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ON ANALYTIC SOLUTIONS OF LINEAR HOMOLOGICAL EQUATION

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Abstract. We study solutions to the difference equation $f(x+\omega)-f(x)=g(x)$, where $x\in\mathbb{R}$ is a variable, $\omega\in(0,1)$ is a given number, and g is a given analytic 1–periodic function. It is well–known that in the case when the number ω is Diophantine (sufficiently poorly approximated by rational numbers), one can construct a 1–periodic solution analytic in a neighbourhood of the real axis. If ω is rational, then in the general case there are no periodic solutions. We have constructed a quasiperiodic analytic solution, which in the case of Diophantine ω decomposes into the sum of the known 1–periodic solution and an analytic ω–periodic function. One can say that in the case of numbers ω that are well–approximated by rational numbers, the Fourier series of each of these two terms diverge, but their sum remains analytic.

Keywords: homological equation, quasiperiodic solution.

Mathematics Subject Classification: 39A06, 39A24, 39A45

1. Introduction

Let $\omega\in(0,1)$ be a given positive number, and let ϕ be a given 1–periodic function of a real or complex variable,

$$\phi(x+1)=\phi(x).$$

The equation

$$\lambda(x+\omega)-\lambda(x)=\phi(x),\quad x\in\mathbb{R}, \tag{1.1}$$

is often called the *linear homological equation*. Such equations are encountered in a wide variety of problems of mathematical physics. They also arise in spectral theory in the study of one–dimensional quasiperiodic Schrödinger operators by means of the ideas of KAM theory (Kolmogorov — Arnold — Moser), see, e.g., [4], in studying difference equations on the real axis with periodic coefficients, see, e.g., [1], and in studying nontrivial model difference quasiperiodic Schrödinger operators on the integer lattice, see, e.g., [3, Sec. 10.3]. Usually 1–periodic solutions of the homological equation are required. They exist only for frequencies ω that are sufficiently poorly approximated by rational numbers. Although it is not difficult to construct such solutions as Fourier series, see Section 2, the substantive study of their subtle properties is of interest, see, e.g., [7]. Our interest is connected with the study of properties of (generalized) eigenfunctions for the Sarnak model, see, e.g., [5], [6]. It turned out to be important for us to construct analytic solutions of the homological equation for all irrational ω, including those well–approximated by rational numbers.

In this short note we construct a particular solution to (1.1) by means of a contour integral, see Section 3. We note that the general solution differs from it only by an ω–periodic term. Then we discuss in more detail the case when ϕ is an analytic function, see Section 4, and

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obtain for the constructed solution a representation as the limit of a sum of suitably chosen partial sums of two generally speaking divergent Fourier series, one of which is 1–periodic and the second is ω–periodic, see Section 4.3. In the cases when the homological equation possesses 1–periodic solution, the solution, which we construct, decomposes into the sum of a 1–periodic solution and an ω–periodic function. In the general case, one can say that the divergences of the two Fourier series (the 1–periodic and the ω–periodic ones) compensate each other. As far as we know, these results are new. In conclusion, in Section 5, by means of the integral representation we obtain the known solutions of the homological equation for Diophantine and rational ω.

In what follows the letter C denotes various positive constants independent of ω, on other parameters, or on integer indices.

2. Standard solutions to homological equation

There are two cases, in which solutions to the homological equation are well–known. We describe their construction by following [1]. We expand the function ϕ into the Fourier series,

$$\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n e^{2\pi i n x}.$$

2.1. The case of Diophantine numbers ω. Let ω be a Diophantine number, i.e., there exist numbers $c > 0$ and $N \in \mathbb{N}$ such that

$$\min_{p\in\mathbb{Z}} \left|\omega - \frac{p}{q}\right| \geqslant \frac{c}{q^{N+1}} \quad \forall q \in \mathbb{N}. \tag{2.1}$$

Then for a sufficiently smooth function ϕ the solution λ can be constructed by the formula

$$\lambda = \frac{\phi_0}{\omega} x + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\phi_n}{e^{2\pi i n\omega} - 1} e^{2\pi i n x}. \tag{2.2}$$

Estimate (2.1) guarantees the convergence of the series in (2.2) if $\sum\limits_{n\in\mathbb{N}} n^N |\phi_n| < \infty$.

2.2. The case of rational numbers ω. Let ω be a rational number, $\omega = \dfrac{p}{q}$, where p and q are coprime natural numbers. Now, if the Fourier series of the function ϕ converges absolutely, λ can be constructed by the formula

$$\lambda = \frac{x}{\omega} \sum_{n\in\mathbb{Z}} \phi_{nq} e^{2\pi i q n x} + \sum_{n\in\mathbb{Z}\setminus(q\mathbb{Z})} \frac{\phi_n}{e^{2\pi i n w} - 1} e^{2\pi i n x}. \tag{2.3}$$

Obviously, the series in this formula converge at the same rate as the Fourier series for ϕ.

2.3. Remark on general solution to homological equation. It is not difficult to see that two functions are simultaneously solutions of (1.1) if and only if they differ by an ω–periodic term. Therefore, both in the case of Diophantine ω and in the case of rational ω the general solution is equal to the sum of the particular solution constructed above and an ω–periodic function.

3. General construction of solutions

Here we propose a general construction of solutions by means of a contour integral.

Let a function $k \to \alpha(k)$ of the complex variable k be analytic in the strip $-K < \operatorname{Im} k < K$ for some $K > 0$, and

$$\left| \frac{\phi_n}{\alpha(n)} \right| \leqslant \frac{C}{1+|n|}, \quad n \in \mathbb{Z}. \tag{3.1}$$

We let

$$\Phi_\alpha(k) = \sum_{n \in \mathbb{Z}} \frac{\sin(\pi(k-n))}{\pi(k-n)} \frac{\phi_n}{\alpha(n)} \tag{3.2}$$

assuming that $\dfrac{\sin(\pi(k-n))}{\pi(k-n)} = 1$ for $k = n$. The next statement holds.

Lemma 3.1. 1. *$\Phi_\alpha(k)$ is an entire function;*

2.

$$\Phi_\alpha(n) = \frac{\phi_n}{\alpha(n)}, \quad \forall n \in \mathbb{Z}; \tag{3.3}$$

3.

$$|\Phi_\alpha(k)| \leqslant C\, e^{\pi |\operatorname{Im} k|}\, \frac{\ln\left(|\operatorname{Re} k| + 1\right) + 1}{|\operatorname{Re} k| + 1}. \tag{3.4}$$

Proof. The second property is obvious. To verify the remaining ones, we estimate the terms in (3.2). Let $|\operatorname{Im} k| < K$, $n_0 \in \mathbb{Z}$ and $|\operatorname{Re} k - n_0| < \dfrac{3}{4}$. It is obvious that

$$\left| \frac{\sin(\pi(k-n_0))}{\pi(k-n_0)} \right| \leqslant C\, e^{\pi |\operatorname{Im} k|}. \tag{3.5}$$

Let $n \neq n_0$. Since

$$|\operatorname{Re} k - n| = |n - n_0| \left| 1 - \frac{\operatorname{Re} k - n_0}{n - n_0} \right| \geqslant \frac{1}{4} |n - n_0|,$$

we have

$$\left| \frac{\sin(\pi(k-n))}{\pi(k-n)} \right| \leqslant C \frac{e^{\pi |\operatorname{Im} k|}}{\sqrt{(\operatorname{Im} k)^2 + (\operatorname{Re} k - n)^2}} \leqslant C \frac{e^{\pi |\operatorname{Im} k|}}{|n - n_0|}. \tag{3.6}$$

Estimates (3.5) and (3.6) imply the uniform convergence of the series in (3.2) on compact sets and hence the function Φ_α is entire.

Now we are going to estimate the function Φ_α. If $n_0 = 0$, then it follows from (3.6) and (3.1) that

$$\Phi_\alpha(k) = \frac{\phi_0}{\alpha(0)} \frac{\sin \pi k}{\pi k} + O\left(e^{\pi |\operatorname{Im} k|} \sum_{n \neq 0} \frac{1}{|n|^2} \right),$$

and this implies (3.4).

Let us consider the case $n_0 \neq 0$. B (3.5) we obtain the estimate for the term with $n = n_0$ in (3.2):

$$\left| \frac{\sin(\pi(k-n_0))}{\pi(k-n_0)} \frac{\phi_{n_0}}{\alpha(n_0)} \right| \leqslant C \frac{e^{\pi |\operatorname{Im} k|}}{n_0}.$$

In view of (3.6) and (3.1), to estimate the sum of the terms with $n \neq n_0$, it sufficient to prove that

$$\sum_{n \neq n_0} \frac{1}{|n - n_0|} \frac{1}{1 + |n|} \leqslant \frac{C}{n_0} (\ln n_0 + 1). \tag{3.7}$$

This estimate implies (3.4). Let us prove (3.7) under the condition $n_0 > 0$. The case $n_0 < 0$ reduces to the considered one by the change of the summation index n to $-n$. For $n_0 > 0$ we obtain

$$\sum_{n=-\infty}^{-2} \frac{1}{|n-n_0|}\frac{1}{1+|n|} < \sum_{n=2}^{\infty} \frac{1}{|n+n_0|}\frac{1}{|n|} \leqslant C \int_1^{\infty} \frac{dn}{(n_0+n)n} = \frac{C}{n_0} \int_{\frac{1}{n_0}}^{\infty} \frac{d\tau}{(\tau+1)\tau}$$

$$\leqslant \frac{C}{n_0} \left(\int_{\frac{1}{n_0}}^{2} \frac{d\tau}{\tau} + \int_2^{\infty} \frac{d\tau}{\tau^2} \right) \leqslant \frac{C}{n_0} (\ln n_0 + 1);$$

$$\sum_{n=n_0+1}^{\infty} \frac{1}{|n-n_0|}\frac{1}{1+|n|} \leqslant \frac{C}{n_0} + C \int_{n_0+1}^{\infty} \frac{dn}{(n-n_0)n} = \frac{C}{n_0} \left(1 + \int_{1+\frac{1}{n_0}}^{\infty} \frac{d\tau}{(\tau-1)\tau} \right)$$

$$\leqslant \frac{C}{n_0} \left(1 + \int_{1+\frac{1}{n_0}}^{2} \frac{d\tau}{\tau-1} + \int_2^{\infty} \frac{d\tau}{\tau^2} \right) \leqslant \frac{C}{n_0} (\ln n_0 + 1);$$

$$\sum_{n=-1}^{n_0-1} \frac{1}{|n-n_0|}\frac{1}{1+|n|} < \frac{2}{n_0} + \sum_{n=1}^{n_0-1} \frac{1}{|n-n_0|\,n}$$

$$\leqslant \frac{C}{n_0} \left(1 + \sum_{n=1}^{\left[\frac{n_0}{2}\right]} \frac{1}{n} + \sum_{n=\left[\frac{n_0}{2}\right]+1}^{n_0-1} \frac{1}{n_0-n} \right) \leqslant \frac{C}{n_0} (\ln n_0 + 1).$$

This completes the proof. □

Let $\varepsilon > 0$. Suppose that in the strip $-K < \operatorname{Im} k < K$

$$|\alpha(k)| \leqslant \frac{C}{1+|k|^{\varepsilon}}. \tag{3.8}$$

Remark 3.1. *The conditions on α implicitly impose restrictions on ϕ. Obviously, one can choose α satisfying conditions (3.1) and (3.8) if*

$$|\phi_n| \leqslant \frac{C}{1+|n|^{1+\varepsilon}}.$$

We also note that condition (3.1) can be noticeably weakened.

Let $0 < K_1 < K$, and let $\Gamma \subset \mathbb{C}$ be a contour consisting of the lines $\operatorname{Im} k = \pm K_1$. The line $\operatorname{Im} k = -K_1$ is oriented from $-\infty$ to ∞, and the line $\operatorname{Im} k = K_1$ is oriented from ∞ to $-\infty$. For $x \in \mathbb{R}$ we let

$$\lambda(x) = \frac{1}{2i} \int_{\Gamma} \alpha(k)\, \Phi_{\alpha}(k)\, e^{2\pi i k x} \frac{\cot \pi k}{e^{2\pi i k \omega} - 1}\, dk. \tag{3.9}$$

In view of (3.4) and (3.8), the integral in (3.9) converges uniformly in x on each finite interval of the real axis, and hence defines a continuous function $x \in \mathbb{R}$. The following theorem holds.

Theorem 3.1. *The function $x \mapsto \lambda(x)$ is a solution to the homological equation.*

Proof. It is obvious that

$$\lambda(x+\omega)-\lambda(x)=\frac{1}{2i}\int\limits_{\Gamma}\alpha(k)\,\Phi_\alpha(k)\,e^{2\pi ikx}\cot\pi k\,dk. \tag{3.10}$$

We recall that by to Lemma 3.1 the function Φ_α is analytic in k. Applying the residue theorem to (3.10), we obtain

$$\lambda(x+\omega)-\lambda(x)=\sum_{n=-N}^{N}\alpha(n)\,\Phi_\alpha(n)\,e^{2\pi inx}+\frac{1}{2i}(I_++I_-),$$

where $I_\pm$ are the integrals of the same integrand as in the integral in (3.10), over the contours $\Gamma_\pm$ corresponding to Figure 1. We note that $\Gamma_\pm$ depends on N and it intersects the real axis at the points $\pm\left(N+\frac{1}{2}\right)$, respectively. It follows from estimates (3.4) and (3.8) that the integrals $I_\pm$ tend to zero as $N\to\infty$. Taking into consideration (3.3), we see that λ solves homological equation (1.1).

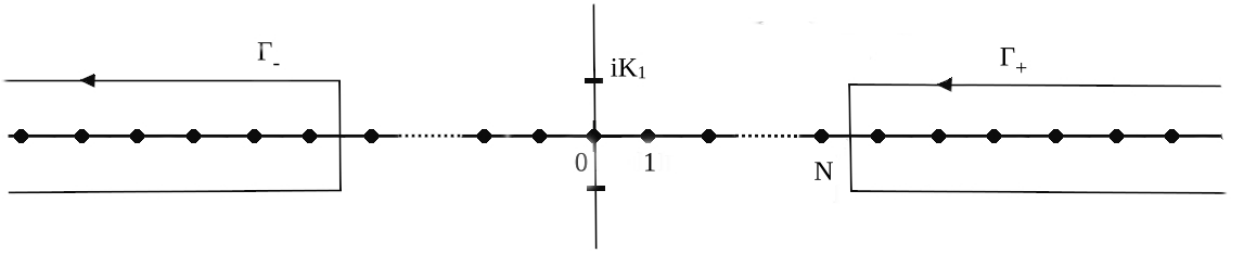


FIGURE 1.

□

Remark 3.2. *The difference of two arbitrary solutions of* (1.1)*, and hence also of solutions corresponding to two different functions* α*, is an* ω*–periodic function.*

4. Analytic case

In the case of an analytic right hand side, by the proven theorem one can construct an analytic solution to the homological equation.

4.1. Integral representation.

The next statement holds.

Theorem 4.1. *Let* $d>0$*,* $0<\tilde{d}<d$*, and let* ϕ *be a* 1*–periodic function analytic in the strip*

$$-d<\operatorname{Im}x<d.$$

Equation (1.1) *has a solution* $\lambda(x)$ *analytic in* x *in the strip* $-\tilde{d}<\operatorname{Im}x<\tilde{d}$ *and infinitely differentiable in* $\omega>0$*.*

Proof. We choose d' and d'' such that

$$0<\tilde{d}<d''<d'<d.$$

Since $d'<d$, the Fourier coefficients of ϕ satisfy the estimate

$$|\phi_n|\leqslant C(d')\,e^{-2\pi d'|n|}.$$

Let

$$\alpha(k) = \frac{1}{\cosh(2\pi d'' k)}, \qquad 0 < K_1 < \frac{1}{4d''}.$$

It is obvious that α is analytic in the strip $|\operatorname{Im} k| < K$ with $K = \dfrac{1}{4d''}$. Thanks to the inequalities $0 < d'' < d'$, the function α satisfies estimates (3.1) and (3.8). Hence, one can construct a solution to the homological equation by formula (3.9). Since $\tilde{d} < d''$, this formula implies that λ is analytic in the strip $-\tilde{d} < \operatorname{Im} x < \tilde{d}$. □

4.2. Convergents. For the further analysis we shall need elementary facts from the theory of Diophantine approximations. Proofs of the facts presented below can be found in [2].

Let $\omega \in (0,1) \setminus \mathbb{Q}$. Then ω decomposes into an infinite continued fraction

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}} \tag{4.1}$$

with natural elements $a_1, a_2, \dots$. The finite continued fraction

$$\cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_L}}}$$

is called a convergent. We represent it as $\dfrac{p_L}{q_L}$, where p_L and q_L are coprime natural numbers. It is known that by choosing the elements $a_1, a_2, \dots, a_l, \dots$ successively sufficiently large, one can make the denominators $q_1, q_2, \dots, q_l, \dots$ respectively arbitrarily large, and that [2, Ch. 1, Sec. 3, Thm. 9]

$$|\omega q_l - p_l| \leqslant \frac{1}{q_{l+1}}, \quad l \in \mathbb{N}. \tag{4.2}$$

4.3. Representation by regularized Fourier series. Let us describe the solution constructed in the proof of Theorem 4.1 as the limit of a subsequence of regularized partial sums of the Fourier series of ϕ. The following statement holds.

Theorem 4.2. *Let $\omega \notin \mathbb{Q}$, then*

$$\lambda(x) = \frac{\phi_0 x}{\omega} + \lambda_0 + \lim_{l\to\infty} \left(\sum_{0<|n|\leqslant q_l} \frac{\phi_n e^{2\pi i n x}}{e^{2\pi i n\omega} - 1} + \frac{1}{2i\omega} \sum_{0<|m|\leqslant p_l} \alpha\left(\frac{m}{\omega}\right) \Phi_\alpha\left(\frac{m}{\omega}\right) \cot\frac{\pi m}{\omega}\, e^{\frac{2\pi i m x}{\omega}} \right), \tag{4.3}$$

where p_l and q_l are the numerators and denominators of the convergents of the number ω, and

$$\lambda_0 = \frac{(\alpha\Phi_\alpha)'|_{k=0}}{2\pi i\omega} - \frac{\phi_0}{2}$$

is a constant.

Note that the first term in the parentheses in (4.3) is a partial sum of Fourier series (2.2) of the standard solution, see Section 2.1, and the second is a segment of the Fourier series of an ω–periodic function. Recall that series (2.2) converges if ω is a Diophantine number. But then it follows from Theorem 4.2 that the limit

$$S_\omega = \frac{1}{2i\omega} \lim_{l\to\infty} \sum_{0<|m|<p_l} \alpha\left(\frac{m}{\omega}\right) \Phi_\alpha\left(\frac{m}{\omega}\right) \cot\frac{\pi m}{\omega}\, e^{\frac{2\pi i m x}{\omega}}, \tag{4.4}$$

exists and is an ω–periodic function. Therefore the solution λ decomposes into the sum of the standard solution and the ω–periodic function $\lambda_0 + S_\omega$.

If the Fourier series for the standard solution diverges, then limit (4.4) does not exist either, but one can say that in (4.3) the divergent sequences of partial sums mutually compensate their divergence.

Remark 4.1. *Fourier series* (2.2) *may diverge. Let, for example,* $\phi_n = e^{-2\pi d|n|}$ *for* $n \in \mathbb{Z}$. *The corresponding function* ϕ *is analytic in the strip* $|\operatorname{Im} x| < d$. *In view of* (4.2) *the* q_l*-th term of series* (2.2) *admits the estimate*

$$\left| \frac{\phi_{q_l} e^{2\pi i q_l x}}{e^{2\pi i q_l \omega} - 1} \right| \geqslant C q_{l+1} e^{-2\pi d|q_l|},$$

but, by a suitable choice of the sequence $\{a_l\}$ *of elements of the continued fraction* (4.1), *one can achieve the chain of inequalities* $q_{l+1} e^{-2\pi d|q_l|} \geqslant 1$, $l \in \mathbb{N}$.

Proof. The proof is carried out by means of the residue theorem. The contour of integration Γ in formula (3.9) is deformed to the union of the contours $\Gamma_\pm$ corresponding to Fig. 2.

Let us describe them in detail. The horizontal parts of the contours $\Gamma_\pm$ are located at the same distance from the real axis in the plane of the variable k as in the proof of Theorem 3.1. Let us describe the vertical parts of $\Gamma_\pm$. The poles of the integrand in (3.9) are located at the integer points and at the points of the set $\dfrac{1}{\omega} \cdot \mathbb{Z}$. In Fig. 2 the points of the first set are depicted by larger circles, and the points of the second by smaller circles, respectively. Since $\omega \notin \mathbb{Q}$, these two sets of poles have only one common point, the zero. In view of (4.2) the distance between the points q_l and $\dfrac{p_l}{\omega}$ does not exceed $\dfrac{1}{\omega q_{l+1}}$, and hence tends to zero as $l \to \infty$; we recall that $0 < \omega < 1$. We consider l sufficiently large and choose $\Gamma_\pm$ so that they intersect the real axis at the points $\pm\left(q_l + \dfrac{1}{2}\right)$. Then the distance from the vertical parts of the contours $\Gamma_\pm$ to the nearest poles turns out to be equal to $\dfrac{1}{2} + O\left(\dfrac{1}{q_l}\right)$.

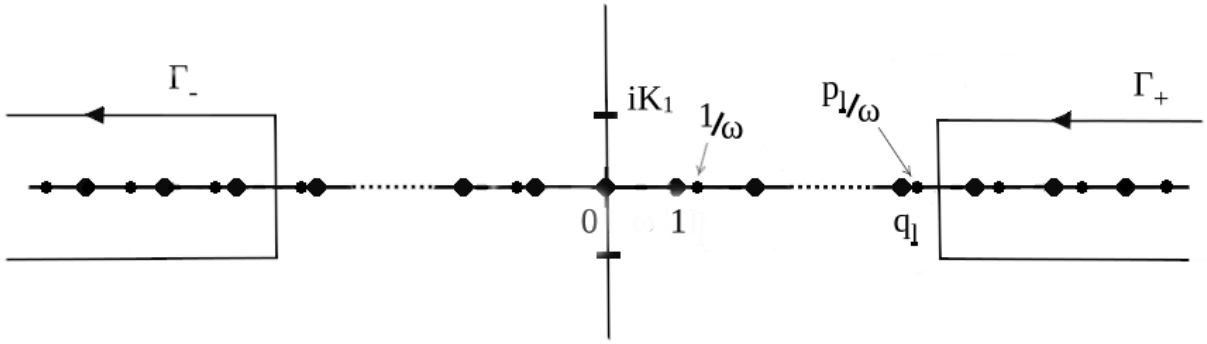


FIGURE 2.

We denote by $f(x,k)\,dk$ the integrand in (3.9). By estimates (3.4) and (3.8), with the described choice of the contours $\Gamma_\pm$ the integrals over $\Gamma_\pm$ of $f(x,k)$ tend to zero as $l \to \infty$, and hence, by the residue theorem,

$$\lambda(x) = 2\pi \lim_{l\to\infty} \left(\operatorname*{Res}_{k=0} f(x,k) + \sum_{n\in\mathbb{Z},\ 0<|n|\leqslant q_l} \operatorname*{Res}_{k=n} f(x,k) + \sum_{m\in\mathbb{Z},\ 0<|m|\leqslant p_l} \operatorname*{Res}_{k=\frac{m}{\omega}} f(x,k) \right). \tag{4.5}$$

The calculation of the residues of f are standard, we note only that in the calculations formula (3.3) is used, and that while calculating the first term in the parentheses in (4.5), a linear

in x term arises, while the calculations of the first and second sums of residues in the parentheses in (4.5) give the first and second sums in the parentheses in (4.3), respectively. The proof is complete. □

5. Construction of standard solutions by means of integral representation

In conclusion we are going to obtain formulas (2.2) and (2.3) from integral representation (3.9). In what follows we assume that ϕ is analytic in the strip $|\operatorname{Im} x| < d$, and $0 < d'' < d' < d$.

5.1. Diophantine ω. Let ω be a Diophantine number, that is, condition (2.1) is satisfied. To construct the solution λ one can choose

$$\alpha(k) = \frac{1}{\cosh(2\pi k d'')} \frac{e^{2\pi i k\omega} - 1}{k}, \qquad 0 < K_1 < \frac{1}{4d''}. \tag{5.1}$$

It follows from estimate (2.1) that α satisfies condition (3.1):

$$\left|\frac{\phi_l}{\alpha(l)}\right| \leqslant C\, \frac{|l|\, e^{-2\pi(d'-d'')\,|l|}}{\inf_{j\in\mathbb{Z}} \big||l|\,\omega - j\big|} \leqslant C\,|l|^{N+1}\, e^{-2\pi(d'-d'')\,|l|}.$$

Under such a choice of α

$$\lambda(x) = \frac{1}{2i} \int\limits_{\Gamma} \frac{\Phi_\alpha(k)}{\cosh(2\pi k d'')}\, e^{2\pi i k x}\, \frac{\cot \pi k}{k}\, dk,$$

and the calculations of the integral by residues gives

$$\lambda = \phi_0 \frac{x}{\omega} + c + \sum_{l\in\mathbb{Z}\setminus\{0\}} \frac{\phi_l e^{2\pi i l x}}{e^{2\pi i l\omega} - 1},$$

where c is a constant. Omitting the standard calculations, we note only that to find $\Phi_\alpha(l)$ one should use formulas (3.3) and (5.1).

5.2. Rational ω. Let ω be a rational number, $\omega = \dfrac{p}{q}$, p and q coprime natural numbers. One can construct λ by choosing

$$\alpha(k) = \frac{1}{\cosh(\pi k d'')} \frac{\sin(k\pi\omega)}{\sin\frac{k\pi}{q}}, \qquad 0 < K_1 < \frac{1}{4d''}.$$

Under such a choice of α, the calculation of the integral by residues gives

$$\lambda = c + \frac{x}{\omega} \sum_{l\in\mathbb{Z}} \phi_{lq}\, e^{2\pi i l q x} + \sum_{l\in\mathbb{Z}\setminus(q\mathbb{Z})} \frac{\phi_l}{e^{2\pi i l\omega} - 1}\, e^{2\pi i l x},$$

where c is a constant. We omit further details.

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