad \text{on} \quad \partial\Omega\cup S. \tag{3.8}$$

This problem is a particular case of (2.9) and this is why, without loss of generality, we suppose that it is solvable for $\operatorname{Re}\lambda \leqslant \lambda_0$. The estimate similar to the second one in (3.1) holds

$$\|w\|_{W_2^2(\Pi^{\tau_0}\setminus S)} \leqslant C\|v^\varepsilon\|_{L_2(\Omega)}. \tag{3.9}$$

We write the integral identity for problem (3.8) with the test function $\chi^\varepsilon v^\varepsilon$ and rewrite it using an identity similar to (3.2). This gives

$$\mathfrak{h}^\varepsilon(u^\varepsilon,\chi^\varepsilon w) - \mathfrak{h}^\varepsilon(u^0\chi^\varepsilon,\chi^\varepsilon w) - \lambda(v^\varepsilon,\chi^\varepsilon w)_{L_2(\Omega)} + G^\varepsilon = (\chi^\varepsilon v^\varepsilon, v^\varepsilon)_{L_2(\Omega)}, \tag{3.10}$$

$$\begin{aligned} G^\varepsilon :=\ & (\mathrm{A}v^\varepsilon\nabla\chi^\varepsilon,\nabla w)_{L_2(\Pi^{R_2\varepsilon})} - (\mathrm{A}\nabla v^\varepsilon, w\nabla\chi^\varepsilon)_{L_2(\Pi^{R_2\varepsilon})} + \frac{1}{2}(v^\varepsilon, w\overline{\mathrm{B}^t}\nabla\chi^\varepsilon)_{L_2(\Pi^{R_2\varepsilon})} \\ & - (V(\,\cdot\,,u^\varepsilon),\chi^\varepsilon w)_{L_2(\Pi^{R_2\varepsilon})} + (V(\,\cdot\,,u^\varepsilon\chi^\varepsilon),w)_{L_2(\Pi^{R_2\varepsilon})}. \end{aligned}$$

Then we reproduce the arguing, which led to identity (3.4) but initially use the test functions $\chi^\varepsilon w$ and $(\chi^\varepsilon)^2 w$ in integral identities (2.3), (2.9). This gives the identity similar to (3.4):

$$\mathfrak{h}^\varepsilon(u^\varepsilon,w\chi^\varepsilon) - \mathfrak{h}^\varepsilon(u^0\chi^\varepsilon,w\chi^\varepsilon) - \lambda(v^\varepsilon,\chi^\varepsilon w)_{L_2(\Omega)} = (f,(1-\chi^\varepsilon)\chi^\varepsilon w)_{L_2(\Pi^{R_2\varepsilon})} + F^\varepsilon(u^0,w\chi^\varepsilon).$$

This identity and (3.10) yields

$$(\chi^\varepsilon v^\varepsilon, v^\varepsilon)_{L_2(\Omega)} = G^\varepsilon + (f,(1-\chi^\varepsilon)\chi^\varepsilon w)_{L_2(\Pi^{R_2\varepsilon})} + F^\varepsilon(u^0,w\chi^\varepsilon).$$

Now we estimate all terms in the right hand side by means of (3.5), (3.6), (3.9) and Lemma 3.1 and we obtain

$$\|v^\varepsilon\|^2_{L_2(\Omega)} = \big((1-\chi^\varepsilon)v^\varepsilon, v^\varepsilon\big)_{L_2(\Omega)} + (\chi^\varepsilon v^\varepsilon, v^\varepsilon)_{L_2(\Omega)} \leqslant C\varepsilon\eta^{-n+2}(\varepsilon)\|v^\varepsilon\|_{L_2(\Omega)}\|f\|_{L_2(\Omega)}.$$

This estimate and the second estimate in (3.5) prove (2.14).

4. Proof of Lemma 3.1

In this section we prove Lemma 3.1. We begin with two auxiliary statements; the proof of first statements reproduces literally the proof of Lemma 4.2 in [4].

Lemma 4.1. *We fix $R_4 \geqslant R_1$. Under Assumption (A2), for each point $x \in S$, the number of balls $B_{R_4\varepsilon}(M^\varepsilon_{k,\pm})$ containing this point is bounded by some constant which is independent of k and ε but depends on R_4.*

Lemma 4.2. *Under Assumption (A3), for all $r > 1$ the functions $X_{\tilde\gamma^\varepsilon_{k,\pm}}$ satisfy uniform in k, ε and r estimate*

$$\|X_{\tilde\gamma^\varepsilon_{k,\pm}} - 1\|^2_{L_2(\theta_r)} \leqslant C\varkappa(r), \qquad \theta_r := B_r(0)\cap\{\xi :\ \xi_n > 0\}, \qquad \varkappa(r) := \begin{cases} r, & n = 3, \\ \ln r, & n = 4, \\ 1, & n \geqslant 5. \end{cases}$$

Proof. To simplify the notations, within the proof we let $\tilde X := X_{\tilde\gamma^\varepsilon_{k,\pm}} - 1$. Since by Assumption (A3) asymptotics (2.5) for $X_{\tilde\gamma^\varepsilon_{k,\pm}}$ is uniform in k and ε, in view of estimate (2.8) there exists a fixed $r_0 > 1$ such that for $r > r_0$ we have

$$\|\tilde X\|^2_{L_2(\theta_r\setminus\theta_{r_0})} \leqslant C \int\limits_{\theta_r\setminus\theta_{r_0}} |\xi|^{-2n+4}\,dx \leqslant C\varkappa(r); \tag{4.1}$$

here and till the end of the proof by C we denote inessential positive constant independent of r, k, ε and ξ. Integrating by parts, in view of boundary value problem (2.4) we find

$$0 = \int\limits_{\theta_r} X_{\tilde\gamma^\varepsilon_{k,\pm}}\Delta_\xi X_{\tilde\gamma^\varepsilon_{k,\pm}} = \int\limits_{\partial\theta_r\cap\{\xi:\ \xi_n>0\}} X_{\tilde\gamma^\varepsilon_{k,\pm}}\frac{\partial}{\partial|\xi|}X_{\tilde\gamma^\varepsilon_{k,\pm}}\,ds - \|\nabla_\xi X_{\tilde\gamma^\varepsilon_{k,\pm}}\|^2_{L_2(\theta_r)},$$

and by (2.5), (2.8) we get

$$\|\nabla_\xi\tilde X\|^2_{L_2(\theta_r)} = \|\nabla_\xi X_{\tilde\gamma^\varepsilon_{k,\pm}}\|^2_{L_2(\theta_r)} \leqslant C|K_{\tilde\gamma^\varepsilon_{k,\pm}}| \leqslant C. \tag{4.2}$$

In the same way we integrate by parts

$$\|\tilde X\|^2_{L_2(\theta_{2r_0})} = \frac12\int\limits_{\theta_{2r_0}}\tilde X^2\Delta_\xi\xi_n^2\,d\xi = \int\limits_{\partial\theta_{2r_0}\cap\{\xi:\ \xi_n>0\}}\left(\tilde X^2\frac{\partial\xi_n^2}{\partial|\xi|} - \xi_n^2\tilde X\frac{\partial\tilde X}{\partial|\xi|}\right)ds - \|\xi_n\nabla_\xi\tilde X\|^2_{L_2(\theta_{2r_0})}.$$

Hence, in view of (2.5), (2.8), (4.2), the norm $\|\tilde X\|^2_{L_2(\theta_{2r_0})}$ is bounded uniformly in k and ε, and together with (4.1) this completes the proof. □

We choose $R_4 \geqslant R_1$ and denote

$$\omega^\varepsilon_{k,\pm} := \bigl\{x:\ |s - s(M^\varepsilon_{k,\pm})|^2 + |\tau|^2 < R_4^2\varepsilon^2,\ \pm\tau > 0\bigr\}.$$

In view of Assumptions (A1), (A2), it is clear that for all sufficiently large R_4 we have

$$\Pi^{R_2\varepsilon}_\pm \subset \bigcup_{k\in\mathbb{N}}\omega^\varepsilon_{k,\pm} =: \omega^\varepsilon_\pm. \tag{4.3}$$

We let

$$\lambda^\varepsilon_{k,\pm} := \inf_{u\in\mathring W^1(\omega^\varepsilon_{k,\pm},\gamma^\varepsilon_{k,\pm})}\frac{\|\nabla u\|^2_{L_2(\omega^\varepsilon_{k,\pm})}}{\|u\|^2_{L_2(\omega^\varepsilon_{k,\pm})}}, \qquad \lambda^\varepsilon_\pm := \inf_{k\in\mathbb{N}}\lambda^\varepsilon_{k,\pm}, \Gamma^\varepsilon_\pm := \bigcup_{k\in\mathbb{N}}\gamma^\varepsilon_{k,\pm}.$$

In view of Lemma 4.1 and embedding (4.3), for all $u \in \mathring W^1_2(\omega^\varepsilon_\pm, \Gamma^\varepsilon_\pm)$ we have

$$\|u\|^2_{L_2(\omega^\varepsilon_\pm)} \leqslant \sum_{k\in\mathbb{N}}\|u\|^2_{L_2(\omega^\varepsilon_{k,\pm})} \leqslant \frac{1}{\lambda^\varepsilon_\pm}\sum_{k\in\mathbb{N}}\|\nabla u\|^2_{L_2(\omega^\varepsilon_{k,\pm})} \leqslant \frac{C}{\lambda^\varepsilon_\pm}\sum_{k\in\mathbb{N}}\|\nabla u\|^2_{L_2(\omega^\varepsilon_{k,\pm})}, \tag{4.4}$$

where C is some constant independent of ε and u. Hence, in order to prove Lemma 3.1, it is sufficient to find an appropriate lower bound for $\lambda^\varepsilon_\pm$, or, equivalently, a lower bound for all $\lambda^\varepsilon_{k,\pm}$ uniformly in k. We shall show that under Assumptions (A1), (A2), (A3) the needed bound reads

$$\lambda^\varepsilon_{k,\pm} \geqslant C\varepsilon^{-2}\eta^{n-2}(\varepsilon), \tag{4.5}$$

where C is a fixed positive constant independent of ε, k. In view of (4.4) such estimate obviously implies the statement of Lemma 3.1, and this is why in what follows we are going to prove estimate (4.5).

In each domain $\omega^\varepsilon_{k,\pm}$, we pass to the variables $\zeta := \big(s(x) - s(M^\varepsilon_{k,\pm}))R_4^{-1}\varepsilon^{-1}, \pm\tau R_4^{-1}\varepsilon^{-1}\big)$. The image of each domain $\omega^\varepsilon_{k,\pm}$ in terms of these variables is obviously $\omega := \big\{\zeta :\ |\zeta| < 1,\ \xi_n > 0\big\}$. Then in view of Assumption (A1) we immediately see that

$$\lambda^\varepsilon_{k,\pm} \geqslant CR_4^{-1}\varepsilon^{-2}\mu^\varepsilon_{k,\pm}, \qquad \mu^\varepsilon_{k,\pm} := \inf_{u\in \mathring{W}^1_2(\omega,\hat\gamma^\varepsilon_{k,\pm})} \frac{\|\nabla_\zeta u\|^2_{L_2(\omega)}}{\|u\|^2_{L_2(\omega)}}, \tag{4.6}$$

where C is some fixed positive constant independent of k and ε and $\hat\gamma^\varepsilon_{k,\pm} := \{\zeta : R_4\zeta \in \tilde\gamma^\varepsilon_{k,\pm},\ \zeta_n = 0\}$. It is clear that $\mu^\varepsilon_{k,\pm}$ is the lowest eigenvalue of the negative Laplacian operator $\mathcal{H}^\varepsilon_{k,\pm}$ in ω subject to the Dirichlet condition on $\hat\gamma^\varepsilon_{k,\pm}$ and to the Neumann condition on $\partial\omega\setminus\overline{\hat\gamma^\varepsilon_{k,\pm}}$.

It follows from the embedding $\gamma^\varepsilon_{k,\pm} \subset B_{\varepsilon\eta}(M^\varepsilon_{k,\pm})$ and the boundedness of η (see (2.6)) that the set $\hat\gamma^\varepsilon_{k,\pm}$ is located inside the ball $B_{CR_4^{-1}\eta(\varepsilon)}(0)$, where C is some constant independent of R_4 and $\eta(\varepsilon)$, and this ball shrinks to the origin as $R_4 \to +\infty$. By $\mathcal{H}$ we denote the negative Neumann Laplacian in ω. Literally reproducing the proof Theorem 1.2 in [9], we can establish that the operator $\mathcal{H}^\varepsilon_{k,\pm}$ converges to $\mathcal{H}$ in the norm resolvent sense as $R_4 \to +\infty$ and the estimate holds

$$\|(\mathcal{H}^\varepsilon_{k,\pm} + 1)^{-1} - (\mathcal{H} + 1)^{-1}\| \leqslant C\rho\big(\eta(\varepsilon)R_4^{-1}\big), \tag{4.7}$$

where C is some fixed constant independent of η, R_4 and k, and $\rho(t) \to 0$ as $t \to +0$. Hence, all lowest eigenvalues $\mu^\varepsilon_{k,\pm}$ tend to zero as $R_4 \to +\infty$ uniformly in k and ε, while other eigenvalues of $\mathcal{H}^\varepsilon_{k,\pm}$ are uniformly separated from zero by some fixed positive distance.

In terms of the functions $X_{\tilde\gamma^\varepsilon_{k,\pm}}(\xi)$ mentioned in Assumption (A3) we introduce the function

$$\begin{aligned}\Phi(\zeta) :=& 1 + \Big(X_{\tilde\gamma^\varepsilon_{k,\pm}}\big(R_4\eta^{-1}(\varepsilon)\zeta\big) - 1\Big)\chi\big(R_4^{\frac12}\eta^{-\frac12}(\varepsilon)|\zeta|\big)\\ &+ K_{\tilde\gamma^\varepsilon_{k,\pm}}R_4^{-n+2}\eta^{n-2}(\varepsilon)\big(1 - \chi\big(R_4^{\frac12}\eta^{-\frac12}(\varepsilon)|\zeta|\big)\big)\bigg(|\zeta|^{-n+2} + \frac{n-2}{2}|\zeta|^2\bigg).\end{aligned} \tag{4.8}$$

It is straightforward to verify that the function Φ satisfies the Dirichlet condition on $\hat\gamma^\varepsilon_{k,\pm}$ and the Neumann condition on $\partial\omega\setminus\overline{\hat\gamma^\varepsilon_{k,\pm}}$ and belongs to the domain of operator $\mathcal{H}^\varepsilon_{k,\pm}$. By direct calculations we find

$$\begin{aligned}&(\mathcal{H}^\varepsilon_{k,\pm} - \hat\mu)\Phi = h := h_1 + h_2 + h_3, \qquad \hat\mu := -n(n-2)K_{\tilde\gamma^\varepsilon_{k,\pm}}R_4^{-n+2}\eta^{n-2}(\varepsilon),\\
&h_1(\zeta) := -\frac{n-2}{2}K_{\tilde\gamma^\varepsilon_{k,\pm}}R_4^{-n+2}\eta^{n-2}(\varepsilon)\Delta_\zeta\chi\big(R_4^{\frac12}\eta^{-\frac12}(\varepsilon)|\zeta|\big)|\zeta|^2,\\
&h_2(\zeta) := -2\nabla_\zeta\chi\big(R_4^{\frac12}\eta^{-\frac12}(\varepsilon)|\zeta|\big)\cdot\nabla_\zeta\Big(K_{\tilde\gamma^\varepsilon_{k,\pm}}R_4^{-n+2}\eta^{n-2}(\varepsilon)|\zeta|^{-n+2} - X_{\tilde\gamma^\varepsilon_{k,\pm}}\big(R_4\eta^{-1}(\varepsilon)\zeta\big)\Big)\Big),\\
&h_3(\zeta) := -\Big(K_{\tilde\gamma^\varepsilon_{k,\pm}}R_4^{-n+2}\eta^{n-2}(\varepsilon)|\zeta|^{-n+2} + 1 - X_{\tilde\gamma^\varepsilon_{k,\pm}}\big(R_4\eta^{-1}(\varepsilon)\zeta\big)\Big)\Delta_\zeta\big(R_4^{\frac12}\eta^{-\frac12}(\varepsilon)|\zeta|\big).\end{aligned}$$

Let us estimate the $L_2(\omega)$–norms of function h_i. We recall that by Assumption (A3) all constants $K_{\tilde\gamma^\varepsilon_{k,\pm}}$ are bounded from above and below, see (2.8); in further calculations we employ this fact not mentioning it explicitly.

Using asymptotics (2.5) for $X_{X_{\tilde\gamma^\varepsilon_{k,\pm}}}$ and the fact that it is uniform in k and ε, by direct calculations we verify the estimates

$$\|h_1\|_{L_2(\omega)} \leqslant C\left(\frac{\eta(\varepsilon)}{R_4}\right)^{n-2+\frac{n}{4}}, \qquad \|h_2\|_{L_2(\omega)} + \|h_3\|_{L_2(\omega)} \leqslant C\left(\frac{\eta(\varepsilon)}{R_4}\right)^{n-1+\frac{n-2}{4}}. \tag{4.9}$$

hereinafter by C we denote inessential positive constants independent of ε, R_4, η and k. Using Lemma 4.2, we can estimate the norm $\|\Phi - 1\|_{L_2(\omega)}$ and we obtain

$$\|\Phi - 1\|_{L_2(\omega)} \leqslant C\left(\frac{\eta(\varepsilon)}{R_4}\right)^{\frac12}, \qquad \|\Phi\|_{L_2(\Omega)} \geqslant C > 0. \tag{4.10}$$

It follows from (2.8) that $|\mu| \leqslant CR_4^{-n+2}\eta^{n-2}(\varepsilon)$. Then, in view of (4.7) and the standard results on the behavior of resolvents of self–adjoint operators in the vicinity of isolated eigenvalues [19, Ch. V, Sect. 3.5], we have

$$\Phi = (\mathcal{H}^\varepsilon_{k,\pm} - \hat{\mu})^{-1}h = \frac{\psi^\varepsilon_{k,\pm}(h, \psi^\varepsilon_{k,\pm})}{\mu^\varepsilon_{k,\pm} - \hat{\mu}} + \mathcal{R}^\varepsilon_{k,\pm}h,$$

where $\psi^\varepsilon_{k,\pm}$ is the eigenfunction of $\mathcal{H}^\varepsilon_{k,\pm}$ associated of $\mu^\varepsilon_{k,\pm}$ normalized in $L_2(\omega)$, and $\mathcal{R}^\varepsilon_{k,\pm}$ is the associated reduced resolvent, which is bounded uniformly in k and ε. It follows from the above representation and (4.9), (4.10) that

$$0 < C \leqslant \|\Phi - \mathcal{R}^\varepsilon_{k,\pm}h\| \leqslant \frac{C}{|\mu^\varepsilon_{k,\pm} - \hat{\mu}|}, \qquad |\mu^\varepsilon_{k,\pm} - \hat{\mu}| \leqslant C\|h\|_{L_2(\omega)}) \leqslant C\left(\frac{\eta(\varepsilon)}{R_4}\right)^{n-2+\frac{n}{4}}.$$

In view of the definition of $\hat{\mu}$, we then get

$$\mu^\varepsilon_{k,\pm} \geqslant CR_4^{-n+2}\eta^{n-2}(\varepsilon),$$

provided R_4 is large enough. We choose and fix such sufficiently large R_4 and substitute then the above estimate into (4.6), which immediately gives (4.5) and completes the proof of Theorem 2.1.

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REFERENCES

1. Y. Amirat, O. Bodart, G.A. Chechkin, A.L. Piatnitski. *Asymptotics of a spectral-sieve problem* // J. Math. Anal. Appl. **435**:2, 1652–1671 (2016).
2. N. Ansini. *The nonlinear sieve problem and applications to thin films* // Asymptotic Anal. **39**:2, 113–145 (2004).
3. D.I. Borisov, T.F. Sharapov. *On the resolvent of multidimensional operators with frequently alternating boundary conditions with the Robin homogenized condition* // J. Math. Sci. **213**:4, 461–503 (2016).
4. D.I. Borisov, A.I. Mukhametrakhimova. *Uniform convergence and asymptotics for problems in domains finely perforated along a prescribed manifold in the case of the homogenized Dirichlet condition* // Sb. Math. **212**:8, 1068–1121 (2021).
5. D. Borisov, G. Cardone, L. Faella, C. Perugia. *Uniform resolvent convergence for strip with fast oscillating boundary* // J. Differ. Equations **255**:12, 4378–4402 (2013).
6. D. I. Borisov, R. R. Suleimanov. *On operator estimates for elliptic operators with mixed boundary conditions in two–dimensional domains with rapidly oscillating boundary* // Math. Notes **116**:2, 182–199 (2024).
7. D.I. Borisov, R.R. Suleimanov. *Operator estimates for elliptic equations in multidimensional domains with strongly curved boundary* // Sb. Math. **216**:1, 25–53 (2025).
8. D.I. Borisov, R.R. Suleimanov. *On operator estimates for elliptic equations in two-dimensional domains with fast oscillating boundary and frequent alternation of boundary conditions* // Sb. Math. **216**:8, 1037–1054 (2025).
9. D.I. Borisov, A.I. Mukhametrakhimova. *On norm resolvent convergence for elliptic operators in multi-dimensional domains with small holes* // J. Math. Sci. **232**:3, 283–298 (2018).
10. D.I. Borisov, A.I. Mukhametrakhimova. *Norm resolvent convergence for problems with perforation alogn a given manifold and with a nonlinear Robin condition on the boundaries of cavities* // St.–Petersburg Math. J. **35**:4, 611–652 (2024).

11. D.I. Borisov. *Operator estimates in two-dimensional problem with a frequent alternation in the case of small parts with the Dirichlet condition* // Proc. Steklov Inst. Math. **321**:Suppl. 1, S33–S52 (2023).
12. D. Borisov, G. Cardone, T. Durante. *Homogenization and uniform resolvent convergence for elliptic operators in a strip perforated along a curve* // Proc. R. Soc. Edinb., Sect. A, Math. **146**:6, 1115–1158 (2016).
13. D. Borisov, R. Bunoiu, G. Cardone. *Waveguide with non–periodically alternating Dirichlet and Robin conditions: homogenization and asymptotics* // Z. Angew. Math. Phys. **64**:3, 439–472 (2013).
14. D. Borisov, R. Bunoiu, and G. Cardone. *On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition* // Ann. H. Poincaré. **11**:8, 1591–1627 (2010).
15. D. Borisov, and G. Cardone. *Homogenization of the planar waveguide with frequently alternating boundary conditions* // J. Phys. A, Math. Theor. **42**:36, 365205 (2009).
16. D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei. *The periodic unfolding method for perforated domains and Neumann sieve models* // J. Math. Pures Appl. **89**:3, 248–277 (2008).
17. A. Damlamian. *Le problème de la passoire de Neumann* // Rend. Semin. Mat. Torino **43**, 427–450, (1985).
18. T. Del Vecchio. *The thick Neumann's sieve* // Ann. Mat. Pura Appl. IV. Ser. **147**, 363–402, (1987).
19. T. Kato. *Perturbation Theory for Linear Operators.* Springer–Verlag, Berlin (1976).
20. A. Khrabustovskyi. *The Neumann sieve problem revisited* // J. Math. Anal. Appl. **554**:1, 129933 (2026).
21. A. Khrabustovskyi. *Operator estimates for the Neumann sieve problem* // Ann. Mat. Pura Appl. (4) **202**:4, 1955–1990 (2023).
22. V.A. Marchenko, E.Y. Khruslov. *Boundary value problems in domains with a fine–grained boundary.* Naukova Dumka, Kiev (1974).
23. F. Murat. *The Neumann sieve* // in "Nonlinear variational problems", Pitman Publishing, London 24–32 (1985).
24. C. Picard. *Analyse limite d'équations variationnelles dans un domaine contenant une grille* // RAIRO, Modélisation Math. Anal. Numér. **21**:2, 293–326 (1987).
25. N.N. Senik. *Homogenization for non–self–adjoint periodic elliptic operators on an infinite cylinder* // SIAM J. Math. Anal. **49**:2, 874–898 (2017).
26. N.N. Senik. *Homogenization for locally periodic elliptic operators* // J. Math. Anal. Appl. **505**:2, 125581 (2022).

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