ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 4 (2026). P. 113-125.

doi:10.13108/2026-18-4-113

ON INTEGRABLE REDUCTIONS OF NONLINEAR EQUATIONS OF TWO–DIMENSIONAL TODA CHAIN TYPE

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Abstract. We consider nonlinear integrable differential–difference equations with three independent variables, two of which are continuous and one is discrete. Such equations, called chains, admit finite–field reductions associated with truncations in the discrete variable. We present reductions in the form of systems of nonlinear hyperbolic partial differential equations possessing complete sets of characteristic integrals in both characteristic directions.

Keywords: nonlinear chains in 3D, characteristic integrals, Darboux integrability, finite–field reductions, Lax pair.

Mathematics Subject Classification: 35Q51, 39A14

1. Introduction

Nowadays nonlinear differential–difference equations of two–dimensional Toda chain type is a well–studied class of equations with three independent variables. Equations of this class appeared in the 18th century in works by Laplace, which were devoted to the study of linear hyperbolic differential equations with variable coefficients. Methods for constructing explicit solutions to the so–called open chains, directly related to the Toda chain, were proposed in the works by Darboux, Goursat, Moutard, Liouville, and others (see, e.g., survey [1]). Interest in this topic was revived in the second half of the last century after the discovery of the inverse scattering method. After works [12], [13] it became clear that integrable equations of Toda chain type can be used in constructing explicit particular solutions of integrable nonlinear partial differential equations of the Davey — Stewartson class. The list of integrable models in 3D of Toda chain type (which seems to be exhaustive) was first obtained by Shabat and Yamilov in [13] as a result of two–dimensionalization in the framework of the symmetry approach of integrable chains of dimension 1+1 from the classification list compiled first by Yamilov, see, e.g., [14]. The problem on constructing characteristic integrals of special finite–field reductions of integrable chains of Toda type was previously investigated in [2], [6], [8], [9], [11].

In the present work we discuss a method for constructing complete sets of characteristic integrals of finite–field reductions by means of the Lax pair. Using this method, we shall construct complete sets of integrals for systems obtained from the following equations of Toda chain type

(E_1) $u_{n,xy} = e^{u_{n+1}-2u_n+u_{n-1}}$,

(E_2) $u_{n,y} = u_n(v_{n+1} - v_n), \quad v_{n,x} = u_n - u_{n-1}$,

(E_3) $u_{n,xy} = (e^{u_{n+1}-u_n} - e^{u_n-u_{n-1}})\, u_{n,x}$,

I.T. Habibullin, A.U. Sakieva, On analytic solutions of linear homological equation.
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Submitted August 26, 2026.

by imposing special truncation conditions on the discrete variable n. We note that there exist various methods for constructing characteristic integrals: by means of characteristic algebras [9], [6], [3], by means of the generalized d'Alembert equation [2], [10], from a Lax pair [8], [11], on the base of Laplace invariants of the linearized system [4], [5], etc.

2. Complete sets of characteristic integrals for system of hyperbolic differential equations

We dwell on some basic facts of the theory of equations integrable in the Darboux sense. We consider a system, which consists of N equations of hyperbolic type

$$u_{i,xy} = F_i(u_1, \dots, u_N, u_{1,x}, \dots, u_{N,x}, u_{1,y}, \dots, u_{N,y}), \qquad i = 1, 2, \dots, N, \tag{2.1}$$

where F_i are analytic functions defined on some domain of the space $\mathbb{C}^{3N}$. Let us recall the necessary definitions (see, e.g., [3]). A smooth function of the form

$$I = I(u_1, \dots, u_N, u_{1,x}, \dots, u_{N,x}, u_{1,xx}, \dots, u_{N,xx}, \dots, u_{1,r}, \dots, u_{N,r}),$$

where we adopt the notation $u_{j,s} := \dfrac{\partial^s}{\partial x^s} u_j$, is called the y-integral of order r of system of equations (2.1) if the following condition holds: $D_y I = 0$, where the derivative with respect to the variable y is calculated by virtue of system (2.1). In other words, the identity must hold

$$D_y I = \sum_i \Big(u_{i,y} \frac{\partial}{\partial u_i} + F_i \frac{\partial}{\partial u_{i,x}} + D_x(F_i) \frac{\partial}{\partial u_{i,xx}} + \cdots \Big) I = Y I = 0.$$

Integrals in the direction x are introduced in the same way.

A set of y-integrals $\{I_i\}_{i=1}^N$ of orders r_i of system of equations (2.1) forms a complete set of independent y-integrals if the following condition holds

$$\det \left(\frac{\partial I_j}{\partial u_{i,m}} \right) = \begin{vmatrix} \dfrac{\partial I_1}{\partial u_{1,r_1}} & \dfrac{\partial I_1}{\partial u_{2,r_1}} & \cdots & \dfrac{\partial I_1}{\partial u_{N,r_1}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial I_N}{\partial u_{1,r_N}} & \dfrac{\partial I_N}{\partial u_{2,r_N}} & \cdots & \dfrac{\partial I_N}{\partial u_{N,r_N}} \end{vmatrix} \neq 0. \tag{2.2}$$

To determine Darboux integrability of a system of hyperbolic equations, there is an effective algebraic criterion based on the notion of characteristic Lie algebra introduced in 1980 by Shabat [9], see also monograph [3].

Theorem 2.1. *System of hyperbolic equations (2.1) admits a complete set of independent y-integrals if and only if the characteristic Lie algebra over the ring of locally analytic functions of the variables $\bar{u}_y$, $\bar{u}$, $\bar{u}_x$, $\bar{u}_{xx}$, ... generated by the characteristic operators*

$$X_j = \frac{\partial}{\partial u_{j,y}}, \quad j = 1, \dots, N;$$
$$Y = \sum_i u_{i,y} \frac{\partial}{\partial u_i} + F_i \frac{\partial}{\partial u_{i,x}} + D_x(F_i) \frac{\partial}{\partial u_{i,xx}} + \cdots$$

has a finite dimension. Here $\bar{u} = (u_1, u_2, \dots, u_N)$.

3. Integrals of finite–field reduction of equation E_1

We consider the system of equations

$$\begin{aligned} &u_{1,xy} = e^{u_2-2u_1},\\ &u_{2,xy} = e^{u_3-2u_2+u_1},\\ &\cdots\cdots\cdots\\ &u_{N-1,xy} = e^{u_N-2u_{N-1}+u_{N-2}},\\ &u_{N,xy} = e^{-2u_N+u_{N-1}}, \end{aligned} \tag{3.1}$$

obtained from E_1 as a result of imposing the truncation conditions $u_0 = 0$ and $u_{N+1} = 0$. System of equations (3.1), called the open chain, was studied by many authors. It was proved in [9] that the characteristic algebras of system (3.1) has a finite dimension, and this implies immediately follows that the system admits complete sets of characteristic integrals. The characteristic integrals were found by means of the generalized d'Alembert equation in [2]. Here we shall use system (3.1) as a model system for explaining the algorithm for constructing complete sets of characteristic integrals by means of the Lax pair. The Lax pair for (3.1) is given in the form of two systems of differential equations determining the dynamics in x

$$\begin{cases} \psi_{0,x} = u_{1,x}\psi_0 + \psi_1,\\ \psi_{k,x} = (u_{k+1,x} - u_{k,x})\psi_k + \psi_{k+1}, \quad 1 \leqslant k \leqslant N \end{cases} \tag{3.2}$$

and the dynamics in y

$$\begin{cases} \psi_{0,y} = 0,\\ \psi_{k,y} = -e^{u_{k+1}-2u_k+u_{k-1}}, \qquad 1 \leqslant k \leqslant N. \end{cases} \tag{3.3}$$

Let us first construct the y-integrals. In order to do this, by using system (3.2) we derive the series of relations

$$\begin{aligned} &\psi_1 = (D_x - u_{1,x})\psi_0,\\ &\psi_2 = (D_x - u_{2,x} + u_{1,x})(D_x - u_{1,x})\psi_0,\\ &\cdots\cdots\cdots\\ &\psi_N = (D_x - u_{N,x} + u_{N-1,x})\cdots(D_x - u_{1,x})\psi_0. \end{aligned}$$

By the identities $\psi_{N+1} = 0$ and $u_{N+1} = 0$ the last identity in system (3.2) reads

$$\psi_{N,x} + u_{N,x}\psi_N = 0.$$

Therefore, the function ψ_0 satisfies the equation $B\psi_0 = 0$, where the differential operator B is given by the relation

$$B = (D_x + u_{N,x})(D_x - u_{N,x} + u_{N-1,x})\cdots(D_x - u_{1,x}).$$

We rewrite the operator B as the polynomial

$$B = D_x^{N+1} + \gamma_{N-1}D_x^{N-1} + \gamma_{N-2}D_x^{N-2} + \cdots + \gamma_1 D_x + \gamma_0.$$

In view of the first equation in system (3.3), the function ψ_0 satisfies the following pair of equations

$$B\psi_0 = 0, \quad D_y\psi_0 = 0. \tag{3.4}$$

System (3.4) obviously implies the relation

$$D_y(\gamma_{N-1})D_x^{N-1}\psi_0 + D_y(\gamma_{N-2})D_x^{N-2}\psi_0 + \cdots + D_y(\gamma_1)D_x\psi_0 + D_y(\gamma_0)\psi_0 = 0,$$

which holds on an arbitrary solution of Lax equations (3.2), (3.3). Hence, the functions $D_x^{N-1}\psi_0$, $D_x^{N-2}\psi_0$, $\ldots$, ψ_0, generally speaking, are not linearly dependent. This yields that the condition

$D_y\gamma_i = 0$ holds for all $i = 0, 1, \ldots, N-1$. Using Theorem 2.1, one can show that the functions γ_0, γ_1, $\ldots$, γ_{N-1} form a complete set of independent characteristic y-integrals of system of equations (3.1). Integrals in the other direction are sought in the same way.

3.1. Examples. For $N = 2$ the system reads

$$\begin{cases} u_{1,xy} = e^{u_2-2u_1}, \\ u_{2,xy} = e^{-2u_2+u_1}. \end{cases} \tag{3.5}$$

The Lax pair for (3.5) is given by the equations

$$\begin{cases} \psi_{0,x} = u_{1,x}\psi_0 + \psi_1, \\ \psi_{1,x} = (u_{2,x} - u_{1x})\,\psi_1 + \psi_2, \\ \psi_{2,x} = -u_{2,x}\psi_2 \end{cases} \tag{3.6}$$

and

$$\begin{cases} \psi_{0,y} = 0, \\ \psi_{1,y} = -e^{u_2-2u_1}, \\ \psi_{2,y} = e^{-2u_2+u_1}. \end{cases} \tag{3.7}$$

By relations (3.6) we obtain

$$(D_x + u_{2,x})\,(D_x - (u_{2,x} - u_{1,x}))\,(D_x - u_{1,x})\,\psi_0 = 0.$$

The first equation in system (3.7) implies $D_y\psi_0 = 0$. In the expanded form the operator B reads

$$\begin{aligned} B =& D_x^3 + \left(-u_{2,xx} - u_{1,xx} + u_{2,x}u_{1,x} - u_{1,x}^2 - u_{2,x}^2\right) D_x \\ &+ \left(-u_{1,xxx} + u_{2,xx}u_{1,x} - 2u_{1,x}u_{1,xx} + u_{2,x}^2u_{1,x} - u_{1,x}^2u_{2,x}\right), \end{aligned}$$

this operator is the required generating function of the integrals. Hence, the functions

$$\begin{aligned} J_1 &= -u_{2,xx} - u_{1,xx} + u_{2,x}u_{1,x} - u_{1,x}^2 - u_{2,x}^2, \\ J_2 &= -u_{1,xxx} + u_{2,xx}u_{1,x} - 2u_{1,x}u_{1,xx} + u_{2,x}^2u_{1,x} - u_{1,x}^2u_{2,x} \end{aligned}$$

are y-integrals of the system.

4. Reductions of polynomial Toda equation

We recall that the infinite system of equations

$$u_{n,y} = u_n(v_{n+1} - v_n), \qquad v_{n,x} = u_n - u_{n-1}, \quad n \in \mathbb{Z} \tag{4.1}$$

is called the polynomial Toda chain. It has a Lax pair given by the system of linear equations

$$\psi_{n,x} = -u_n\psi_{n+1}, \qquad \psi_{n,y} = -v_n\psi_n + \psi_{n-1}. \tag{4.2}$$

In what follows we study the Darboux integrability of the system of hyperbolic equations

$$\begin{aligned} u_{1,y} &= u_1(v_2 - v_1), \\ u_{2,y} &= u_2(v_3 - v_2), \\ &\cdots\cdots\cdots \\ u_{N-2,y} &= u_{N-2}(v_{N-1} - v_{N-2}), \\ u_{N-1,y} &= -u_{N-1}\Big(2v_{N-1} + \sum_{j=1}^{N-2} v_j\Big) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &v_{1,x} = u_1, \\ &v_{2,x} = u_2 - u_1, \\ &\cdots\cdots\cdots \\ &v_{N-1,x} = u_{N-1} - u_{N-2}, \\ &v_{N-1,x} = -u_{N-1}, \end{aligned} \tag{4.4}$$

obtained from the chain (4.1) as a result of imposing the truncation conditions

$$u_0 = 0, \qquad u_N = 0, \qquad v_1 + v_2 + \cdots + v_N = 0. \tag{4.5}$$

As far as the authors know, such reductions of chain (4.1) have not been discussed before in the literature. We note that once the conditions

$$\Psi_0 = 0, \qquad \Psi_{N+1} = 0$$

are satisfied, linear system (4.2) becomes the Lax pair for system (4.3), (4.4). The dynamics of the eigenfunctions with respect to the variable x is determined by the system of equations

$$\begin{aligned} &\psi_{1,x} = -u_1\psi_2, \\ &\psi_{2,x} = -u_2\psi_3, \\ &\cdots\cdots\cdots \\ &\psi_{N-1,x} = -u_{N-1}\psi_N, \\ &\psi_{N,x} = 0 \end{aligned} \tag{4.6}$$

while the dynamics in y is determined by the system

$$\begin{aligned} &\psi_{1,y} = -v_1\psi_1, \\ &\psi_{2,y} = -v_2\psi_2 + \psi_1, \\ &\cdots\cdots\cdots \\ &\psi_{N-1,y} = -v_{N-1}\psi_{N-1} + \psi_{N-2}, \\ &\psi_{N,y} = \sum_{j=1}^{N-1} v_j\psi_N + \psi_{N-1}. \end{aligned} \tag{4.7}$$

To find the x-integrals of system (4.3), (4.4), we use equations (4.7), which imply the series of relations

$$\begin{aligned} &\psi_{N-1} = \Big(D_y - \sum_{j=1}^{N-1} v_j\Big)\psi_N, \\ &\psi_{N-2} = (D_y + v_{N-1})\Big(D_y - \sum_{j=1}^{N-1} v_j\Big)\psi_N, \\ &\cdots\cdots\cdots \\ &\psi_1 = (D_y + v_2)(D_y + v_3)\cdots(D_y + v_{N-1})\Big(D_y - \sum_{j=1}^{N-1} v_j\Big)\psi_N, \end{aligned} \tag{4.8}$$

By the last relation in (4.8), first equation in system (4.7) yields the identity $B\psi_N = 0$, where the operator B reads

$$B = (D_y + v_1)(D_y + v_2)(D_y + v_3)\cdots(D_y + v_{N-1})\Big(D_y - \sum_{j=1}^{N-1} v_j\Big).$$

On the other hand we have $\psi_{N,x}=0$. Therefore, as a result, we arrive at the pair of relations for the function ψ_N:

$$B\psi_N=0, \qquad D_x\psi_N=0, \tag{4.9}$$

which obviously imply

$$D_x(B)\psi_N=0. \tag{4.10}$$

We rewrite the operator B in expanded form

$$B=D_y^N+\gamma_N D_y^{N-1}+\cdots+\gamma_2 D_y+\gamma_1.$$

Then the relation $D_x(B)\psi_N=0$ becomes

$$D_x(\gamma_N)D_y^{N-1}(\psi_N)+D_x(\gamma_{N-1})D_y^{N-2}(\psi_N)+\cdots+D_x(\gamma_2)D_y(\psi_N)+D_x(\gamma_1)\psi_N=0.$$

Next we observe that the function ψ_N is an arbitrary solution to the linear equation $B\psi_N=0$ of order N, therefore the functions ψ_N, $D_y\psi_N$, …, $D_y^{N-1}\psi_N$, generally speaking, are not linearly dependent. Hence, the coefficients of the linear combination above, namely, the functions $D_x(\gamma_1)$, $D_x(\gamma_2)$, …, $D_x(\gamma_N)$, vanish. In other words, γ_1, γ_2, …, γ_N are x-integrals of system (4.3), (4.4).

We construct the y-integrals of the system (4.3), (4.4) using the relation $R_1\psi_1=0$, where the operator R_1 reads

$$R_1=D_x\frac{1}{u_{N-1}}D_x\frac{1}{u_{N-2}}D_x\dots\frac{1}{u_1}D_x. \tag{4.11}$$

This relation is a simple consequence of system of equations (4.6). On the other hand, by (4.7) we have the equation $\psi_{1,y}=-v_1\psi_1$, which we need to rewrite in a form convenient for solving. In order to do this, we use the substitutions

$$u_1=e^{q_2-q_1}, \qquad u_2=e^{q_3-q_2}, \qquad \dots, \qquad u_{N-1}=e^{q_N-q_{N-1}}, \tag{4.12}$$

and the identities

$$v_1=q_{1y}, \qquad v_2=q_{2y}, \qquad \dots, \qquad v_N=q_{Ny}, \tag{4.13}$$

which relate system (4.3), (4.4) with the following system of hyperbolic equations

$$q_{1,xy}=e^{q_2-q_1}, \qquad q_{2,xy}=e^{q_3-q_2}-e^{q_2-q_1}, \qquad \dots, \qquad q_{N,xy}=-e^{q_N-q_{N-1}}, \tag{4.14}$$

which is a reduction of the exponential Toda chain (see [13]). It is easy to verify that the identity holds

$$(q_1+q_2+\cdots+q_N)_{xy}=0,$$

and therefore we have $q_1+q_2+\cdots+q_N=a(x)+b(y)$, where $a(x)$ and $b(y)$ are arbitrary functions. To find the functions q_1, q_2, …, q_N from the known u_1, u_2, …, u_N one can use the system of equations

$$\begin{aligned}
&q_1+q_2+\cdots+q_N=a+b,\\
&q_2-q_1=\ln u_2,\\
&\cdots\cdots\cdots\\
&q_{N-1}-q_{N-2}=\ln u_{N-2},\\
&q_N-q_{N-1}=\ln u_{N-1}.
\end{aligned} \tag{4.15}$$

By means of elementary transformations one can find explicit expressions for the required functions

$$q_1 = \frac{1}{N}\big(a+b-(N-1)\ln u_1-(N-2)\ln u_2\cdots-3\ln u_{N-3}-2\ln u_{N-2}-\ln u_{N-1}\big),$$
$$q_2 = \frac{1}{N}\big(a+b+\ln u_1-(N-2)\ln u_2\cdots-3\ln u_{N-3}-2\ln u_{N-2}-\ln u_{N-1}\big),$$
$$\cdots\cdots\cdots$$
$$q_{N-1} = \frac{1}{N}\big(a+b+\ln u_1+2\ln u_2\cdots+(N-3)\ln u_{N-3}+(N-2)\ln u_{N-2}-\ln u_{N-1}\big),$$
$$q_N = \frac{1}{N}\big(a+b+\ln u_1+2\ln u_2\cdots+(N-3)\ln u_{N-3}+(N-2)\ln u_{N-2}+(N-1)\ln u_{N-1}\big).$$

In the relations

$$R_1\psi_1 = 0, \qquad \psi_{1,y} = -v_1\psi_1 \tag{4.16}$$

we pass to the variables q_1 and ϕ_1 by letting $\psi_1 = e^{-q_1}\phi_1$ and using the identity $v_1 = q_{1y}$ (see above). As a result we obtain $D_y\phi_1 = 0$, $Re^{-q_1}\phi_1 = 0$, where the functions $e^{\pm q_1}$ are assumed to be expressed in terms of u_j. Next we transform the operator R_1 according to the rule

$$R = e^{q_1}u_1u_2\dots u_{N-1}R_1e^{-q_1}.$$

It is not difficult to verify that the obtained operator

$$R = e^{q_1}u_1u_2\dots u_{N-1}\Big(D_x\frac{1}{u_{N-1}}D_x\frac{1}{u_{N-2}}D_x\dots\frac{1}{u_1}D_x\Big)e^{-q_1}$$

can be represented in the form of a polynomial in the operator D_x

$$R = D_x^N + \beta_{N-1}D_x^{N-1} + \cdots + \beta_1D_x + \beta_0,$$

where the functions β_j depend only on the variables u_1, u_2, …, u_{N-1} and their derivatives with respect to x. The pair of relations

$$D_y\phi_1 = 0, \qquad R_1\phi_1 = 0$$

implies immediately that $D_y(R_1)\phi_1 = 0$, or in expanded form

$$D_y(\beta_{N-1})(D_x^{N-1}\phi_1) + D_y(\beta_{N-2})(D_x^{N-2}\phi_1) + \cdots + D_y(\beta_0)\phi_1 = 0.$$

Due to the fact that the functions $D_x^{N-1}\phi_1$, $D_x^{N-2}\phi_1$, …, $D_x\phi_1$, ϕ_1, generally speaking, are linearly independent, all coefficients in the linear expression above are equal to zero. Therefore, for each $j = 0,\dots,N-1$ the function β_j is a characteristic y-integral of system (4.3), (4.4).

4.1. Examples. Let us provide illustrative examples.

Example 1. For $N = 2$ system (4.3), (4.4) reads

$$u_y = -2uv, \qquad v_x = u. \tag{4.17}$$

Its Lax pair is given by the equations

$$\psi_{1,x} = -u\psi_2, \qquad \psi_{2,x} = 0,$$
$$\psi_{1,y} = -v\psi_1, \qquad \psi_{2,y} = v\psi_2 + \psi_1.$$

In order to find the x-integrals, we employ the generating function of the integrals

$$B = (D_y + v)(D_y - v) = D_y^2 - v_y - v^2.$$

Therefor, the function $I = v_y + v^2$ is an x-integral of system (4.17).

To construct the integral in the direction y, we use the operator

$$R_1 = e^{-q_1} D_x e^{2q_1} D_x e^{-q_1} = D_x^2 + q_{1,xx} - q_{1,x}^2.$$

Passing to the variables u in the expression $q_{1,xx} + q_{1,x}^2$, we obtain

$$q_{1,xx} + q_{1,x}^2 = -\frac{1}{2}\Big(\frac{u_{xx}}{u} - \frac{3}{2}\frac{u_x^2}{u^2}\Big).$$

Hence, the function

$$J = \frac{u_{xx}}{u} - \frac{3}{2}\frac{u_x^2}{u^2}$$

is a y-integral of system (4.17). We note that system (4.17) is easily reduced to the equation

$$v_{xy} + 2vv_x = 0,$$

found in [7], where its Darboux integrability was proved.

Example 2. For $N = 3$ system of equations (4.3), (4.4) reads

$$u_{1,y} = u_1(v_2 - v_1), \qquad u_{2,y} = -u_2(2v_2 + v_1), \tag{4.18}$$

$$v_{1,x} = u_1, \qquad v_{2,x} = u_2 - u_1. \tag{4.19}$$

We represent the operator

$$B = (D_y + v_1)(D_y + v_2)(D_y - v_1 - v_2)$$

in the expanded form

$$B = D_y^3 + (h - 2w_y - w^2)D_y - w_{yy} - ww_y - hw,$$

where $w = v_1 + v_2$, $h = v_{2,y} + v_1v_2$. After some simplifications we obtain a complete set of independent integrals for system (4.18), (4.19)

$$I_1 = 2v_{1,y} + v_{2,y} + v_1^2 + v_2^2 + v_1v_2,$$
$$I_2 = v_{1,yy} + v_{2,yy} + (v_1 + v_2)(v_{1,y} + 2v_{2,y} + v_1v_2).$$

The generating function of the y-integrals is the operator

$$R = e^{q_1} u_1u_2 D_x \frac{1}{u_2} D_x \frac{1}{u_1} D_x e^{-q_1}.$$

In view of truncation conditions (4.5), by relations (4.12) we find the representations $u_1 = e^{q_2 - q_1}$, $u_2 = e^{-2q_2 - q_1}$, which imply

$$R = e^{-q_2 - q_1} D_x e^{2q_2 + q_1} D_x e^{q_1 - q_2} D_x e^{-q_1}.$$

Direct calculations show that the operator R can be represented as

$$R = D_x^3 + J_1 D_x + J_0,$$

where the coefficients J_1, J_2 read

$$J_1 = -2q_{1,xx} - q_{2,xx} + q_{1,x}q_{2,x} - (q_{1,x} + q_{2,x})^2,$$
$$J_2 = -q_{1,xxx} + q_{1,x}q_{2,xx} - q_{1,x}q_{1xx} + q_{1,x}^2 q_{2,x} + q_{1,x}q_{2,x}^2,$$

at the same time, the conditions $D_yJ_1 = 0$, $D_yJ_2 = 0$ are satisfied. Passing to the initial variables u_1, u_2 by means of the substitution

$$q_1 = -\frac{1}{3}\ln(u_2u_1^2), \qquad q_2 = \frac{1}{3}\ln\Big(\frac{u_1}{u_2}\Big)$$

we obtain a complete set of y-integrals of system (4.18), (4.19):

$$\begin{aligned}
J_1 =& \frac{u_{1,xx}}{u_1} + \frac{u_{2,xx}}{u_2} - \frac{4}{3}\frac{u_{1,x}^2}{u_1^2} - \frac{1}{3}\frac{u_{1,x}u_{2,x}}{u_1u_2} - \frac{4}{3}\frac{u_{2,x}^2}{u_2^2},\\
J_2 =& \frac{2}{3}\frac{u_{1,xxx}}{u_1} + \frac{u_{2,xxx}}{3u_2} - \left(\frac{8}{3}\frac{u_{1,x}}{u_1} + \frac{u_{2,x}}{3u_2}\right)\frac{u_{1,xx}}{u_1} - \frac{u_{2,x}u_{2,xx}}{u_2^2}\\
&+ \frac{56}{27}\frac{u_{1,x}^3}{u_1^3} + \frac{4}{9}\frac{u_{1,x}^2u_{2,x}}{u_1^2u_2} - \frac{u_{2,x}^2u_{1,x}}{9u_2^2u_1} + \frac{16}{27}\frac{u_{2,x}^3}{u_1^3}.
\end{aligned}$$

5. Characteristic integrals of finite reduction of equation E_3

We consider the system of equations

$$\begin{aligned}
&u_{1,xy} = u_{1,x}e^{u_2-u_1},\\
&u_{2,xy} = u_{2,x}\left(e^{u_3-u_2} - e^{u_2-u_1}\right),\\
&\cdots\cdots\cdots\\
&u_{N-1,xy} = u_{N-1,x}\left(e^{u_N-u_{N-1}} - e^{u_{N-1}-u_{N-2}}\right),\\
&u_{N,xy} = -u_{N,x}e^{u_N-u_{N-1}},
\end{aligned} \tag{5.1}$$

obtained from E_3 by imposing the truncation conditions $u_0 = +\infty$ and $u_{N+1} = -\infty$. The Lax pair for (5.1) is introduced as two systems of differential equations determining the dynamics in x:

$$\begin{cases}
\psi_{k,x} = u_{k,x}\left(\psi_k - \psi_{k+1}\right), \quad 1 \leqslant k \leqslant N-1,\\
\psi_{N,x} = u_{N,x}\psi_N,
\end{cases} \tag{5.2}$$

and the dynamics in y :

$$\begin{cases}
\psi_{1,y} = 0,\\
\psi_{k,y} = e^{u_k-u_{k-1}}\left(\psi_{k-1} - \psi_k\right), \quad 2 \leqslant k \leqslant N-1.
\end{cases} \tag{5.3}$$

We first construct the y-integrals. We express ψ_2 from the first equation in system (5.2):

$$\psi_2 = \left(1 - \frac{1}{u_{1,x}}D_x\right)\psi_1.$$

Then we express ψ_3, using identity (5.2) once again,

$$\psi_3 = \left(1 - \frac{1}{u_{2,x}}D_x\right)\left(1 - \frac{1}{u_{1,x}}D_x\right)\psi_1.$$

Then

$$\psi_N = \left(1 - \frac{1}{u_{N-1,x}}D_x\right)\left(1 - \frac{1}{u_{N-2,x}}D_x\right)\ldots\left(1 - \frac{1}{u_{1,x}}D_x\right)\psi_1.$$

The last equation in (5.2) implies $D_x\left(e^{-u_N}\psi_N\right) = 0$, and hence, the function ψ_1 belongs to the kernel of the operator

$$B = D_xe^{-u_N}\left(1 - \frac{1}{u_{N-1,x}}D_x\right)\left(1 - \frac{1}{u_{N-2,x}}D_x\right)\ldots\left(1 - \frac{1}{u_{1,x}}D_x\right).$$

By of $\psi_{1,y} = 0$ this leads us to the system of relations for ψ_1:

$$B\psi_1 = 0, \quad D_y\psi_1 = 0,$$

which implies $D_y(B)\psi_1 = 0$ or in the expanded form

$$D_y(\beta_{N-1})(D_x^{N-1}\psi_1) + D_y(\beta_{N-2})(D_x^{N-2}\psi_1) + \cdots + D_y(\beta_0)\psi_1 = 0.$$

Since the functions $D_x^{N-1}\psi_1$, $D_x^{N-2}\psi_1$, $\dots D_x\psi_1$, ψ_1, generally speaking, are linearly independent, all coefficients in the linear expression above are equal to zero. Therefore, for each $j = 0, \dots, N-1$ the function β_j is a characteristic y-integral of system (5.1).

5.1. Examples. Let us consider illustrative examples.

Example 1. For $N = 2$ system (5.1) reads

$$\begin{cases} u_{1,xy} = u_{1,x}e^{u_2-u_1}, \\ u_{2,xy} = -u_{2,x}e^{u_2-u_1}. \end{cases} \tag{5.4}$$

The Lax pair for (5.4) is given by the equations

$$\begin{aligned} &\psi_{1,x} = u_{1,x}\left(\psi_1 - \psi_2\right), && \psi_{2,x} = u_{2,x}\psi_2, \\ &\psi_{1,y} = 0, && \psi_{2,y} = e^{u_2-u_1}\left(\psi_1 - \psi_2\right). \end{aligned}$$

The relations imply

$$\left(D_x - u_{2,x}\right)\frac{1}{u_{1,x}}\left(D_x - u_{1,x}\right)\psi_1 = 0$$

or

$$\left(D_x - \left(\frac{u_{1,xx}}{u_{1,x}} + u_{2,x}\right)\right)\left(D_x - u_{1,x}\right)\psi_1 = 0.$$

In expanded form the operator B reads

$$B = D_x^2 + \left(\frac{u_{1,xx}}{u_{1,x}} + u_{2,x} + u_{1,x}\right)D_x + u_{2,x}u_{1,x},$$

and it is the generating function for the integrals. Hence, the functions

$$J_1 = \frac{u_{1,xx}}{u_{1,x}} + u_{2,x} + u_{1,x}, \qquad J_2 = u_{2,x}u_{1,x}$$

are y-integrals of system (5.4). Calculating determinant (2.2), we verify that the integrals are independent:

$$\begin{vmatrix} \dfrac{1}{u_{1,x}} & 0 \\ u_{2,x} & u_{1,x} \end{vmatrix} = 1.$$

Example 2. For $N = 3$ system (5.1) reads

$$\begin{cases} u_{1,xy} = u_{1,x}e^{u_2-u_1}, \\ u_{2,xy} = u_{2,x}\left(e^{u_3-u_2} - e^{u_2-u_1}\right), \\ u_{3,xy} = -u_{3,x}e^{u_3-u_2}. \end{cases} \tag{5.5}$$

The Lax pair for (5.5) is given by the equations

$$\begin{cases} \psi_{1,x} = u_{1,x}\left(\psi_1 - \psi_2\right), \\ \psi_{2,x} = u_{2,x}\left(\psi_2 - \psi_3\right), \\ \psi_{3,x} = u_{3,x}\psi_3, \end{cases} \tag{5.6}$$

and

$$\begin{cases} \psi_{1,y} = 0, \\ \psi_{2,y} = e^{u_2-u_1}\left(\psi_1 - \psi_2\right), \\ \psi_{3,y} = e^{u_3-u_2}\left(\psi_2 - \psi_3\right). \end{cases} \tag{5.7}$$

These relations yield

$$B\psi_1 = \left(D_x - u_{3,x}\right)\left(\frac{1}{u_{2,x}}\right)\left(D_x - u_{2,x}\right)\left(\frac{1}{u_{1,x}}\right)\left(D_x - u_{1,x}\right)\psi_1 = 0.$$

We write the operator B, which is the generating function of the integrals, in the expanded form

$$\begin{aligned} B =& D_x^3 + \left(r_1 + r_2 + r_3\right) D_x^2 + \left(r_{2,x} + 2r_{1,x} + r_1 r_2 + r_2 r_3 + r_1 r_3\right) D_x \\ &+ r_{1,xx} + r_{1,x} r_2 + r_1 r_{2,x} + r_3 r_{1,x} + r_1 r_2 r_3, \end{aligned}$$

where

$$r_1 = u_{1,x}, \qquad r_2 = \frac{u_{1,xx}}{u_{1,x}} + u_{2,x}, \qquad r_3 = \frac{u_{1,xx}}{u_{1,x}} + \frac{u_{2,xx}}{u_{2,x}} + u_{3,x}.$$

Therefore, the functions

$$\begin{aligned} J_1 =& -\frac{u_{1,xxx}}{u_{1,x}} + 2\frac{{u_{1,xx}}^2}{{u_{1,x}}^2} + \frac{u_{1,xx}u_{2,xx}}{u_{1,x}u_{2,x}} + \frac{u_{1,xx}}{u_{1,x}}\left(u_{3,x} + u_{2,x}\right) \\ &+ \frac{u_{2,xx}u_{1,x}}{u_{2,x}} + u_{1,x}u_{2,x} + u_{2,x}u_{3,x} + u_{1,x}u_{3,x}, \\ J_2 =& u_{1,x} + u_{2,x} + u_{3,x} + 2\frac{u_{1,xx}}{u_{1,x}} + \frac{u_{2,xx}}{u_{2,x}}, \\ J_3 =& u_{3,x}u_{2,x}u_{1,x} \end{aligned}$$

are y-integrals for system (5.5). We calculate determinant (2.2) for the obtained integrals:

$$\begin{vmatrix} \dfrac{-1}{u_{1,x}} & 0 & 0 \\ \dfrac{2}{u_{1,x}} & \dfrac{1}{u_{2,x}} & 0 \\ u_{3,x}u_{2,x} & u_{1,x}u_{3,x} & u_{1,x}u_{2,x} \end{vmatrix} = -1.$$

It is non–zero, therefore, the found functions form a complete set of independent integrals.

Let us construct the integrals in the direction x. In order to do this, we use system (5.3) and express ψ_{N-1}, ψ_{N-2}, …, ψ_1 in terms of ψ_N:

$$\begin{aligned} &\psi_{N-1} = \left(e^{-\omega_{N-1}}D_y + 1\right)\psi_N, \qquad \omega_{N-1} = u_n - u_{N-1}, \\ &\psi_{N-2} = \left(e^{-\omega_{N-2}}D_y + 1\right)\left(e^{-\omega_{N-1}}D_y + 1\right)\psi_N, \\ &\cdots\cdots\cdots \\ &\psi_1 = \left(e^{-\omega_1}D_y + 1\right)\cdots\left(e^{-\omega_{N-1}}D_y + 1\right)\psi_N. \end{aligned}$$

Using the first equation in (5.2), we obtain

$$H\psi_N = 0,$$

where

$$H = D_y\left(e^{-\omega_1}D_y + 1\right)\cdots\left(e^{-\omega_{N-1}}D_y + 1\right).$$

On the other hand, by the last equation in system (5.3) we obtain

$$D_x\left(e^{-u_N}\psi_N\right) = 0.$$

Therefore, the function $R = e^{-u_N}\psi_N$ satisfies the system of equations

$$\tilde{H}R = 0, \qquad D_x R = 0.$$

Therefore, in view of the above arguing, the operator $\tilde{H}$ is the generating function of characteristic x-integrals for system (5.1).

Example 3. For $N = 2$ the system has form (5.4), the Lax pair is given by the equations

$$\begin{aligned} &\psi_{1,x} = u_{1,x}\left(\psi_1 - \psi_2\right), && \psi_{2,x} = u_{2,x}\psi_2 \\ &\psi_{1,y} = 0, && \psi_{2,y} = e^{u_2-u_1}\left(\psi_1 - \psi_2\right), \end{aligned}$$

and we thus obtain

$$D_y\left(e^{-u_2+u_1}D_y + 1\right)\psi_2 = 0, \qquad \left(D_x - u_{2,x}\right)\psi_2 = 0$$

or

$$D_y\left(e^{-u_2+u_1}D_y + 1\right)e^{u_2}\xi = 0, \qquad D_x\xi = 0, \qquad \xi = e^{-u_2}\psi_2.$$

Then the operator $\bar{H}$ becomes

$$D_y^2 + \left(u_{2,y} + u_{1,y} + e^{u_2-u_1}\right)D_y + \left(u_{2,yy} + u_{1,y}u_{2,y} + e^{u_2-u_1}u_{2,y}\right),$$

and the functions

$$J_1 = u_{2,y} + u_{1,y} + e^{u_2-u_1}, \qquad J_2 = u_{2,yy} + u_{1,y}u_{2,y} + e^{u_2-u_1}u_{2,y}$$

are x-integrals of system (5.4).

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