ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 4 (2026). P. 126-136.

doi:10.13108/2026-18-4-126

REDUCTION OF THE LAPLACE SEQUENCE OF DIFFERENTIAL–DIFFERENCE EQUATIONS

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Abstract. The method for constructing a Lax pair and a recursion operator for nonlinear integrable equations of sine–Gordon type, developed in the works of I.T. Habibullin et al. (J. Phys. A, Math. Theor. **49**, 035202 (2016)), is adapted to the case of differential–difference equations of soliton type. As an illustration of the method, we consider the equation

$$u_{n+1,x} = -u_{n,x} - (u_{n+1} - u_n)^2,$$

construct for this equation a Lax pair and a recursion operator that describes the hierarchy of higher symmetries.

Keywords: Laplace sequence, nonlinear integrable equation, soliton-type equation, Lax pair, higher symmetry, recursion operator.

Mathematics Subject Classification: 37K10, 35Q51, 35Q53

1. Introduction

In the present work, we study the reduction of the Laplace sequence of differential–difference equations

$$u_{n+1,x} = f(u_{n,x}, u_{n+1}, u_n), \tag{1.1}$$

where the function $u_n(x)$ depends on a continuous variable x and a discrete variable n. We consider equations (1.1) possessing the following properties:

1) they admit higher symmetries of order three and higher;

2) they admit a Lax representation;

3) they have no nontrivial characteristic integrals.

These are equations of soliton type, which are also called equations of the semi–discrete sine–Gordon equation type (see, e.g., [30], [29], [22]). A list of equations of the form (1.1) possessing properties 1)–3) can be found in works [11], [3].

The construction of solutions to equations, like that of any nonlinear equation, faces certain difficulties and requires the use of auxiliary objects such as the Lax pair ([24], [7], [8], [13], [31]), higher symmetries [28], [26], invariant manifolds (differential constraints) [12], [10], generalized invariant manifolds [19], [20], [21], Dubrovin–type equations [5], [21], and others. In [18], [15], [16] we showed that for constructing Lax pairs, recursion operators and generalized invariant manifolds of nonlinear integrable equations of sine–Gordon type

$$u_{xy} = F(u, u_x, u_y), \tag{1.2}$$

one can use the method of cascade Laplace integration [23]. Let us recall the essence of the cascade Laplace method, which is known as one of the classical methods for constructing the

A.R. Khakimova, Reduction of the Laplace sequence of differential–difference equations.
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Submitted August 28, 2026.

general solution of a linear hyperbolic equation

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{1.3}$$

The main idea of this method is connected with the sequence of Laplace invariants

$$\dots, h_{[-2]}, h_{[-1]}, h_{[0]}, h_{[1]}, h_{[2]}, \dots, \tag{1.4}$$

the coefficients of which are computed by the recurrence formula

$$h_{[i+1]} = 2h_{[i]} - h_{[i-1]} - (\ln(h_{[i]}))_{xy}, \quad i \in \mathbb{Z},$$

where $h_{[-1]} = b_y + ab - c$, $h_{[0]} = a_x + ab - c$.

If the sequence of Laplace invariants (1.4) terminates with zero at least on one side, i.e., if $h_{[k]} = 0$ or $h_{[-k]} = 0$, then equation (1.3) is integrable in quadratures. If the sequence of Laplace invariants terminates with zero on both sides, then one can construct the general solution of the given equation.

Later the cascade Laplace method was adapted to the case of nonlinear integrable hyperbolic equations of Liouville type (see, e.g., [25], [14], [9], [17], [6], [2]). In work [6] it was proved that if the sequence of Laplace invariants $h_{[i]}$, $i \in \mathbb{Z}$ of the linearization of a nonlinear hyperbolic equation of the form (1.2)

$$v_{xy} + av_x + bv_y + cv = 0, \tag{1.5}$$

where

$$a = -\frac{\partial F}{\partial u_x}, \qquad b = -\frac{\partial F}{\partial u_y}, \qquad c = -\frac{\partial F}{\partial u}, \tag{1.6}$$

is finite, then the given nonlinear equation is Darboux integrable. Moreover, it was shown in that work that the cascade Laplace method can be used for the classification of nonlinear equations.

In [1] the method of cascade Laplace integration was adapted to the case of nonlinear differential–difference and fully discrete equations. In [4] Laplace transformations are used to construct discrete analogues of integrable systems.

Despite the fact that our equations are of a different type, in [18], [15], [16] we showed that Laplace sequences can be useful in constructing recursion operators and Lax pairs of nonlinear integrable equations of hyperbolic type. We also note that recently recursion operators arising from Laplace transformations of the linearization of the sine–Gordon equation received a geometric interpretation in [27].

In [18], [15], [16] we proved that the Laplace sequences of all known nonlinear differential equations of sine–Gordon type admit finite–field reductions (see [18], [15], [16]) and the following theorem is proved.

Theorem 1.1 ([18], [15], [16]). *The Laplace sequences*

$$\begin{aligned} &\left(\frac{\partial}{\partial y} + a_{[i]}\right) v_{[i]} = v_{[i+1]}, && \left(\frac{\partial}{\partial x} + b_{[0]}\right) v_{[i+1]} - h_{[i]}v_{[i]} = 0, \\ &\left(\frac{\partial}{\partial x} + b_{[-i]}\right) v_{[-i]} = v_{[-i-1]}, && \left(\frac{\partial}{\partial y} + a_{[0]}\right) v_{[-i-1]} - k_{[-i]}v_{[-i]} = 0, \end{aligned} \tag{1.7}$$

corresponding to the linearized equations of the following five equations

$$\begin{aligned} &u_{xy} = \sin(u);\\ &u_{xy} = f(u)\sqrt{1+u_x^2}, \quad f'' = \gamma f, \quad \gamma = \text{const};\\ &u_{xy} = \sqrt{u_x}\sqrt{u_y^2+1};\\ &u_{xy} = \sqrt{u_x^2+1}\sqrt{u_y^2+1};\\ &u_{xy} = \sqrt{\wp(u)-\mu}\sqrt{u_x^2+1}\sqrt{u_y^2+a}, \end{aligned} \tag{1.8}$$

admit reductions of the form

$$\begin{aligned} v_{[m+1]} &= \alpha_{[-k]}v_{[-k]} + \alpha_{[-k+1]}v_{[-k+1]} + \cdots + \alpha_{[m]}v_{[m]},\\ v_{[-k-1]} &= \beta_{[-k]}v_{[-k]} + \beta_{[-k+1]}v_{[-k+1]} + \cdots + \beta_{[m]}v_{[m]}, \end{aligned} \tag{1.9}$$

where $m = 1$, $k = 1$.

On the base of the found reductions, for all five equations (1.8) recursion operators and Lax pairs were constructed, some of which were not previously known.

The aim of the present article is to adapt the method of constructing a Lax pair and a recursion operator, based on reductions of the Laplace sequence and developed in [18], to the case of differential–difference equations (1.1). In Section 2 reductions of the Laplace sequence of differential–difference equations are discussed and the necessary formulas are presented. In Section 3, as an illustration of the method, a known differential–difference equation is considered, for which the corresponding reductions, recursion operators and Lax pair are constructed.

2. Laplace sequences for differential–difference equations and their reductions

We first provide necessary formulas and definitions related to the cascade Laplace method for differential–difference equations (1.1) (see also [1]). We focus on the linearization (or Fréchet derivative) of equation (1.1)

$$v_{n+1,x} + a_n v_{n,x} + b_n v_{n+1} + c_n v_n = 0, \tag{2.1}$$

where

$$a_n = -\frac{\partial f}{\partial u_{n,x}}, \qquad b_n = -\frac{\partial f}{\partial u_{n+1}}, \qquad c_n = -\frac{\partial f}{\partial u_n}.$$

We note that equation (2.1) can be represented in one of the following forms

$$\begin{aligned} &(D_n + a_n)(D_x + b_{n-1})v_n + H_{n,0}v_n = 0,\\ &(D_x + b_n)(D_n + a_n)v_n + K_{n,0}v_n = 0, \end{aligned}$$

where D_n is the shift operator acting by the rule $D_n w(n) = w(n+1)$, D_x is the operator of total differentiation with respect to x, and the functions $H_{n,0}$ and $K_{n,0}$ are calculated by the formulas

$$H_{n,0} = c_n - a_n b_{n-1}, \qquad K_{n,0} = c_n - a_n b_n - D_x(a_n).$$

We introduce new variables $v_{n,[1]}$, $v_{n,[-1]}$ as

$$\begin{aligned} &(D_x + b_{n-1})v_n = v_{n,[1]}, \qquad && (D_n + a_n)v_{n,[1]} = -H_{n,0}v_n,\\ &(D_n + a_n)v_n = v_{n,[-1]}, \qquad && (D_x + b_n)v_{n,[-1]} = -K_{n,0}v_n. \end{aligned}$$

If the functions $H_{n,0}$ and $K_{n,0}$ are non–zero, then the introduced functions $v_{n,[1]}$, $v_{n,[-1]}$ satisfy the equations

$$v_{n+1,[1],x} + a_{n,[1]}v_{n,[1],x} + b_{n,[1]}v_{n+1,[1]} + c_{n,[1]}v_{n,[1]} = 0, \tag{2.2}$$
$$v_{n+1,[-1],x} + a_{n,[-1]}v_{n,[-1],x} + b_{n,[-1]}v_{n+1,[-1]} + c_{n,[-1]}v_{n,[-1]} = 0, \tag{2.3}$$

where

$$\begin{aligned}
&a_{n,[1]} = a_n, \qquad b_{n,[1]} = b_{n-1} - D_x(\ln H_{n,[0]}),\\
&c_{n,[1]} = D_x(a_n) + a_n b_{n-1} - a_n D_x(\ln H_{n,[0]}) + H_{n,[0]},\\
&a_{n,[-1]} = \frac{a_n K_{n+1,[0]}}{K_{n,[0]}}, \qquad b_{n,[-1]} = b_{n+1},\\
&c_{n,[-1]} = a_n b_n \frac{K_{n+1,[0]}}{K_{n,[0]}} + K_{n+1,[0]}.
\end{aligned}$$

The transformations mapping equation (2.1) into equations (2.2) are called the x- and respectively the n-Laplace transformations. A multiple repetition of the described procedure gives a sequence of Laplace transformations

$$v_{n+1,[i],x} + a_{n,[i]}v_{n,[i],x} + b_{n,[i]}v_{n+1,[i]} + c_{n,[i]}v_{n,[i]} = 0, \quad i = 0, 1, 2, \dots, \tag{2.4}$$
$$v_{n+1,[-i],x} + a_{n,[-i]}v_{n,[-i],x} + b_{n,[-i]}v_{n+1,[-i]} + c_{n,[-i]}v_{n,[-i]} = 0, \quad i = 1, 2, \dots, \tag{2.5}$$

where the coefficients read

$$\begin{aligned}
&a_{n,[0]} = a_n, \qquad b_{n,[0]} = b_n, \qquad c_{n,[0]} = c_n,\\
&a_{n,[i]} = a_{n,[i-1]} = a_{n,[0]}, \qquad b_{n,[i]} = b_{n-1,[i-1]} - D_x(\ln H_{n,[i-1]}),\\
&c_{n,[i]} = D_x(a_{n,[i-1]}) + a_{n,[i-1]}b_{n-1,[i-1]} - a_{n,[i-1]}D_x(\ln H_{n,[i-1]}) + H_{n,[i-1]},\\
&H_{n,[i]} = H_{n,[i-1]} + D_x(a_{n,[0]}) + a_{n,[0]}(b_{n,[i]} - b_{n-1,[i]}),\\
&a_{n,[-i]} = \frac{a_{n,[-i+1]}K_{n+1,[-i+1]}}{K_{n,[-i+1]}}, \qquad b_{n,[-i]} = b_{n+1,[-i+1]} = b_{n+i,[0]},\\
&c_{n,[-i]} = a_{n,[-i+1]}b_{n,[-i+1]}\frac{K_{n+1,[-i+1]}}{K_{n,[-i+1]}} + K_{n+1,[-i+1]},\\
&K_{n,[-i]} = K_{n+1,[-i+1]} - D_x(a_{n,[-i]}) - a_{n,[-i]}(b_{n+i,[0]} - b_{n+i-1,[0]}).
\end{aligned} \tag{2.6}$$

Equations (2.4) can be represented as the systems

$$(D_x + b_{n-1,[i]})v_{n,[i]} = v_{n,[i+1]}, \qquad (D_n + a_{n,[i]})v_{n,[i+1]} = -H_{n,[i]}v_{n,[i]}, \tag{2.7}$$
$$(D_n + a_{n,[-i]})v_{n,[-i]} = v_{n,[-i-1]}, \qquad (D_x + b_{n,[-i]})v_{n,[-i-1]} = -K_{n,[-i]}v_{n,[-i]}. \tag{2.8}$$

We formulate a conjecture.

Conjecture 2.1. *For each nonlinear integrable equation (1.1) having no nontrivial integrals, the sequence of Laplace transformations associated with its linearization admits a finite–field reduction, i.e., is compatible with a constraint*

$$v_{n,[m+1]} = \alpha_{n,[-k]}v_{n,[-k]} + \alpha_{n,[-k+1]}v_{n,[-k+1]} + \dots + \alpha_{n,[m]}v_{n,[m]}, \tag{2.9}$$
$$v_{n,[-k-1]} = \beta_{n,[-k]}v_{n,[-k]} + \beta_{n,[-k+1]}v_{n,[-k+1]} + \dots + \beta_{n,[m]}v_{n,[m]}, \tag{2.10}$$

where $m = 1, 2, 3, \dots$, $k = 1, 2, 3, \dots$, the coefficients $\alpha_{n,[i]}$, $\beta_{n,[i]}$ depend on the dynamical variables u_n, u_{n+1}, $u_{n,x}$, u_{n+2}, $u_{n,xx}$, ... and the spectral parameter λ, and moreover, $\alpha_{n,[-k]} \neq 0$, $\beta_{n,[-k]} \neq 0$.

Proposition 2.1. *Suppose that the functions $H_{n,[i]}$ and $K_{n,[-i]}$ are non–zero for each i and the sequence of Laplace transformations (2.7), (2.8) is compatible with the constraint (2.9). Then for an arbitrary j the function $v_{n,[j]}$ can be represented as*

$$v_{n,[j]} = \alpha^{j}_{n,[-k]} v_{n,[-k]} + \alpha^{j}_{n,[-k+1]} v_{n,[-k+1]} + \ldots + \alpha^{j}_{n,[m]} v_{n,[m]}.$$

In other words, constraint (2.9) determines a reduction of the entire Laplace sequence.

Proof. We apply the operator $(D_x + b_{n-1,[m+1]})$ to the first equation of (2.9) and, using representation (2.7), we find

$$v_{n,[m+2]} = \tilde{\alpha}^{1}_{n,[-k]} v_{n,[-k]} + \tilde{\alpha}^{1}_{n,[-k+1]} v_{n,[-k+1]} + \ldots + \tilde{\alpha}^{1}_{n,[m+1]} v_{n,[m+1]}.$$

In the obtained identity we eliminate the variable $v_{n,[m+1]}$ by virtue of the first equation in (2.9) and as a result we arrive at

$$v_{n,[m+2]} = \alpha^{1}_{n,[-k]} v_{n,[-k]} + \alpha^{1}_{n,[-k+1]} v_{n,[-k+1]} + \ldots + \alpha^{1}_{n,[m]} v_{n,[m]}.$$

Continuing in the same way, we obtain the required representation for $v_{n,[m+s]}$, $s \geqslant 2$. The representation in the opposite direction is proved similarly, by applying the operator $(D_n + a_{n,[-k-1]})$ to the second equation of (2.9). □

As it was shown in works [18], [15], [16], reductions (2.9) allow one to construct recursion operators and a Lax pair for a differential equation of hyperbolic type. Let us show that the approach also works in the case of differential–difference equations.

3. Example

In this section we test the method of constructing recursion operators and a Lax pair for differential–difference equations (1.1) at the example of the following equation

$$u_{n+1,x} = -u_{n,x} - (u_{n+1} - u_n)^2. \tag{3.1}$$

This equation is a known integrable equation of soliton type (see, e.g., [11], [3]). Its higher symmetries of minimal order read

$$u_t = u_{xxx} + 6u_x^2,$$
$$u_\tau = \frac{1}{u_{n+1} - u_{n-1}}.$$

3.1. Reduction of sequence of Laplace transformations. We focus on the linearization of equation (3.1), which reads

$$v_{n+1,x} + v_{n,x} + 2(u_{n+1} - u_n)v_{n+1} - 2(u_{n+1} - u_n)v_n = 0. \tag{3.2}$$

We consider the finite system obtained from (2.7), (2.8) for $i = 0, 1$

$$\begin{aligned}
&(D_x + b_{n-1,[0]})v_{n,[0]} = v_{n,[1]}, && (D_n + a_{n,[0]})v_{n,[1]} = -H_{n,[0]}v_{n,[0]},\\
&(D_x + b_{n-1,[1]})v_{n,[1]} = v_{n,[2]}, && (D_n + a_{n,[1]})v_{n,[2]} = -H_{n,[1]}v_{n,[1]},\\
&(D_n + a_{n,[0]})v_{n,[0]} = v_{n,[-1]}, && (D_x + b_{n,[0]})v_{n,[-1]} = -K_{n,[0]}v_{n,[0]},\\
&(D_n + a_{n,[-1]})v_{n,[-1]} = v_{n,[-2]}, && (D_x + b_{n,[-1]})v_{n,[-2]} = -K_{n,[-1]}v_{n,[-1]}.
\end{aligned} \tag{3.3}$$

Theorem 3.1. *The sequences of Laplace transformations (3.3) corresponding to equation (3.2) admit reduction (2.9) with parameters $k = 1$, $m = 1$:*

$$\begin{aligned}
v_{n,[2]} &= \alpha_{n,[-1]}v_{n,[-1]} + \alpha_{n,[0]}v_{n,[0]} + \alpha_{n,[1]}v_{n,[1]},\\
v_{n,[-2]} &= \beta_{n,[-1]}v_{n,[-1]} + \beta_{n,[0]}v_{n,[0]} + \beta_{n,[1]}v_{n,[1]},
\end{aligned} \tag{3.4}$$

where the coefficients read

$$\begin{aligned}
&\alpha_{n,[-1]} = \frac{2}{\lambda},\\
&\alpha_{n,[0]} = -2(u_{n+1}-u_{n-1})^2 - \frac{4}{\lambda},\\
&\alpha_{n,[1]} = 2(u_{n+1}-u_{n-1}) + u_n - u_{n-2},\\
&\beta_{n,[-1]} = \lambda(u_n-u_{n+2})^2 - \frac{u_n-u_{n+2}}{u_{n+1}-u_n},\\
&\beta_{n,[0]} = \lambda(u_n-u_{n+2})(2u_{n+1}-2u_{n-1}-u_n+u_{n+2}),\\
&\beta_{n,[1]} = -\lambda(u_n-u_{n+2}).
\end{aligned} \tag{3.5}$$

Proof. The theorem is proved by direct computation, so we present only the main milestones.

We substitute representations (3.4) into system (3.3) and obtain six equations for the unknown functions $\alpha_{n,[j]}$ and $\beta_{n,[j]}$

$$\begin{aligned}
&v_{n,[1],x} = v_{n,[2]} - b_{n-1,[1]}v_{n,[1]} = \alpha_{n,[-1]}v_{n,[-1]} + \alpha_{n,[0]}v_{n,[0]} + \alpha_{n,[1]}v_{n,[1]} - b_{n-1,[1]}v_{n,[1]},\\
&v_{n,[0],x} = v_{n,[1]} - b_{n-1,[0]}v_{n,[0]},\\
&v_{n,[-1],x} = -b_{n,[0]}v_{n,[-1]} - K_{n,[0]}v_{n,[0]},
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&v_{n+1,[1]} = -a_{n,[0]}v_{n,[1]} - H_{n,[0]}v_{n,[0]},\\
&v_{n+1,[0]} = v_{n,[-1]} - a_{n,[0]}v_{n,[0]},\\
&v_{n+1,[-1]} = v_{n,[-2]} - a_{n,[-1]}v_{n,[-1]} = \beta_{n,[-1]}v_{n,[-1]} + \beta_{n,[0]}v_{n,[0]} + \beta_{n,[1]}v_{n,[1]} - a_{n,[-1]}v_{n,[-1]},
\end{aligned} \tag{3.7}$$

where the coefficients are calculated by formulas (2.6):

$$\begin{aligned}
&a_{n,[0]} = 1, && b_{n,[0]} = -2(u_{n+1}-u_n),\\
&H_{n,[0]} = -2(u_{n+1}-u_{n-1}), && K_{n,[0]} = -4(u_{n+1}-u_n),\\
&a_{n,[-1]} = \frac{u_{n+2}-u_{n+1}}{u_{n+1}-u_n}, && b_{n,[1]} = u_{n+1}-u_{n-1}.
\end{aligned}$$

We do not write out explicitly the remaining two equations in system (3.3) since they do not add conditions for determining the required coefficients.

The compatibility conditions

$$D_n(v_{n,[j],x}) = D_x(v_{n+1,[j]}), \qquad j=-1,0,1,$$

lead to a system of nonlinear equations for the unknown functions $\alpha_{n,[j]}$ and $\beta_{n,[j]}$

1) $\alpha_{n+1,[0]} - \alpha_{n,[0]} - \alpha_{n+1,[-1]}\beta_{n,[0]} - 2\alpha_{n+1,[1]}(u_{n+1}-u_{n-1}) = 0,$

2) $\alpha_{n+1,[0]} + \alpha_{n,[-1]} + \alpha_{n+1,[-1]}\beta_{n,[-1]} - \dfrac{u_{n+2}-u_{n+1}}{u_{n+1}-u_n}\alpha_{n+1,[-1]} = 0,$

3) $\alpha_{n+1,[1]} - \alpha_{n,[1]} - \alpha_{n+1,[-1]}\beta_{n,[1]} + u_{n+1} + u_n - u_{n-1} - u_{n-2} = 0,$

4) $(\beta_{n,[0]})_x + \alpha_{n,[0]}\beta_{n,[1]} + 2(u_{n+2} - u_{n+1} - u_n + u_{n-1})\beta_{n,[0]} + 4(u_{n+1}-u_n)\beta_{n,[-1]} = 0,$

5) $(\beta_{n,[-1]})_x + \alpha_{n,[-1]}\beta_{n,[1]} + 2(u_{n+2} - 2u_{n+1} + u_n)$
$- 2\dfrac{u_{n+2}-u_n}{(u_{n+1}-u_n)^2}u_{n,x} - \dfrac{(u_{n+2}-u_n)(u_{n+2}+u_{n+1}-2u_n)}{u_{n+1}-u_n} = 0,$

6) $(\beta_{n,[1]})_x + \beta_{n,[0]} + \alpha_{n,[1]}\beta_{n,[1]} + 2u_{n+2} - 2u_{n+1} - u_n + u_{n-2} = 0.$

By analysing these equations 1)–6) we are convinced that the unknown functions $\alpha_{n,[j]}$ and $\beta_{n,[j]}$ depend on the dynamical variables as follows

$$\alpha_{n,[j]} = \alpha_{n,[j]}(u_{n-2}, u_{n-1}, u_n, u_{n+1}), \qquad \beta_{n,[j]} = \beta_{n,[j]}(u_{n-1}, u_n, u_{n+1}, u_{n+2}). \tag{3.8}$$

Moreover, substituting representations (3.8) into equations 1)–3), we immediately see that the functions $\alpha_{n,[j]}$ depend on the variable u_{n-2}, i.e., we obtain

$$\alpha_{n,[1]} = -u_{n-2} + \tilde{\alpha}_{n,[1]}(u_{n-1}, u_n, u_{n+1}),$$
$$\alpha_{n,[0]} = \tilde{\alpha}_{n,[0]}(u_{n-1}, u_n, u_{n+1}),$$
$$\alpha_{n,[-1]} = \tilde{\alpha}_{n,[-1]}(u_{n-1}, u_n, u_{n+1}).$$

Continuing the analysis of equations 1)–6) in the same way, we arrive at formulas (3.5). The proof is complete. □

We rewrite the system of equations (3.6), (3.7) as the matrix systems

$$\Psi_{n,x} = A\Psi_n, \qquad \Psi_{n+1} = B\Psi_n \tag{3.9}$$

for the vector $\Psi_n = (v_{n,[1]}, v_{n,[0]}, v_{n,[-1]})^T$, where

$$A = \begin{pmatrix} 2(u_{n+1}-u_{n-1}) & \frac{2}{\lambda} & -\frac{4}{\lambda} - 2(u_{n+1}-u_{n-1})^2 \\ 1 & -2(u_n - u_{n-1}) & 0 \\ 0 & 4(u_{n+1}-u_n) & -2(u_{n+1}-u_n) \end{pmatrix},$$

$$B = \begin{pmatrix} -1 & 2(u_{n+1}-u_{n-1}) & 0 \\ 0 & -1 & 1 \\ (u_{n+2}-u_n)\lambda & -(u_{n+2}-u_n)(u_{n+2}+2u_{n+1}-u_n-2u_{n-1})\lambda & 1+(u_{n+2}-u_n)^2\lambda \end{pmatrix}.$$

Corollary 3.1. *System of equations (3.9) is compatible if and only if the function $u_n = u_n(x)$ is a solution of equation (3.1).*

This implies that system (3.9) defines a Lax pair for equation (3.1). It should be noted that this pair differs from the usual one since it has a higher order. However, system (3.9) can be reduced to the usual Lax pair by means of suitable transformations.

3.2. Lax pair and recursion operators of equation (3.1). Here we derive the usual Lax pair and the recursion operator of equation (3.1) from system of equations (3.9). We begin with the recursion operator. In order to do this, we rewrite the matrix equations in (3.9) in the form of scalar differential and third order difference equations order for the unknown function $v_{n,[0]}$:

$$\begin{aligned} &v_{n,[0],xxx} + \left(8u_{n,x} + \frac{4}{\lambda}\right) v_{n,[0],x} + 4u_{n,xx} v_{n,[0]} = 0, \\ &v_{n+2,[0]} + \left(\frac{u_{n+2}-u_n}{u_{n+1}-u_{n-1}} - \lambda(u_{n+2}-u_n)^2\right) v_{n+1,[0]} \\ &\qquad + \left(\lambda(u_{n+1}-u_{n-1})(u_{n+2}-u_n) - 1\right) v_{n,[0]} - \frac{u_{n+2}-u_n}{u_{n+1}-u_{n-1}} v_{n-1,[0]} = 0. \end{aligned} \tag{3.10}$$

We equations (3.10) in operator form as

$$\begin{aligned} &\left(D_x^3 + 8u_{n,x}D_x + 4u_{n,xx}\right) v_{n,[0]} = -\frac{4}{\lambda} D_x\left(v_{n,[0]}\right), \\ &\left(D_n^2 + \frac{u_{n+2}-u_n}{u_{n+1}-u_{n-1}} D_n - 1 - \frac{u_{n+2}-u_n}{u_{n+1}-u_{n-1}} D_n^{-1}\right) v_{n,[0]} = \\ &\qquad \lambda(u_{n+2}-u_n)\left(D_n - 1\right)\left((u_{n+1}-u_{n-1}) v_{n,[0]}\right). \end{aligned} \tag{3.11}$$

We apply the operator D_x^{-1} to the first equation, and, respectively, the operator $\frac{1}{(u_{n+1}-u_{n-1})}\left(D_n - 1\right)^{-1} \frac{1}{(u_{n+2}-u_n)}$ to the second equation (3.11); then we obtain

$$R_x v_{n,[0]} = -\frac{4}{\lambda} v_{n,[0]}, \qquad R_n v_{n,[0]} = \lambda v_{n,[0]},$$

where the operators R_x and R_n read

$$\begin{aligned} R_x =& D_x^2 + 8u_{n,x} - 4D_x^{-1}u_{n,xx}, \\ R_n =& \frac{1}{(u_{n+1}-u_{n-1})^2}(D_n + D_n^{-1}) + \frac{2}{(u_{n+1}-u_{n-1})(u_n - u_{n-2})} \\ &+ \frac{2}{(u_{n+1}-u_{n-1})}(D_n - 1)^{-1}\left(\frac{1}{u_n - u_{n-2}} - \frac{1}{u_{n+2}-u_n}\right). \end{aligned} \tag{3.12}$$

Proposition 3.1. *The operators R_x and R_n given in (3.12) define recursion operators describing the hierarchies of higher symmetries in the directions x and n, respectively, for equation (3.1).*

We proceed to constructing the Lax pair for equation (3.1). In order to do this, we shall use the following four equations: equation (3.1), its linearization (3.2), and also two equations (3.10) obtained in the previous section. We begin with the first equation (3.10), which we rewrite as

$$v_{n,xxx} = -\left(8u_{n,x} - \frac{4}{\lambda}\right)v_{n,x} - 4u_{n,xx}v_n. \tag{3.13}$$

First we need to reduce the order of this equation. In order to do this, we seek a constraint of the form

$$v_{n,xx} = F(v_{n,x}, v_n, u_{n,x}, u_n), \tag{3.14}$$

which satisfies the condition

$$\left(8u_{n,x} - \frac{4}{\lambda}\right)v_{n,x} + 4u_{n,xx}v_n + D_x(F(v_{n,x}, v_n, u_{n,x}, u_n))\Big|_{(3.13),(3.14)} = 0$$

or in expanded form

$$(F_{u_{n,x}} + 4v_n)u_{n,xx} + F_{v_{n,x}}F + F_{v_n}v_{n,x} + F_{u_n}u_{n,x} + (8u_{n,x} + 4\lambda^{-1})v_{n,x} = 0. \tag{3.15}$$

Thus, to find the unknown function $F = F(v_{n,x}, v_n, u_{n,x}, u_n)$ we use equation (3.15). Note that the variables $u_n, u_{n,x}, u_{n,xx}, \dots$ are independent, therefore the problem of finding the function F reduces to solving an overdetermined system of equations. First of all, in equation (3.15) we equate the coefficients of the highest derivative $u_{n,xx}$ and find

$$F(v_{n,x}, v_n, u_{n,x}, u_n) = -4u_{n,x}v_n + F_1(v_{n,x}, v_n, u_n). \tag{3.16}$$

Thus, at the first step we have determined the dependence of the function $F(v_{n,x}, v_n, u_{n,x}, u_n)$ on the variable $u_{n,x}$, therefore, after substituting (3.16) into equation (3.15), we can equate the coefficients of this variable. Omitting further computations, we present only the answer. Continuing to analyze equation (3.15) in the same r way, we obtain that the function $F(v_{n,x}, v_n, u_{n,x}, u_n)$ reads

$$F(v_{n,x}, v_n, u_{n,x}, u_n) = \frac{v_{n,x}^2}{2v_n} - 4u_{n,x}v_n - 2\lambda^{-1}v_n + \frac{C}{v_n},$$

where C is an arbitrary constant. Then we obtain the constraint (3.14):

$$v_{n,xx} = \frac{v_{n,x}^2}{2v_n} - 4u_{n,x}v_n - 2\lambda^{-1}v_n + \frac{C}{v_n}. \tag{3.17}$$

To construct the Lax pair of equation (3.1), it is necessary to derive one more identity relating the variables v_{n+1}, $v_{n,x}$, v_n, as well as the variable u_n and its derivatives with respect to x and shifts in n. Let us apply the shift operator D_n to equation (3.17) and replace the variables $v_{n+1,x}$ and $v_{n+1,xx}$ by virtue of linearized equation (3.2), and the variables $u_{n+1,x}$ and $u_{n+1,xx}$ by virtue of equation (3.1). As a result we obtain the required identity

$$v_{n+1} = \frac{\lambda}{4}\frac{v_{n,x}^2}{v_n} - \lambda(u_{n+1} - u_n)v_{n,x} + \lambda(u_{n+1} - u_n)^2 v_n + \frac{C\lambda}{2v_n}. \tag{3.18}$$

Using the above relations, we rewrite linearized equation (3.2)

$$\begin{aligned} v_{n+1,x} = &-\frac{1}{2}(u_{n+1}-u_n)\lambda\frac{v_{n,x}^2}{v_n} + (2\lambda(u_{n+1}-u_n)^2-1)v_{n,x} \\ &+ 2(u_{n+1}-u_n)(1-\lambda(u_{n+1}-u_n)^2)v_n - (u_{n+1}-u_n)\frac{C\lambda}{v_n}. \end{aligned} \tag{3.19}$$

Note that the resulting triple of equations (3.17), (3.18), (3.19) is compatible on solutions of given equation (3.1). Now, using this compatible triple of equations, let us derive the standard Lax pair for equation (3.1). Let $C = 0$ and make the following change of variables in these equations:

$$v_n = \lambda^n \varphi_n^2.$$

Then we obtain the Lax pair

$$\begin{aligned} &\varphi_{n,xx} = -(2u_{n,x} + \lambda^{-1})\varphi_n, \\ &\varphi_{n+1} = \varphi_{n,x} - (u_{n+1}-u_n)\varphi_n, \\ &\varphi_{n+1,x} = -(u_{n+1}-u_n)\varphi_{n,x} + ((u_{n+1}-u_n)^2 - \lambda^{-1})\varphi_n \end{aligned} \tag{3.20}$$

or in matrix form

$$\Phi_{n,x} = \tilde{A}\Phi_n, \qquad \Phi_{n+1} = \tilde{B}\Phi_n, \tag{3.21}$$

where $\Phi_n = (\varphi_{n,x}, \varphi_n)^T$,

$$\tilde{A} = \begin{pmatrix} 0 & -2u_{n,x} - \lambda^{-1} \\ 1 & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} -u_{n+1}+u_n & (u_{n+1}-u_n)^2 - \lambda^{-1} \\ 1 & -u_{n+1}+u_n \end{pmatrix}.$$

One can verify that the pair of equations (3.21) is compatible if and only if the function u_n satisfies equation (3.1). Thus, (3.21) defines a Lax pair for equation (3.1).

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