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RECOVERY OPERATOR FOR HAAR WAVELET SPACES

S.Zh. BASSAROV, N.T. TLEUKHANOVA

Abstract. In this paper we present a constructive approximation of a function by using its values at certain points. We consider the approximation of functions, the their Fourier — Haar coefficients of which decay rapidly.

We provide a method for calculating the Fourier — Haar coefficients and analyse the convergence rate of the error. If the function is a Haar polynomial with a degree at most $2^m - 1$ (meaning that $\prod_{i=1}^{d} k_i < 2^m$, where k_i is a degree of ith variable), then the method approximates Fourier — Haar coefficients with the error $O(m^{d-1}2^m)$, where d is the amount of points used for the approximation. The method is based on a sparse grid with weights and nodes given explicitly.

We construct a recovery operator based on our method. The recovery operator is a Haar polynomial with coefficients expressed in terms of the values of function. The recovery operator is exact for the corresponding Haar polynomial. We analyse the error made by the recovery operator in the metric of L_∞ and obtain an upper bound for the error in terms of m.

Keywords: multi–dimension, function approximation, Fourier coefficient, Fourier series, Haar function, Haar system.

Mathematics Subject Classification: 41A10, 41A25

Dedicated to the memory of professor M.D. Ramazanov.

1. Introduction

Let $m = j + 2^{k-1}$, $j = 0, 1, \ldots, 2^{k-1} - 1$, $k \in \mathbb{N}$. Then $m \geqslant 1$. For $k = 0$ we let $j = 0$ and $m = 0$. We define

$$\Delta_m = \Delta_k^j = \left[\frac{j}{2^{k-1}}, \frac{j+1}{2^{k-1}}\right), \qquad \Delta_0 = \Delta_0^0 = [0, 1).$$

Let $k, l \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. The following statements are obviously true.

1. If $k = l$ and $i \neq j$, then $\Delta_k^i \cap \Delta_l^j = \emptyset$.
2. If $\Delta_k^i \cap \Delta_l^j \neq \emptyset$, then either $\Delta_k^i \subset \Delta_l^j$ or $\Delta_l^j \subset \Delta_k^i$.

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Definition 1.1. *Let $m = j + 2^{k-1}$, $j = 0, 1, \dots, 2^{k-1} - 1$, $k \in \mathbb{N}$, $x \in [0, 1)$. The functions*

$$h_0(x) = 1,$$
$$h_m(x) = h_k^j(x) = \begin{cases} 2^{\frac{k-1}{2}}, & x \in \Delta_m^+ = \Delta_{k+1}^{2j} = \left[\dfrac{j}{2^{k-1}}, \dfrac{2j+1}{2^k}\right), \\ -2^{\frac{k-1}{2}}, & x \in \Delta_m^- = \Delta_{k+1}^{2j+1} = \left[\dfrac{2j+1}{2^k}, \dfrac{j+1}{2^{k-1}}\right), \\ 0, & \textit{othwerwise}, \end{cases} \tag{1.1}$$

are called the Haar functions.

Definition 1.2. *Let $d \in \mathbb{N}$. The function*

$$h_{\mathbf{m}}(\mathbf{x}) = h_{m_1,\dots,m_d}(x_1, \dots, x_d) = h_{k_1}^{j_1}(x_1) h_{k_2}^{j_2}(x_2) \cdot \ldots \cdot h_{k_d}^{j_d}(x_d), \tag{1.2}$$

is called the multivariate (multiple) Haar function, where $m_i = j_i + 2^{k_i - 1}$, $j_i = 0, 1, \dots, 2^{k_i-1} - 1$, $k_i \in \mathbb{Z}_+$, $x_i \in [0, 1)$, $i = 1, \dots, d$.

The question on approximating a function by a polynomial goes back to the famous Weierstrass theorem. After this theorem, the study developed from the univariate case to the multivariate case in various function spaces.

In this paper we consider the following problem. Let $\alpha > \frac{1}{q}$, $1 \leqslant q < \infty$, $1 < \tau < \infty$ and $f \in HA_{q,\tau}^{\alpha}$ (see the definition below). We are interesting in an upper bound for the quantity

$$\sup_{\|f\|_{HA_{q,\tau}^{\alpha}}=1} \|f - F_m(\,\cdot\,; f)\|_{L_\infty},$$

in terms of m, which is the logarithm of the highest degree involved in the polynomial $F_m(\mathbf{x}; f)$, which will be defined later in (2.2), (2.3).

An interesting property of the spaces $HA_{q,\tau}^{\alpha}$ is the embeddings

$$B_{q,\tau}^{\alpha} \hookrightarrow HA_{q,\tau}^{\alpha}, \tag{1.3}$$
$$W_q^{\alpha} \hookrightarrow HA_{q,\max(q,2)}^{\alpha}, \tag{1.4}$$

where $B_{q,\tau}^{\alpha}$, W_q^{α} are Besov and Sobolev spaces with a dominating mixed derivative, $0 < \alpha < 1$, $1 \leqslant q, \tau \leqslant \infty$.

Embedding (1.3) are due to [9, Thm. 1.52] with $A = 0$ and $B = 1$; we note that $B_{q,\tau}^{\alpha} = B_{q,\tau}^{\alpha}[0, 1)^d$ is the restriction of $B_{q,\tau}^{\alpha}(\mathbb{R}^d)$. Embedding (1.4) is implied by the known embedding $W_q^{\alpha} \hookrightarrow B_{q,\max(q,2)}^{\alpha}$. Here $d \in \mathbb{N}$ is the dimension. In what follows vectors will be denoted by the **bold** font; we also let

$$\mathbf{1} = (1, \dots, 1), \qquad \mathbf{0} = (0, \dots, 0), \qquad \mathbf{2^k} = (2^{k_1}, \dots, 2^{k_d}), \qquad \operatorname{sgn}(\mathbf{x}) = (\operatorname{sgn}(x_1), \dots, \operatorname{sgn}(x_d)).$$

The constant c may vary from line to line.

2. Main results

Definition 2.1. *Let $\alpha > 0$ and $1 \leqslant q, \tau \leqslant \infty$. We define a Haar wavelet space $HA_{q,\tau}^{\alpha} = HA_{q,\tau}^{\alpha}[0, 1)^d$ as a set of functions*

$$f \in L_1 = \left\{ f : \int\limits_{[0,1)^d} |f(\mathbf{x})| \mathbf{dx} < \infty \right\}$$

such that

$$\|f\|_{HA^{\alpha}_{q,\tau}} = \left(\sum_{k=0}^{\infty} 2^{\left(\alpha+\frac{1}{2}-\frac{1}{q}\right)k\tau} \sum_{\substack{|\mathbf{s}|_1=k,\\ \mathbf{s}\in\mathbb{Z}^d_+}} \left(\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{2^{s-1}-1}} |a^{\mathbf{j}}_{\mathbf{s}}|^q \right)^{\frac{\tau}{q}} \right)^{\frac{1}{\tau}} < \infty, \tag{2.1}$$

where

$$a^{\mathbf{j}}_{\mathbf{s}} = \int\limits_{[0,1)^d} f(\mathbf{x}) h^{\mathbf{j}}_{\mathbf{s}}(\mathbf{x})\, d\mathbf{x}$$

is the Haar coefficient of the function f.

In case of $q=\infty$ or $\tau=\infty$ the norm is to be modified in the usual way as for l_r.

Remark 2.1. *In case $\alpha=\frac{1}{q}$, $1\leqslant q<\infty$ and $\tau=1$ we have $HA^{\frac{1}{q}}_{q,1}=S_q$, where S_q was defined by Sobol' [6]. In case $\alpha>0$, $q=\infty=\tau$ we have $HA^{\alpha}_{\infty,\infty}=E_d^{\alpha+\frac{1}{2}}$, where E^{α}_d was defined by Entacher [2].*

The properties of the space $HA^{\alpha}_{q,\tau}$ will be discussed in Section 3.
We define a recovery operator as follows

$$F_m(\mathbf{x};f) = \sum_{\substack{|\mathbf{p}|_1\leqslant m,\\ \mathbf{p}\in\mathbb{Z}^d_+}} \sum_{j_1=0}^{2^{p_1-1}-1} \cdots \sum_{j_d=0}^{2^{p_d-1}-1} T_{m,\mathbf{p}}\left(f \prod_{i=1}^{d} h^{j_i}_{p_i} \right) \prod_{i=1}^{d} h^{j_i}_{p_i}(x_i), \tag{2.2}$$

$$T_{m,\mathbf{p}}(f) = \frac{1}{2^m} \sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ \mathbf{k}\in\mathbb{Z}^d_+}} \sum_{r_1=0}^{2^{k_1+p_1}-1} \cdots \sum_{r_d=0}^{2^{k_d+p_d}-1} (-1)^{\sum\limits_{i=1}^{d-1}(r_i+1)\operatorname{sgn}(k_i)} f\left(\frac{r_1}{2^{k_1+p_1}}, \ldots, \frac{r_d}{2^{k_d+p_d}} \right). \tag{2.3}$$

Remark 2.2. *The recovery operator (2.2), (2.3) with respect to the trigonometric system was considered in [4].*

Remark 2.3. *Similar approaches were considered in [7, 3] for functions with moderate smoothness. In [7, 3] they used rank one lattice rule for calculating the Fourier coefficients and approximation of function.*

Theorem 2.1. *Let $m\in\mathbb{N}$, $\mathbf{p}\in\mathbb{Z}^d_+$, $|\mathbf{p}|_1\leqslant m$, f converge to its Fourier — Haar series $\sum\limits_{\mathbf{m}\in\mathbb{Z}^d_+} a_{\mathbf{m}} h_{\mathbf{m}}(\mathbf{x})$ uniformly and the series converges absolutely. Then*

$$\begin{aligned} T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d} h^{j_i}_{p_i} \right) - a^{\mathbf{j}}_{\mathbf{p}} =& 2^{\frac{|(\mathbf{p}-\mathbf{1})\operatorname{sgn}(\mathbf{p})|_1}{2}} \sum_{\substack{|\mathbf{s}|_1\geqslant m+1,\\ s_i\geqslant p_i}} 2^{\frac{|(\mathbf{s}-\mathbf{1})\operatorname{sgn}(\mathbf{s})|_1}{2}} \\ &\cdot \left(\frac{1}{2^m} \sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ s_i\geqslant k_i+p_i,\\ s_d>k_d+p_d}} \sum_{\mathbf{r}=\mathbf{2^{k+1}j}}^{\mathbf{2^{k+1}(j+1)-1}} c_{\mathbf{s},\mathbf{k},\mathbf{p},\mathbf{r}} a^{\mathbf{r}\mathbf{2^{s-k-p-1}}}_{\mathbf{s}} \right) \end{aligned} \tag{2.4}$$

where

$$c_{\mathbf{s},\mathbf{k},\mathbf{p},\mathbf{r}} = \prod_{i=1}^{d} \tau_{j_i,k_i,r_i} g_{s_i,k_i,p_i,r_i},$$

$$g_{s_i,k_i,p_i,r_i} = \begin{cases} (-1)^{(r_i+1)\operatorname{sgn} k_i}, & s_i > k_i + p_i, \\ 2, & s_i = k_i + p_i, \ i = 1, \dots, d-1, \end{cases}$$
$$g_{s_d,k_d,p_d,r_d} = 1,$$
$$\tau_{j_i,k_i,r_i} = \begin{cases} 1, & r_i < 2^{k_i+1}\Big(j_i + \dfrac{1}{2}\Big), \\ -1, & r_i \geqslant 2^{k_i+1}\Big(j_i + \dfrac{1}{2}\Big). \end{cases}$$

In case $p_i = 0$ we assume that $0 \leqslant r_i \leqslant 2^{k_i} - 1$, $\tau_{j_i,k_i,r_i} = 1$ and g_{s_i,k_i,p_i,r_i} remain unchanged.

Corollary 2.1. *Let $f \in L_1[0,1)^d$, $m \in \mathbb{N}$ and*

$$P_m(\mathbf{x}) = \sum_{\substack{|\mathbf{k}|_1 \leqslant m, \\ \mathbf{k} \in \mathbb{Z}_+^d}} \sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{2^{k-1}-1}} a_{\mathbf{k}}^{\mathbf{j}} \prod_{i=1}^{d} h_{k_i}^{j_i}(x_i), \quad a_{\mathbf{k}}^{\mathbf{j}} = \int\limits_{[0,1)^d} f(\mathbf{x}) h_{\mathbf{k}}^{\mathbf{j}}(\mathbf{x})\, d\mathbf{x} \tag{2.5}$$

be a Haar polynomial with a degree not exceeding $2^m - 1$. Then

$$F_m(\mathbf{x}; P_m) = P_m(\mathbf{x}). \tag{2.6}$$

Proof. The statement follows from Theorem 2.1 with $T_{m,\mathbf{p}}\left(P_m \prod_{i=1}^d h_{p_i}^{j_i}\right) - a_{\mathbf{p}}^{\mathbf{j}} = 0$, where $\mathbf{p} \in \mathbb{Z}_+^d$, $|\mathbf{p}|_1 \leqslant m$, $j_i = 0, 1, \dots, 2^{p_i - 1} - 1$, $i = 1, \dots, d$. □

Theorem 2.2. *Let*

$$\alpha > \frac{1}{q}, \qquad 1 \leqslant q < \infty, \qquad 1 < \tau < \infty, \qquad m \in \mathbb{N},$$
$$\mathbf{p} \in \mathbb{Z}_+^d, \qquad |\mathbf{p}|_1 \leqslant m, \qquad m > d, \qquad f \in HA_{q,\tau}^{\alpha}.$$

Then

$$\left| a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f \prod_{i=1}^{d} h_{p_i}^{j_i}\right) \right| \leqslant c \frac{m^{(1-\frac{1}{\tau})(d-1)}}{2^{\alpha m}} 2^{(\frac{1}{q} - \frac{1}{2})|\mathbf{p}|_1} \|f\|_{HA_{q,\tau}^{\alpha}}, \tag{2.7}$$

where c depends on α, q, τ, d. If $\alpha > 0$, $q = \infty = \tau$ and $f \in HA_{\infty,\infty}^{\alpha}$, then

$$\left| a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f \prod_{i=1}^{d} h_{p_i}^{j_i}\right) \right| \leqslant c \frac{\max(m - |\mathbf{p}|_1, 1)^{d-1}}{2^{\alpha m}} 2^{-\frac{|\mathbf{p}|_1}{2}} \|f\|_{HA_{\infty,\infty}^{\alpha}}, \tag{2.8}$$

where c depends on α, q, τ, d.

Theorem 2.3. *Let*

$$\alpha > \frac{1}{q}, \qquad 1 \leqslant q < \infty, \qquad 1 < \tau < \infty, \qquad m \in \mathbb{N}, \qquad m > d, \qquad f \in HA_{q,\tau}^{\alpha}.$$

Then

$$\sup_{\|f\|_{HA_{q,\tau}^{\alpha}} = 1} \|f - F_m(\cdot; f)\|_{L_\infty} \leqslant c \frac{m^{(2-\frac{1}{\tau})(d-1)}}{2^{(\alpha - \frac{1}{q})m}}, \tag{2.9}$$

where c depends on α, q, τ, d. If $\alpha > 0$, $q = \infty = \tau$ and $f \in HA_{\infty,\infty}^{\alpha}$, then

$$\sup_{\|f\|_{HA_{\infty,\infty}^{\alpha}} = 1} \|f - F_m(\cdot; f)\|_{L_\infty} \leqslant c \frac{m^{2d-1}}{2^{\alpha m}}, \tag{2.10}$$

where c depends on α, q, τ, d.

Due to (1.3), (1.4) we have the following corollary.

Corollary 2.2. *Let*

$$\frac{1}{q} < \alpha < 1, \qquad 1 < q, \tau < \infty, \qquad m \in \mathbb{N}, \qquad m > d, \qquad \frac{1}{\tilde{q}} = \min\left(\frac{1}{q}, \frac{1}{2}\right).$$

Then

$$\sup_{\|f\|_{B^{\alpha}_{q,\tau}}=1} \|f - F_m(\cdot; f)\|_{L_\infty} \leqslant c\frac{m^{(2-\frac{1}{\tau})(d-1)}}{2^{(\alpha-\frac{1}{q})m}}, \tag{2.11}$$

$$\sup_{\|f\|_{W^{\alpha}_{q}}=1} \|f - F_m(\cdot; f)\|_{L_\infty} \leqslant c\frac{m^{(2-\frac{1}{\tilde{q}})(d-1)}}{2^{(\alpha-\frac{1}{q})m}}, \tag{2.12}$$

where c depends on α, q, τ, d. If $0 < \alpha < 1$, $q = \infty = \tau$, then

$$\sup_{\|f\|_{B^{\alpha}_{\infty,\infty}}=1} \|f - F_m(\cdot; f)\|_{L_\infty} \leqslant c\frac{m^{2d-1}}{2^{\alpha m}}, \tag{2.13}$$

where c depends on α, q, τ, d.

Remark 2.4. *The case $\alpha > \frac{1}{q}$, $1 < q, \tau < \infty$ for Besov $B^{\alpha}_{q,\tau}$ and Sobolev spaces W^{α}_{q} in the metric L_∞ was considered in* [1, 8]*, see also the references therein. Estimates* (2.11)–(2.13) *are not optimal with respect to m. However, due to a simple structure of $F_m(\mathbf{x}; f)$ the recovery operator might be of interest for the application.*

3. Haar wavelet space $HA^{\alpha}_{q,\tau}$

Lemma 3.1. *Let $\alpha > 0$, $1 \leqslant q, \tau \leqslant \infty$.*

1. *If $q \geqslant q_1$ and $\tau \leqslant \tau_1$, then*
$$HA^{\alpha}_{q,\tau} \hookrightarrow HA^{\alpha}_{q_1,\tau_1}.$$
2. *If $\alpha_0 < \alpha_1$ and $1 \leqslant q, \tau_0, \tau_1 \leqslant \infty$, then*
$$HA^{\alpha_1}_{q,\tau_1} \hookrightarrow HA^{\alpha_0}_{q,\tau_0}.$$
3. *If $\alpha_0 - \frac{1}{q_0} = \alpha_1 - \frac{1}{q_1}$ and $\alpha_0 > \alpha_1$, then*
$$HA^{\alpha_0}_{q_0,\tau} \hookrightarrow HA^{\alpha_1}_{q_1,\tau}.$$
4. *If $\alpha_0 > \alpha_1$, $1 \leqslant \tau_0, \tau_1 \leqslant \infty$, then*
$$\left(HA^{\alpha_0}_{q,\tau_0}, HA^{\alpha_1}_{q,\tau_1}\right)_{\theta,\tau} = HA^{\alpha}_{q,\tau},$$
where $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

The proof is analogous to the case of approximation spaces $A^{\alpha}_{p,q}$, see, for example, [5].

Lemma 3.2. *Assume that $\alpha > \frac{1}{q}$, $\tau, q > 1$ ($\alpha > 0$ for $q = \infty = \tau$). Then the functions $f \in HA^{\alpha}_{q,\tau}$ ($f \in HA^{\alpha}_{\infty,\infty}$) converge uniformly to their Fourier — Haar series and the Fourier — Haar series converges absolutely.*

Proof. Lemma 3.1 implies the embedding $HA^{\alpha}_{q,\tau} \hookrightarrow HA^{\frac{1}{q}}_{q,1} = S_q$. Due to [6] the functions $f \in S_q$ converge uniformly to their Fourier — Haar series and these Fourier — Haar series converges absolutely. The limiting case $\alpha > 0$ when $q = \infty = \tau$ is implies by [2]. The proof is complete. □

4. Proofs of Theorems 2.1–2.3

Before we proceed to the proofs, we prove the following lemmas.

Lemma 4.1. *Let $k \in \mathbb{N}$, $s \in \mathbb{Z}_+$, $j = 0, 1, \dots, 2^{s-1} - 1$. Then*

$$\sum_{r=0}^{2^k-1} \operatorname{sgn}\left(h_s^j\left(\frac{r}{2^k}\right)\right) = \begin{cases} 1, & s > k, \quad \frac{j}{2^{s-k-1}} \text{ is integer}, \\ 0, & \text{else}. \end{cases} \tag{4.1}$$

$$\sum_{r=0}^{2^k-1} (-1)^r \operatorname{sgn}\left(h_s^j\left(\frac{r}{2^k}\right)\right) = \begin{cases} (-1)^t, & t = \dfrac{j}{2^{s-k-1}} \text{ is integer}, \quad s \geqslant k+1, \\ 2, & s = k,\ r = 2j,\ 2j+1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.2}$$

Proof. Let us prove (4.1). Let $s \leqslant k$. Then $\Delta_s^j = \operatorname{supp}(h_s^j)$ has the same amount of points 2^{k-s} in the "positive" part of the interval $(\Delta_s^j)^+$ and in the "negative" one $(\Delta_s^j)^-$. Thus,

$$\sum_{r=0}^{2^k-1} \operatorname{sgn}\left(h_s^j\left(\frac{r}{2^k}\right)\right) = 0, \quad s \leqslant k.$$

In the case $s > k$ only one point $\frac{r}{2^k}$ lies in the support of function h_s^j provided $j = 2^{s-k-1}r$. Thus,

$$\sum_{r=0}^{2^k-1} \operatorname{sgn}\left(h_s^j\left(\frac{r}{2^k}\right)\right) = 1, \quad s > k, \quad \frac{j}{2^{s-k-1}} \text{ is integer}.$$

The proof of (4.2) follows the same lines. The proof is complete. □

Lemma 4.2. *Let*

$$k, s \in \mathbb{Z}_+, \qquad k \leqslant s, \qquad i = 0, 1, \dots, 2^{k-1} - 1, \qquad j = 0, 1, \dots, 2^{s-1} - 1, \qquad x \in [0, 1).$$

Then

$$h_k^i(x) h_s^j(x) = \begin{cases} 2^{\frac{k-1}{2}} h_s^j(x), & k < s, \quad \Delta_s^j \subset \left(\Delta_k^i\right)^+, \\ -2^{\frac{k-1}{2}} h_s^j(x), & k < s, \quad \Delta_s^j \subset \left(\Delta_k^i\right)^-, \\ 2^{k-1} \operatorname{sgn}\left(h_k^i(x)\right), & k = s, \quad i = j, \\ 0, & \text{otherwise}. \end{cases} \tag{4.3}$$

Proof. The statement is implied immediately by the definition of a Haar function and the properties of Δ_k^i. □

Lemma 4.3. *Assume that $m, d \in \mathbb{N}$, then for fixed d we have*

$$\sum_{\substack{|\mathbf{s}|_1 \leqslant m, \\ \mathbf{s} \in \mathbb{Z}_+^d}} (m - |\mathbf{s}|_1)^{d-1} \leqslant c_d m^{2d-1}, \tag{4.4}$$

where c_d is a constant depended on d.

Proof. We consider

$$\sum_{\substack{|\mathbf{s}|_1 \leqslant m, \\ \mathbf{s} \in \mathbb{Z}_+^d}} (m - |\mathbf{s}|_1)^{d-1} = \sum_{n=0}^{m} (m-n)^{d-1} \sum_{\substack{|\mathbf{s}|_1 = n, \\ \mathbf{s} \in \mathbb{Z}_+^d}} 1 = \sum_{n=0}^{m} (m-n)^{d-1} \binom{n+d-1}{d-1}.$$

For fixed d we have

$$\binom{n+d-1}{d-1} \sim n^{d-1}.$$

Therefore,

$$\sum_{n=0}^{m}(m-n)^{d-1}\binom{n+d-1}{d-1}\leqslant m^{d-1}+c_1\sum_{n=1}^{m}(n(m-n))^{d-1}.$$

The function $x(m-x)$ reaches its maximum at the point $x=[m/2]$. We thus get

$$\sum_{\substack{|\mathbf{s}|_1\leqslant m,\\ \mathbf{s}\in\mathbb{Z}_+^d}}(m-|\mathbf{s}|_1)^{d-1}\leqslant m^{d-1}+c_2m^{2d-2}\sum_{n=1}^{m}=O(m^{2d-1}).$$

The proof is complete. □

Lemma 4.4. *Assume that $m,d\in\mathbb{N}$, then for fixed d we have*

$$\sum_{\substack{|\mathbf{s}|_1=m,\\ \mathbf{s}\in\mathbb{Z}_+^d}}1\leqslant c_dm^{d-1}. \tag{4.5}$$

Proof. The inequality is implied by the identity

$$\sum_{\substack{|\mathbf{s}|_1=m,\\ \mathbf{s}\in\mathbb{Z}_+^d}}1=\binom{m+d-1}{d-1}=O(m^{d-1}).$$

The proof is complete. □

Proof of Theorem 2.1. Using the uniform convergence of the Fourier series $\sum\limits_{\mathbf{m}\in\mathbb{Z}_+^d}a_{\mathbf{m}}h_{\mathbf{m}}(\mathbf{x})$, the definition $h_k^j(x)=2^{\frac{k-1}{2}}\operatorname{sgn}(h_k^j(x))$ and (4.3), we get

$$T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)=\sum_{\mathbf{s}\in\mathbb{Z}_+^d}\sum_{\mathbf{z}=\mathbf{0}}^{\mathbf{2^{s-1}-1}}a_{\mathbf{s}}^{\mathbf{z}}T_{m,\mathbf{p}}\left(\prod_{i=1}^{d}h_{p_i}^{j_i}h_{s_i}^{z_i}\right)$$

$$=2^{\frac{|(\mathbf{p}-\mathbf{1})\operatorname{sgn}(\mathbf{p})|_1}{2}}\sum_{\substack{\mathbf{s}\in\mathbb{Z}_+^d,\\ s_i\geqslant p_i}}2^{\frac{|(\mathbf{s}-\mathbf{1})\operatorname{sgn}(\mathbf{s})|_1}{2}}\sum_{\mathbf{z}=\mathbf{2^{s-p}j}}^{\mathbf{2^{s-p}(j+1)-1}}\prod_{i=1}^{d}c_{s_i,p_i,j_i,z_i}a_{\mathbf{s}}^{\mathbf{z}}T_{m,\mathbf{p}}\left(\prod_{i=1}^{d}\operatorname{sgn}(h_{s_i}^{z_i})\right),$$

where

$$c_{s_i,p_i,j_i,z_i}=\begin{cases}1, & z_i<2^{s_i-p_i}\left(j_i+\dfrac{1}{2}\right),\\ -1, & z_i\geqslant 2^{s_i-p_i}\left(j_i+\dfrac{1}{2}\right).\end{cases}$$

As $s_i=p_i$, $i=1,\dots,d$, we get $z_i=j_i$ and $T_{m,\mathbf{p}}=2^{-|\mathbf{p}+\operatorname{sgn}(\mathbf{p})|_1}$. Thus,

$$T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)=a_{\mathbf{p}}^{\mathbf{j}}+2^{\frac{|(\mathbf{p}-\mathbf{1})\operatorname{sgn}(\mathbf{p})|_1}{2}}\sum_{\substack{\mathbf{s}\in\mathbb{Z}_+^d,\\ s_i\geqslant p_i}}2^{\frac{|(\mathbf{s}-\mathbf{1})\operatorname{sgn}(\mathbf{s})|_1}{2}}$$

$$\cdot\sum_{\mathbf{z}=\mathbf{2^{s-p}j}}^{\mathbf{2^{s-p}(j+1)-1}}\prod_{i=1}^{d}c_{s_i,p_i,j_i,z_i}a_{\mathbf{s}}^{\mathbf{z}}T_{m,\mathbf{p}}\left(\prod_{i=1}^{d}\operatorname{sgn}(h_{s_i}^{z_i})\right).$$

We consider

$$T_{m,\mathbf{p}}\left(\prod_{i=1}^{d}\operatorname{sgn}(h_{s_i}^{z_i})\right)=\frac{1}{2^m}\sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ \mathbf{k}\in\mathbb{Z}_+^d}}(-1)^{\sum\limits_{i=1}^{d-1}\operatorname{sgn}(k_i)}\sum_{r_1=0}^{2^{k_1+p_1}-1}(-1)^{r_1\operatorname{sgn}(k_1)}\operatorname{sgn}\left(h_{s_1}^{z_1}\left(\frac{r_1}{2^{k_1+p_1}}\right)\right)$$

$$\cdots \sum_{r_d=0}^{2^{k_d+p_d}-1} \operatorname{sgn}\left(h_{s_d}^{z_d}\left(\frac{r_d}{2^{k_d+p_d}}\right)\right).$$

Due to Lemma 4.1 the functional is nonzero as

$$s_i > k_i + p_i, \qquad z_i = 2^{s_i-k_i-p_i-1}r_i, \quad i = 1,\dots,d-1, \qquad s_d \geqslant k_d + p_d, \qquad z_d = 2^{s_d-k_d-p_d-1}r_d.$$

But considering $|\mathbf{k}+\mathbf{p}|_1 = m$ and $2^{s_i-p_i}j_i \leqslant z_i \leqslant 2^{s_i-p_i}(j_i+1)-1$ we get

$$|\mathbf{s}|_1 \geqslant m+1, \qquad 2^{k_i+1}j_i \leqslant r_i \leqslant 2^{k_i+1}(j_i+1)-1.$$

This leads us to identity (2.4) and completes the proof. □

Proof of Theorem 2.2. Using Theorem 2.1 and the Hölder inequality, we get

$$\left|a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)\right| \leqslant c\frac{2^{(\frac{1}{q}-\frac{1}{2})|\mathbf{p}|_1}}{2^{\frac{m}{q}}}\sum_{n=m+1}^{\infty}2^{\frac{n}{2}}\sum_{\substack{|\mathbf{s}|_1=n,\\ \mathbf{s}\in\mathbb{Z}_+^d}}\sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ k_i\leqslant s_i-p_i,\\ k_d\leqslant s_d-p_d-1}}\left(\sum_{\mathbf{z}=\mathbf{0}}^{\mathbf{2^{s-1}-1}}|a_{\mathbf{p}}^{\mathbf{j}}|^q\right)^{\frac{\tau}{q}}.$$

By the change of variables $r_i = s_i - k_i - p_i$, $r_d = s_d - k_d - p_d - 1$ we obtain the inequality

$$\sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ k_i\leqslant s_i-p_i,\\ k_d\leqslant s_d-p_d-1}} 1 = \sum_{\substack{|\mathbf{r}|_1=|\mathbf{s}|_1-m,\\ r_i\leqslant s_i-p_i,\\ r_d\leqslant s_d-p_d-1}} 1 \leqslant \binom{|\mathbf{s}|_1-m+d-1}{d-1}.$$

Using again the Hölder inequality and the condition $\alpha > \frac{1}{q}$, we obtain

$$\left|a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)\right| \leqslant c\frac{2^{(\frac{1}{q}-\frac{1}{2})|\mathbf{p}|_1}}{2^{\frac{m}{q}}}\left(\sum_{n=m+1}^{\infty}2^{(\frac{1}{q}-\alpha)n\tau'}\binom{n-m+d-1}{d-1}^{\tau'}\sum_{\substack{|\mathbf{s}|_1=n,\\ \mathbf{s}\in\mathbb{Z}_+^d}}\right)^{\frac{1}{\tau'}}\|f\|_{HA_{q,\tau}^{\alpha}}.$$

Using Lemma 4.4 and letting $n = t+m$, $t = 1,2,\dots$, we get

$$\left|a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)\right| \leqslant c\frac{m^{\frac{d-1}{\tau'}}}{2^{\alpha m}}\left(\sum_{t=1}^{\infty}2^{(\frac{1}{q}-\alpha)t\tau'}\binom{t+d-1}{d-1}^{\tau'}\right)^{\frac{1}{\tau'}}2^{(\frac{1}{q}-\frac{1}{2})|\mathbf{p}|_1}\|f\|_{HA_{q,\tau}^{\alpha}}$$
$$+\frac{c}{2^{\alpha m}}\left(\sum_{t=1}^{\infty}2^{(\frac{1}{q}-\alpha)t\tau'}\binom{t+d-1}{d-1}^{\tau'}t^{d-1}\right)^{\frac{1}{\tau'}}2^{(\frac{1}{q}-\frac{1}{2})|\mathbf{p}|_1}\|f\|_{HA_{q,\tau}^{\alpha}}.$$

The sums depend only on α, q, d, τ, where $\tau' = \frac{\tau}{\tau-1}$, and comparing these two terms

$$\frac{1}{2^{\alpha m}} \leqslant \frac{m^{\frac{d-1}{\tau'}}}{2^{\alpha m}}, \qquad \left(\sum_{t=1}^{\infty}2^{(\frac{1}{q}-\alpha)t\tau'}\binom{t+d-1}{d-1}^{\tau'}\right)^{\frac{1}{\tau'}} \leqslant \left(\sum_{t=1}^{\infty}2^{(\frac{1}{q}-\alpha)t\tau'}\binom{t+d-1}{d-1}^{\tau'}t^{d-1}\right)^{\frac{1}{\tau'}},$$

we get

$$\left|a_{\mathbf{p}}^{\mathbf{j}} - T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h_{p_i}^{j_i}\right)\right| \leqslant c\frac{m^{\frac{d-1}{\tau'}}}{2^{\alpha m}}\left(\sum_{t=1}^{\infty}2^{(\frac{1}{q}-\alpha)t\tau'}\binom{t+d-1}{d-1}^{\tau'}t^{d-1}\right)^{\frac{1}{\tau'}}2^{\left(\frac{1}{q}-\frac{1}{2}\right)|\mathbf{p}|_1}\|f\|_{HA_{q,\tau}^{\alpha}}.$$

This leads us to inequality (2.7).

We rewrite the error in (2.4) as

$$\frac{2^{\frac{|(\mathbf{p}-\mathbf{1})\operatorname{sgn}(\mathbf{p})|_1}{2}}}{2^m}\sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ k_i\in\mathbb{Z}_+}}\sum_{s_1=k_1+p_1}^{\infty}\cdots\sum_{s_d=k_d+p_d+1}^{\infty}2^{\frac{|(\mathbf{s}-\mathbf{1})\operatorname{sgn}(\mathbf{s})|_1}{2}}\left(\sum_{\mathbf{r}=\mathbf{2^{k+1}j}}^{\mathbf{2^{k+1}(j+1)-1}}c_{\mathbf{s},\mathbf{k},\mathbf{p},\mathbf{r}}a_{\mathbf{s}}^{\mathbf{r}\mathbf{2^{s-k-p-1}}}\right).$$

We thus get

$$\left|a^{\mathbf{j}}_{\mathbf{p}} - T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h^{j_i}_{p_i}\right)\right| \leqslant c\frac{2^{\frac{|\mathbf{p}|_1}{2}}}{2^m}\sum_{\substack{|\mathbf{k}+\mathbf{p}|_1=m,\\ k_i\in\mathbb{Z}_+}}\sum_{s_1=k_1+p_1}^{\infty}\cdots\sum_{s_d=k_d+p_d+1}^{\infty}2^{\frac{|\mathbf{s}|_1}{2}}2^{|\mathbf{k}|_1}\max_{\mathbf{z}=\mathbf{0},\ldots,\mathbf{2^{s-1}-1}}|a^{\mathbf{z}}_{\mathbf{s}}|$$

$$\leqslant c_3\frac{\max(m-|\mathbf{p}|_1,1)^{d-1}}{2^{\alpha m}}2^{-\frac{|\mathbf{p}|_1}{2}}\|f\|_{HA^{\alpha}_{\infty,\infty}},$$

where c_3 depends on α, q, d. The proof is complete. □

Proof of Theorem 2.3. Assume that $S_m(\mathbf{x};f)$ is of the type (2.5). Using Minkowski inequality, we get

$$\|f-F_m(\cdot;f)\|_{L_\infty}\leqslant\|f-S_m(\cdot;f)\|_{L_\infty}+\|S_m(\cdot;f)-F_m(\cdot;f)\|_{L_\infty}=I_1+I_2.$$

We consider I_2 and apply Theorem 2.2:

$$\left\|\sum_{\substack{|\mathbf{p}|_1\leqslant m,\\ \mathbf{s}\in\mathbb{Z}^d_+}}\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{2^{p-1}-1}}\left(a^{\mathbf{j}}_{\mathbf{p}}-T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h^{j_i}_{p_i}\right)\right)h^{\mathbf{j}}_{\mathbf{p}}(\cdot)\right\|_{L_\infty}$$

$$\leqslant c\sum_{\substack{|\mathbf{p}|_1\leqslant m,\\ \mathbf{s}\in\mathbb{Z}^d_+}}2^{\frac{|\mathbf{p}|_1}{2}}\max_{\mathbf{j}=\mathbf{0},\mathbf{1},\ldots,\mathbf{2^{p-1}-1}}\left|a^{\mathbf{j}}_{\mathbf{p}}-T_{m,\mathbf{p}}\left(f\prod_{i=1}^{d}h^{j_i}_{p_i}\right)\right|$$

$$\leqslant c\frac{m^{(1-\frac{1}{\tau})(d-1)}}{2^{\alpha m}}\|f\|_{HA^{\alpha}_{q,\tau}}\sum_{\substack{|\mathbf{p}|_1\leqslant m,\\ \mathbf{s}\in\mathbb{Z}^d_+}}2^{\frac{|\mathbf{p}|_1}{q}}.$$

Due to (4.5) we have

$$\sum_{\substack{|\mathbf{p}|_1\leqslant m,\\ \mathbf{s}\in\mathbb{Z}^d_+}}2^{\frac{|\mathbf{p}|_1}{q}}=\sum_{n=0}^{m}2^{\frac{n}{q}}\binom{n+d-1}{d-1}\leqslant c\binom{m+d-1}{d-1}2^{\frac{m}{q}}=cm^{d-1}2^{\frac{m}{q}}.$$

Thus,

$$I_2\leqslant c\frac{m^{(2-\frac{1}{\tau})(d-1)}}{2^{(\alpha-\frac{1}{q})m}}\|f\|_{HA^{\alpha}_{q,\tau}}.$$

We consider I_1 and apply the Hölder inequality. This gives

$$I_1=\|f-S_m(\cdot;f)\|_{L_\infty}=\left\|\sum_{\substack{|\mathbf{p}|_1>m,\\ \mathbf{p}\in\mathbb{Z}^d_+}}\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{2^{p-1}-1}}a^{\mathbf{j}}_{\mathbf{p}}h^{\mathbf{j}}_{\mathbf{p}}(\cdot)\right\|_{L_\infty}\leqslant c\sum_{\substack{|\mathbf{p}|_1>m,\\ \mathbf{p}\in\mathbb{Z}^d_+}}2^{\frac{|\mathbf{p}|_1}{2}}\max_{\mathbf{j}=\mathbf{0},\ldots,\mathbf{2^{p-1}}}|a^{\mathbf{j}}_{\mathbf{p}}|$$

$$\leqslant c\sum_{\substack{|\mathbf{p}|_1>m,\\ \mathbf{p}\in\mathbb{Z}^d_+}}2^{\frac{|\mathbf{p}|_1}{2}}\left(\sum_{\mathbf{j}=\mathbf{0}}^{\mathbf{2^{p}-1}}|a^{\mathbf{j}}_{\mathbf{p}}|^q\right)^{\frac{1}{q}}\leqslant c\left(\sum_{\substack{|\mathbf{p}|_1>m,\\ \mathbf{p}\in\mathbb{Z}^d_+}}2^{(\frac{1}{q}-\alpha)|\mathbf{p}|_1\tau}\right)^{\frac{1}{\tau}}\|f\|_{HA^{\alpha}_{q,\tau}}\leqslant C_{\alpha,d}\frac{m^{\frac{d-1}{\tau'}}}{2^{(\alpha-\frac{1}{q})m}}\|f\|_{HA^{\alpha}_{q,\tau}}.$$

The proof of inequality (2.10) follows the same lines. The proof is complete. □

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Serzhan Zhandosuly Bassarov,
Private Institution "Nazarbayev University Research Administration" (NURA),
Kabanbai batyr av. 53,
010017, Astana, Kazakhstan
Geometry Limited Liability Partnership,
Kabanbai batyr ave. 49
010017, Astana, Kazakhstan
E-mail: bassarov.serzhan98@gmail.com

Nazerke Tulekovna Tleukhanova,
L.N. Gumilyov Eurasian National university,
Satpayev str. 2,
010010, Astana, Kazakhstan
Geometry Limited Liability Partnership,
Kabanbai batyr ave. 49
010017, Astana, Kazakhstan
E-mail: tleukhanova@rambler.ru