ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 4 (2026). P. 1-27.

doi:10.13108/2026-18-4-1

# METHOD OF ESSENTIAL FUNCTIONS AND ITS APPLICATION IN THEORY OF WIENER — HOPF FACTORIZATION

**V.M. ADUKOV**

**Abstract.** By analogy with the previously introduced notions of indices and essential polynomials of a finite sequence of matrices, we introduce indices and essential functions for an infinite sequence of matrix Fourier coefficients of an invertible matrix function, which belongs to the Wiener matrix algebra. Based on these notions, we propose a new proof of the existence of the Wiener — Hopf factorization of matrix functions by using only the methods of functional analysis. This method is applied to finding an explicit solution for the problem on the canonical factorization of a Laurent matrix polynomial of a special form. We obtain an effective sufficient criterion for the existence of a canonical factorization of strictly nonsingular second order matrix functions.

**Keywords:** Wiener — Hopf factorization of matrix functions, Toeplitz operators, factorization of matrix polynomials

**Mathematics Subject Classification:** 47A68, 47B35, 30E25

## 1. Introduction

The Riemann boundary value problem for a vector is one of the most demanded problems in the complex analysis. It has numerous applications in the topology (classification of holomorphic vector bundles on the Riemann sphere), the operator theory (singular integral operators with Cauchy kernel, Toeplitz operators, Wiener — Hopf operators), in the analytic theory of differential equations (the Riemann — Hilbert problem). The Wiener — Hopf technique, based on the Riemann boundary value problem, is an indispensable working tool in various problems of mathematical physics and mechanics, namely, in soliton theory [10], in diffraction theory [16], [17], in elasticity theory [11]. A survey of the current state–of–art of the theory and experience of applying the Wiener–Hopf technique can be found in [19].

The solving of vector Riemann boundary value problem (homogeneous or inhomogeneous) reduces to constructing a canonical matrix of the problem, which is equivalent to the following problem on factorization of matrix functions.

Let $\Gamma$ be a simple piecewise–smooth closed contour in the complex plane $\mathbb{C}$, which encloses a domain $D_+$. We can suppose that $0 \in D_+$. The complement of $D_+ \cup \Gamma$ in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is denoted by $D_-$. Let $A(t)$ be a continuous and invertible matrix function of order $p$ on $\Gamma$.

V.M. Adukov, Method of essential functions and its application in theory of Wiener — Hopf factorization.

© Adukov V.M. 2026.

The work is supported by the Russian Science Foundation, project no. 25-21-20134, https://rscf.ru/project/25-21-20134/.

*Submitted May 4, 2025.*

The *left Wiener — Hopf factorization* of $A(t)$ with respect to the contour $\Gamma$ is the following representation for $A(t)$ :

$$A(t) = A_+^l(t)D^l(t)A_-^l(t), \quad t \in \Gamma. \tag{1.1}$$

Here $A_\pm^l(t)$ are continuous and invertible matrix functions on $\Gamma$, which admit analytic continuation into the domain $D_\pm$, and moreover these continuations are invertible there; $D^l(t) = \operatorname{diag}[t^{\lambda_1}, \dots, t^{\lambda_p}]$. The integers $\lambda_1$, ..., $\lambda_p$ are called the *left partial indices* of $A(t)$. They are uniquely determined by $A(t)$. We can assume that the partial indices are taken in the decreasing order $\lambda_1 \geqslant \dots \geqslant \lambda_p$. It is clear that $\lambda_1 + \dots + \lambda_p = \varkappa$, where $\varkappa = \operatorname{ind}\det A(t)$ is the index of the determinant of $A(t)$.

The *right Wiener — Hopf factorization* of $A(t)$ is defined in the same way

$$A(t) = A_-^r(t)D^r(t)A_+^r(t). \tag{1.2}$$

Here $D^r(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$, $\rho_1 \leqslant \dots \leqslant \rho_p$, are the *right partial indices* of $A(t)$. They are also uniquely determined by $A(t)$ and they satisfy the identity $\rho_1 + \dots + \rho_p = \varkappa$. The factorization problem can also be considered on unbounded contours $\Gamma$, in particular, on the real line.

In what follows we shall consider only the right Wiener — Hopf factorization. This is related with the fact that the main method of our investigation is the theory of Toeplitz operators. Results on the left factorization can be obtained by resorting to transposition or inversion of the matrix function $A(t)$.

Not each continuous matrix function $A(t)$ admits a right (left) factorization with factors $A_\pm^r(t)$ ($A_\pm^l(t)$) continuous on the contour. It is known that the factorization problem has a solution for each invertible matrix function from the algebra $H_\alpha^{p\times p}(\Gamma)$, $0 < \alpha < 1$, of Hölder matrix functions, and from the matrix Wiener algebra $W^{p\times p}$ on the unit circle or on the real line. Moreover, the factorization factors $A_\pm^{l,r}(t)$ also belong to the corresponding algebra.

There are many works, which established the possibility of factorization in these algebras; we mention the main ones[2], [5]–[7], [9].

All of them are based on similar ideas and their essence is existence theorems. Nowadays, in contrast to the scalar functions, there are no explicit formulas for the factors $A_\pm^{l,r}(t)$ and partial indices in the general case. And this is the main problem in factorization theory. The second problem is that the factorization problem for matrix functions is unstable in the general case. Moreover, the factorization factors $A_\pm^{l,r}(t)$ are not determined uniquely. Therefore, a normalization of factorizations is required, which in the general case is still an open problem. All this complicates significantly the development of approximate factorization methods.

In this work we propose a new way of proving the factorizability of matrix functions from the Wiener algebra on the unit circle. The proposed method from complex analysis explicitly uses only the maximum modulus principle. One can say that factorizability is proved only by means of functional analysis. Another feature of our approach is that it is an infinite–dimensional analogue of the method of indices and essential polynomials of a finite sequence of matrices [1], [13], which was specially developed for the explicit solution of the factorization problem for matrix polynomials, analytic and meromorphic matrix functions.

Let $a(t)$ be an invertible matrix function from the matrix Wiener algebra $W^{p\times p}$ on the unit circle and $\{a_n\}_{n=-\infty}^{\infty}$, $a_n \in \mathbb{C}^{p\times p}$, be the sequence of its matrix Fourier coefficients. In what follows, when speaking of an infinite sequence of matrices, we shall have in mind precisely this sequence. It turned out that the proposed method of indices and essential functions for an infinite sequence is completely equivalent to the method of Wiener — Hopf factorization of matrix functions. This means that, knowing the indices and essential functions of the sequence $\{a_n\}_{n=-\infty}^{\infty}$, one can construct the right Wiener — Hopf factorization of $a(t)$ and, conversely, the right factorization of the matrix function allows one to determine the indices and essential functions of the corresponding sequence. In particular, this equivalence allows one to study the structure of kernel of the Toeplitz operator with the matrix symbol $a(t)$ and to obtain

inversion formulas or generalized inversion of Toeplitz operators in terms of indices and essential functions in the same way as it was done in the finite–dimensional case [13]. This indicates that there exists a unified approach to the study of Toeplitz operators in the finite–dimensional and infinite–dimensional cases.

We note that the previously known methods for proving factorizability were essentially pure existence theorems.

The proposed method allows one to obtain an explicit solution of the factorization problem for one class of Laurent matrix polynomials arising in the theory of factorization of strictly nonsingular second order matrix functions [12]. In its turn, this explicit solution is used to obtain an effective condition for the existence of a canonical factorization for strictly nonsingular matrix functions of a special form.

The work is organized as follows. In Section 2 we provide the necessary information about Toeplitz operators $T_a$ and the associated operators $U_a$. In the next section, by analogy with the finite–dimensional case, we include these operators into the families of operators $\{T_k\}_{k=-\infty}^{\infty}$, $\{U_k\}_{k=-\infty}^{\infty}$ and prove the Noetherian property of the operators belonging to these families. In the finite–dimensional case this holds immediately. In Section 4 we introduce the notion of indices and essential functions for the infinite sequence $\{a_n\}_{n=-\infty}^{\infty}$. In Section 5 the criterion of essentiality is proved. As in the finite–dimensional case, this is the cornerstone of the theory. Then in Section 6 we present a new proof of the factorizability of matrix functions by the method of indices and essential functions and prove the equivalence of this method with the method of Wiener — Hopf factorization. In Section 7 this method is used to construct explicitly the canonical factorization of some Laurent matrix polynomials. We also provide the applications of obtained factorization.

## 2. Preliminary information on discrete Wiener — Hopf operators

In foundational work [6], the system of discrete Wiener — Hopf equations was considered in the entire family $E^{p\times 1}$ of Banach spaces of vector sequences: $l_s^{p\times 1}$ $(1 \leqslant s \leqslant \infty)$, $c^{p\times 1}$, $c_0^{p\times 1}$.

Since our goal is to propose a new method for proving the possibility of factorization of matrix functions, we shall restrict ourselves to the minimal collection of spaces necessary for this. We shall use the following notation: $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_{\pm}$ is the set of nonnegative/nonpositive integers, and $\mathbb{Z}_{\pm}^0$ is the set of positive/negative integers.

We introduce the spaces of one–sided vector (column) sequences

$$l_1^{p\times 1}(\mathbb{Z}_+) = \Big\{x_+ = (x_0, x_1, x_2, \ldots) : x_k \in \mathbb{C}^{p\times 1}, \sum_{k=0}^{\infty} \|x_k\|_1 < \infty\Big\},$$

equipped with the norm $\|x_+\|_1 := \sum\limits_{k=0}^{\infty} \|x_k\|_1$, and

$$l_\infty^{p\times 1}(\mathbb{Z}_+) = \Big\{x_+ = (x_0, x_1, x_2, \ldots) : x_k \in \mathbb{C}^{p\times 1}, \sup_{k\geqslant 0} \|x_k\|_\infty < \infty\Big\},$$

with the norm $\|x_+\|_\infty := \sup\limits_{k\geqslant 0} \|x_k\|_\infty$. Here $\|\cdot\|_1$, $\|\cdot\|_\infty$ are the Hölder norms on $\mathbb{C}^{p\times 1}$. Any of the spaces $l_1^{p\times 1}(\mathbb{Z}_+)$, $l_\infty^{p\times 1}(\mathbb{Z}_+)$ is denoted by $E^{p\times 1}(\mathbb{Z}_+)$.

We shall also need similar spaces $E^{1\times p}(\mathbb{Z}_-)$ of one–sided row sequences

$$l_1^{1\times p}(\mathbb{Z}_-) = \Big\{f_- = (\ldots, f_{-2}, f_{-1}, f_0) : f_k \in \mathbb{C}^{1\times p}, \sum_{k=-\infty}^{0} \|f_k\|_1 < \infty\Big\},$$

with the norm $\|f_-\|_1 := \sum\limits_{k=-\infty}^{0} \|f_k\|_1$, and

$$l_\infty^{1\times p}(\mathbb{Z}_-) = \Big\{ f_- = (\dots, f_{-2}, f_{-1}, f_0) : f_k \in \mathbb{C}^{1\times p}, \sup_{k\leqslant 0} \|f_k\|_\infty < \infty \Big\},$$

$\|f_-\|_\infty := \sup\limits_{k\leqslant 0} \|f_k\|_\infty$.

The spaces $E^{p\times 1}(\mathbb{Z}_-^0)$, $E^{1\times p}(\mathbb{Z}_+^0)$ and $E^{p\times 1}(\mathbb{Z})$, $E^{1\times p}(\mathbb{Z})$ are defined in the same way. In the last two cases we are dealing with spaces of two–sided sequences. These two spaces admit the decompositions

$$E^{p\times 1}(\mathbb{Z}) = E^{p\times 1}(\mathbb{Z}_+) \oplus E^{p\times 1}(\mathbb{Z}_-^0), \qquad E^{1\times p}(\mathbb{Z}) = E^{1\times p}(\mathbb{Z}_-) \oplus E^{1\times p}(\mathbb{Z}_+^0).$$

The projection from $E^{p\times 1}(\mathbb{Z})$ ($E^{1\times p}(\mathbb{Z})$) into $E^{p\times 1}(\mathbb{Z}_+)$ ($E^{1\times p}(\mathbb{Z}_-)$) parallel to $E^{p\times 1}(\mathbb{Z}_-^0)$ ($E^{1\times p}(\mathbb{Z}_+^0)$) is denoted by $\mathcal{P}_+$ ($\Pi_-$).

We consider an infinite two–sided sequence of matrices

$$a = \{a_j\}_{j=-\infty}^{\infty}, \quad a_j \in \mathbb{C}^{p\times p},$$

belonging to the Banach space $l_1^{p\times p}(\mathbb{Z})$. The norm in this space is defined as $\|a\| = \sum\limits_{j=-\infty}^{\infty} \|a_j\|_1$, where $\|\cdot\|_1$ is the maximum column norm on $\mathbb{C}^{p\times p}$. In some cases we shall use the maximum row norm $\|\cdot\|_\infty$.

For $a = \{a_j\}_{j=-\infty}^{\infty} \in l_1^{p\times p}(\mathbb{Z})$ and $x = \{x_j\}_{j=-\infty}^{\infty} \in E^{p\times 1}(\mathbb{Z})$ we define the convolution

$$(a * x)_i = \sum_{j=-\infty}^{\infty} a_{i-j} x_j, \quad i \in \mathbb{Z},$$

which is an element of the space $E^{p\times 1}(\mathbb{Z})$. The convolution $x * a$ is defined in the same way, where $x \in E^{1\times p}(\mathbb{Z})$.

The set $l_1^{p\times p}(\mathbb{Z})$ is a Banach algebra, in which the convolution serves as the multiplication. The sets $l_1^{p\times p}(\mathbb{Z}_\pm)$, $l_1^{p\times p}(\mathbb{Z}_-^0)$ are closed subalgebras of $l_1^{p\times p}(\mathbb{Z})$ and

$$l_1^{p\times p}(\mathbb{Z}) = l_1^{p\times p}(\mathbb{Z}_+) \oplus l_1^{p\times p}(\mathbb{Z}_-^0).$$

A linear bounded operator acting in $E^{p\times 1}(\mathbb{Z}_+)$ by the formula

$$T_a x_+ = \mathcal{P}_+(a * x_+),$$

is called the *discrete Wiener — Hopf operator*. Besides the operator $T_a$ we shall also need the operator $U_a$, acting in $E^{1\times p}(\mathbb{Z}_-)$ by the formula

$$U_a f_- = \Pi_-(f_- * a),$$

which we call the operator *associated* with $T_a$. In more detail,

$$(T_a x_+)_i = \sum_{j=0}^{\infty} a_{i-j} x_j, \quad i \geqslant 0, \qquad (U_a f_-)_i = \sum_{j=-\infty}^{0} f_j a_{i-j}, \quad i \leqslant 0.$$

If we introduce the infinite block Toeplitz matrix generated by the sequence $a$, then the action of these operators can be visualized as follows:

$$T_a x_+ = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix},$$

$$U_a f_- = \begin{pmatrix} f_0 & f_{-1} & f_{-2} & \dots \end{pmatrix} \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is not difficult to verify that the dual of $l_1^{p\times 1}(\mathbb{Z}_+)$ $\Big(l_1^{1\times p}(\mathbb{Z}_-)\Big)$ is isometrically isomorphic to $l_\infty^{1\times p}(\mathbb{Z}_-)$ $\Big(l_\infty^{p\times 1}(\mathbb{Z}_+)\Big)$. This isomorphism $\varphi \mapsto f_-$ is given by the formula

$$\varphi(x_+) = \sum_{k=0}^{\infty} f_{-k} x_k.$$

Identifying these spaces, we obtain the following obvious result.

**Proposition 2.1.** 1) *The adjoint operator to $T_a$ in $l_1^{p\times 1}(\mathbb{Z}_+)$ is the operator $U_a$ acting in the space $l_\infty^{1\times p}(\mathbb{Z}_-)$.*

2) *The adjoint operator to $U_a$ in $l_1^{1\times p}(\mathbb{Z}_-)$ is the operator $T_a$ acting in $l_\infty^{p\times 1}(\mathbb{Z}_+)$.*

In the definition of operator $T_a$ we assign the generating matrix function $a(t) = \sum\limits_{j=-\infty}^{\infty} a_j t^j$, $|t| = 1$ with the two–sided matrix sequence $a = \{a_j\}_{j=-\infty}^{\infty} \in l_1^{p\times p}(\mathbb{Z})$, and call it the *matrix symbol* of the operator. The matrix symbol is expanded into a matrix series absolutely convergent in the norm $l_1^{p\times p}(\mathbb{Z})$, which is its matrix Fourier series.

The space of Fourier transforms of elements of $l_1^{p\times p}(\mathbb{Z})$ is denoted by $W^{p\times p}$. This space is equipped with the norm in $l_1^{p\times p}(\mathbb{Z})$. Under the correspondence $a \mapsto a(t)$ the convolution $a * b$ passes into the product of matrix functions $a(t)\cdot b(t)$. Thus, the convolution Banach algebra $l_1^{p\times p}(\mathbb{Z})$ becomes isometrically isomorphic to the Banach algebra $W^{p\times p}$ of matrix functions. This algebra is called the *matrix Wiener algebra*. The norm in $W^{p\times p}$ is denoted by $\|\cdot\|_W$. We shall also use the notation $\|\cdot\|_{W,1}$, $\|\cdot\|_{W,\infty}$ once we need to specify which norm is used for the matrix Fourier coefficients.

Similarly, the Fourier transform of the subalgebras $l_1^{p\times p}(\mathbb{Z}_\pm)$, $l_1^{p\times p}(\mathbb{Z}_-^0)$ generates closed subalgebras $W_\pm^{p\times p}$ and $(W_-^{p\times p})^0$, and moreover $W^{p\times p}$ is a decomposable matrix algebra:

$$W^{p\times p} = W_+^{p\times p} \oplus (W_-^{p\times p})^0.$$

The projection $\mathcal{P}_+$ generates a projection (still denoted by the same symbol) from $W^{p\times p}$ into $W_+^{p\times p}$ parallel to $(W_-^{p\times p})^0$ acting by the formula

$$\mathcal{P}_+\Big(\sum_{j=-\infty}^{\infty} a_j t^j\Big) = \sum_{j=0}^{\infty} a_j t^j.$$

If in the definition of discrete Wiener — Hopf operator in the space $l_1^{p\times 1}(\mathbb{Z}_+)$ we pass from sequences to their generating functions (Fourier transforms), we obtain the definition of the *Toeplitz operator*: $T_a x_+(t) = \mathcal{P}_+\Big(a(t)\cdot x_+(t)\Big)$, acting in the space $W_+^{p\times 1}$. We shall often use the term "Toeplitz operator" as a synonym for the term "discrete Wiener — Hopf operator". Everything said above is extended in the obvious way to the associated operator $U_a$.

The groups of invertible elements of the algebras $W^{p\times p}$, $W_\pm^{p\times p}$ are denoted by $GW^{p\times p}$, $GW_\pm^{p\times p}$, respectively.

A matrix function $a(t) \in W^{p\times p}$ is invertible in this algebra if and only if $\det a(t) \neq 0$ on the unit circle. It is obvious that if $a_+(t) \in W_+^{p\times p}$ ($a_-(t) \in W_-^{p\times p}$), then it admits analytic continuation from the unit circle into the interior (into the exterior) of the unit disk. The condition $a_+(t) \in GW_+^{p\times p}$ ($a_-(t) \in GW_-^{p\times p}$) is equivalent to the inequality $\det a_\pm(z) \neq 0$ for $|z| \leqslant 1$ (for $|z| \geqslant 1$). A short proof of these facts can be obtained by means of the theory of commutative Banach algebras.

The operators $T_a$, $U_a$ possess a property critically important for what follows: the mappings $a \mapsto T_a$ and $a \mapsto U_a$ are *partially multiplicative*:

$$T_{a_-a} = T_{a_-}T_a, \qquad T_{aa_+} = T_aT_{a_+}, \tag{2.1}$$

$$U_{a_-a} = U_aU_{a_-}, \qquad U_{aa_+} = U_{a_+}U_a, \tag{2.2}$$

for all $a(t) \in W^{p\times p}$ and $a_\pm(t) \in W_\pm^{p\times p}$.

This property is well–known in the theory of Toeplitz operators (see, e.g., [18, Formula (2.19)]).

An important consequence of partial multiplicativity is the following result.

**Proposition 2.2.** *If $a_\pm(t) \in W_\pm^{p\times p}$ and are invertible in the algebra $W^{p\times p}$, then the operators $T_{a_\pm}$, $U_{a_\pm}$ are one–sidedly invertible, namely $T_{a_+}$, $U_{a_-}$ are left invertible and $T_{a_-}$, $U_{a_+}$ are right invertible. Moreover, the operators $T_{a_\pm}^{(-1)} := T_{a_\pm^{-1}}$, $U_{a_\pm}^{(-1)} := U_{a_\pm^{-1}}$ are one–sided inverses to $T_{a_\pm}$, $U_{a_\pm}$.*

*If, however, $a_\pm(t) \in GW_\pm^{p\times p}$, then the operators $T_{a_\pm}$, $U_{a_\pm}$ are invertible.*

These statements can be proved by the direct verification.

## 3. Basic properties of operators $T_k$, $U_k$

Our further arguing will in a certain sense be a carbon copy of the corresponding constructions for the case of finite Toeplitz matrices [1], [13]. Those constructions led to the notions of indices and essential polynomials for a finite sequence of complex matrices. This is exactly what we are going to do for an infinite sequence of complex matrices. In this section we shall study those properties of Toeplitz operators and of the operators associated with them that we need, and that hold immediately in the finite–dimensional case.

We need some information about generalized invertible and Noetherian operators (see [4]); let us recall it. Let $B_1$, $B_2$ be Banach spaces. We denote by $\mathcal{L}(B_1, B_2)$ the Banach space of all linear bounded operators acting from $B_1$ to $B_2$. An operator $\mathcal{A} \in \mathcal{L}(B_1, B_2)$ is called *generalized invertible* if there exists an operator $\mathcal{A}^\dagger \in \mathcal{L}(B_2, B_1)$ such that $\mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{A}$. In particular, invertible and one–sidedly invertible operators are generalized invertible.

An operator $\mathcal{A}$ is generalized invertible if and only if the following conditions hold:

1) $\mathcal{A}$ is a normally solvable operator, i.e., the subspace $\operatorname{Im}\mathcal{A}$ is closed;

2) the subspace $\ker\mathcal{A}$ can be complemented in $B_1$;

3) the subspace $\operatorname{Im}\mathcal{A}$ can be complemented in $B_2$.

In particular, the operator $\mathcal{A}$ is invertible on the right (left) if and only if it is normally solvable, $\ker\mathcal{A}$ can be complemented in $B_1$ and $\operatorname{Im}\mathcal{A} = B_2$ ($\ker\mathcal{A} = \{0\}$ and $\operatorname{Im}\mathcal{A}$ can be complemented in $B_2$).

An operator $\mathcal{A} \in \mathcal{L}(B_1, B_2)$ is called *Noetherian* or *Fredholm* if it is normally solvable and its kernel and cokernel are finite–dimensional. The *cokernel* $\operatorname{coker}\mathcal{A}$ of a normally solvable operator $\mathcal{A}$ is the quotient space $B_2/\operatorname{Im}\mathcal{A}$. If the cokernel is finite–dimensional, then $\dim\operatorname{coker}\mathcal{A} =$

$\dim\ker\mathcal{A}^* = \operatorname{codim}\operatorname{Im}\mathcal{A}$. Here $\mathcal{A}^*$ is the adjoint operator and $\operatorname{codim}\operatorname{Im}\mathcal{A}$ is the dimension of the direct complement of $\operatorname{Im}\mathcal{A}$. The number $\operatorname{ind}\mathcal{A} = \dim\ker\mathcal{A} - \dim\operatorname{coker}\mathcal{A}$ is called the *index* of the Noetherian operator $\mathcal{A}$. Obviously, each Noetherian operator is generalized invertible.

We shall also use the well-known result, which is the lemma on factorization of an element of a Banach algebra close to the identity element [5, Lm. 2.2]. Let us state it for the particular case of the Banach algebra $W^{p\times p}$. The identity matrix of order $p$ is denoted by $I_p$.

**Lemma 3.1.** *There exists a value $\varepsilon > 0$ such that each matrix function $\alpha(t)$ from $W^{p\times p}$ satisfying the inequality $\|I_p - \alpha\|_W < \varepsilon$ is invertible in the algebra $W^{p\times p}$ and admits the representations $\alpha(t) = \beta_+(t)\beta_-(t) = \gamma_-(t)\gamma_+(t)$. Here $\beta_\pm(t)$ and $\gamma_\pm(t)$ belong to $GW_\pm^{p\times p}$.*

By $T_k$, $k\in\mathbb{Z}$, we denote the Toeplitz operator with the matrix symbol $t^{-k}a(t)$, $a(t)\in W^{p\times p}$, acting in the space $E^{p\times 1}(\mathbb{Z}_+)$. It is convenient to treat $T_k := T_{t^{-k}a}$ as the operator generated by the infinite block Toeplitz matrix

$$T_k := \begin{pmatrix} a_k & a_{k-1} & a_{k-2} & \cdots \\ a_{k+1} & a_k & a_{k-1} & \cdots \\ a_{k+2} & a_{k+1} & a_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad -\infty < k < \infty. \tag{3.1}$$

This sequence of operators is the infinite–dimensional analogue of the finite sequence $\{T_k\}_{k=m}^n$, $m<n$, of finite block Toeplitz matrices

$$T_k := \begin{pmatrix} a_k & a_{k-1} & \cdots & a_m \\ a_{k+1} & a_k & \cdots & a_{m+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & \cdots & a_{n+m-1} \end{pmatrix}, \quad m\leqslant k\leqslant n,$$

generated by the sequence $\{a_m, a_{m+1}, \dots, a_{n-1}, a_n\}$. Consideration of the structure of the right and left kernels of the matrices from this sequence led us to the notions of indices and essential polynomials of a finite sequence of matrices [1], [13]. In order to be able to consider the left kernels of the infinite block Toeplitz matrices $T_k$, we also introduce the sequence of associated operators $\{U_k := U_{t^{-k}a}\}_{k\in\mathbb{Z}}$.

The study of the properties of the families of operators $\{T_k\}_{k\in\mathbb{Z}}$ and $\{U_k\}_{k\in\mathbb{Z}}$ in the general case seems to be an unmanageable problem. To apply the developed method to the problem of right Wiener — Hopf factorization (1.2), it is sufficient to consider the case with the following additional assumption:

A. The sequence $a := \{a_j\}_{j=-\infty}^{\infty}$ is the sequence of matrix Fourier coefficients of an invertible matrix function $a(t)\in W^{p\times p}$.

In what follows we assume that condition A is satisfied.

Now we proceed to studying the operators $T_k$, $U_k$. Our goal is to prove that they are Noetherian. We shall prove this in three steps. First we show that these operators are generalized invertible, then we prove the finite–dimensionality of their kernels, and then we shall be able to prove that they are Noetherian.

**Theorem 3.1.** *Let $a(t)\in GW^{p\times p}$. Then the operators $T_k$ ($U_k$) are generalized invertible in any space $E^{p\times 1}(\mathbb{Z}_+)$ ($E^{1\times p}(\mathbb{Z}_-)$) for all $k\in\mathbb{Z}$.*

*Moreover, there exists an integer $n>0$ such that the operator $T_k$ ($U_k$) is left (right) invertible for $k\leqslant -n$ and right (left) invertible for $k\geqslant n$.*

*Proof.* We being with proving the one–sided invertibility of the operators $T_k$.

We choose the value $\varepsilon$ in accordance with Lemma 3.1. For the given invertible matrix function $a(t) = \sum\limits_{j=-\infty}^{\infty} a_j t^j$ we choose $n>0$ so that for its Laurent matrix polynomial $a_n(t) = \sum\limits_{j=-n}^{n} a_j t^j$

satisfies the inequality

$$\|a - a_n\|_W < \frac{\varepsilon}{\|a^{-1}\|_W}.$$

Then $\|I_p - a_n a^{-1}\|_W < \varepsilon$ and, in accordance with Lemma 3.1, $a_n(t)$ is invertible in the algebra $W^{p\times p}$ and $a_n(t)a^{-1}(t)$ admits the factorization $a_n(t)a^{-1}(t) = \beta_+(t)\beta_-(t)$. Thus,

$$a(t) = b_-(t)b_+(t)a_n(t). \tag{3.2}$$

Here $b_\pm(t) = \beta_\pm^{-1}(t)$.

Since $t^{-k}a_n(t) \in W_+^{p\times p}$ for $k \leqslant -n$, by the partial multiplicativity property we obtain:

$$T_k = T_{b_-}T_{b_+}T_{t^{-k}a_n},$$

where, by Proposition 2.2, the operator $T_{t^{-k}a_n}$ is left invertible, and the operators $T_{b_\pm}$ are invertible. Thus, the operator $T_k$ is left invertible for $k \leqslant -n$.

To prove the right invertibility of the operator $T_k$ for $k \geqslant n$, we use the inequality $\|I_p - a^{-1}a_n\|_W < \varepsilon$ and represent $a(t)$ as

$$a(t) = a_n(t)\widetilde{b}_-(t)\widetilde{b}_+(t).$$

Then $T_k = T_{t^{-k}a_n}T_{\widetilde{b}_-}T_{\widetilde{b}_+}$ and $T_{t^{-k}a_n}$ is invertible on the right for $k \geqslant n$.

Let us proceed to the proof of generalized invertibility of the operators $T_k$ for $-n < k < n$. We first consider the case $0 \leqslant k < n$. We define the operator $\mathcal{K}_1 = T_k - T_{t^{n-k}I_p}T_n$. Since $\mathcal{K}_1 = T_{t^{-k}I_p}(\mathcal{I} - T_{t^nI_p}T_{t^{-n}I_p})T_a$ and the projection $\mathcal{I} - T_{t^nI_p}T_{t^{-n}I_p}$ onto $\ker T_{t^{-n}I_p}$ is finite–dimensional, $\mathcal{K}_1$ is finite–dimensional. Thus, $T_k = T_{t^{n-k}I_p}T_n + \mathcal{K}_1$. The operator $T_{t^{n-k}I_p}$ is left invertible for $n - k > 0$, and by the proven part of theorem the operator $T_n$ is right invertible. It is easy to verify that in this case their product $T_{t^{n-k}I_p}T_n$ is a generalized invertible operator. Thus, the operator $T_k$ is represented as the sum of a generalized invertible and a finite–dimensional operator. By [4, Thm. 5.3] it is generalized invertible.

Let $-n < k < 0$. We introduce the finite–dimensional operator

$$\mathcal{K}_2 = T_k - T_{-n}T_{t^{-n-k}I_p} = T_a(\mathcal{I} - T_{t^nI_p}T_{t^{-n}I_p})T_{t^{-k}I_p}.$$

Since $T_{-n}$ is left invertible and $T_{t^{-n-k}I_p}$ is right invertible, $T_{-n}T_{t^{-n-k}I_p}$ is generalized invertible. Applying [4, Thm. 5.3] again, we see that $T_k$ is generalized invertible.

The part of theorem on the operator $U_k$ can be proved in the same way by using the representation $\alpha(t) = \gamma_-(t)\gamma_+(t)$ from Lemma 3.1 and the partial multiplicativity (2.2) for the operators $U_a$. The proof is complete. □

This theorem implies two obvious corollaries.

**Corollary 3.1.** *If $a(t) \in GW^{p\times p}$, then all operators $T_k$ $(U_k)$ are normally solvable in any of the spaces $E^{p\times 1}(\mathbb{Z}_+)$ $(E^{1\times p}(\mathbb{Z}_-))$.*

The criterion of the one–sided invertibility of operators yields the following corollary.

**Corollary 3.2.** *In each space $E^{p\times 1}(\mathbb{Z}_+)$*

$$\ker T_k = 0 \quad \textit{for} \quad k \leqslant -n, \qquad \operatorname{Im} T_k = E^{p\times 1}(\mathbb{Z}_+) \quad \textit{for} \quad k \geqslant n.$$

*Similarly, in each space $E^{1\times p}(\mathbb{Z}_-)$*

$$\ker U_k = 0 \quad \textit{for} \quad k \geqslant n, \qquad \operatorname{Im} U_k = E^{1\times p}(\mathbb{Z}_-) \quad \textit{for} \quad k \leqslant -n.$$

At the second stage we prove that the kernels of operators $T_k$, $U_k$ are finite–dimensional.

**Theorem 3.2.** *If $a(t) \in GW^{p\times p}$, then for any fixed $k$ the kernel $\ker T_k$ $\big(\ker U_k\big)$ is finite–dimensional and it is the same in all spaces $E^{p\times 1}(\mathbb{Z}_+)$ $\big(E^{1\times p}(\mathbb{Z}_-)\big)$.*

*Proof.* By Corollary 3.2 for the operator $T_k$ we need to consider only the case $k > -n$. We represent $t^{-k}a_n(t)$ as

$$t^{-k}a_n(t) = t^{-(k+n)}\pi_{2n}(t),$$

where $\pi_{2n}(t)$ is a matrix polynomial in $t$ of degree at most $2n$. Then, by the partial multiplicativity we obtain $T_k = T_{b_-}T_{t^{-(n+k)}I_p}T_{b_+\pi_{2n}}$. Since $\pi_{2n}(t)$ is a matrix polynomial belonging to $GW^{p\times p}$, the operator $T_{b_+\pi_{2n}}$ is left invertible.

Let $x_+ \in E^{p\times 1}(\mathbb{Z}_+)$ be an arbitrary element in the space $\ker T_k$. Since $T_{b_-}$ is an invertible operator, $T_{b_+\pi_{2n}}x_+ \in \ker T_{t^{-(n+k)}I_p}$, i.e., $x_+ \in T^{\dagger}_{b_+\pi_{2n}} \ker T_{t^{-(n+k)}I_p}$. Here $T^{\dagger}_{b_+\pi_{2n}}$ is the left inverse operator for $T_{b_+\pi_{2n}}$. Thus, $\ker T_k \subseteq T^{\dagger}_{b_+\pi_{2n}} \ker T_{t^{-(n+k)}I_p}$.

In each space $E^{p\times 1}(\mathbb{Z}_+)$ the operator $T_{t^{-(k+n)}I_p}$ acts as a shift to the left by $k+n$ positions. Therefore, the space $\ker T_{t^{-(k+n)}I_p}$ consists of sequences of the form $y = (y_0, \ldots, y_{n+k-1}, 0, 0, \ldots)$, where $y_j$ are arbitrary vectors from $\mathbb{C}^{p\times 1}$. Hence this space is finite–dimensional and is a subspace of $l_1^{p\times 1}(\mathbb{Z}_+)$. We denote the space of such sequences by $l_1^{p\times 1}([0, n+k-1])$. Then in all spaces $E^{p\times 1}(\mathbb{Z}_+)$

$$\ker T_k \subseteq T^{\dagger}_{b_+\pi_{2n}} l_1^{p\times 1}([0, k+m)).$$

This yields that $\ker T_k$ is a finite-dimensional subspace of $l_1^{p\times 1}(\mathbb{Z}_+)$. The corresponding part of the theorem on the operator $U_k$ can be proved in the same way. The proof is complete. □

Now we are in position to prove the main result of this section.

**Theorem 3.3.** *If $a(t) \in GW^{p\times p}$, then for each fixed $k$ the operator $T_k$ ($U_k$) is Noetherian and has the same index in each space $E^{p\times 1}(\mathbb{Z}_+)$ ($E^{1\times p}(\mathbb{Z}_-)$).*

*Proof.* The normal solvability of operators $T_k$ and $U_k$ is implied by their generalized invertibility (see Corollary 3.1). By Theorem 3.2, $\alpha_k := \dim\ker T_k < \infty$ and is the same in all spaces $E^{p\times 1}(\mathbb{Z}_+)$. Similarly, $\beta_k := \dim\ker U_k < \infty$ and it is the same in all spaces $E^{1\times p}(\mathbb{Z}_-)$. It remains to prove that

$$\alpha_k^* := \dim\operatorname{coker} T_k = \dim\ker T_k^* < \infty$$
$$(\beta_k^* := \dim\operatorname{coker} U_k = \dim\ker U_k^* < \infty)$$

and it is the same in all spaces $E^{p\times 1}(\mathbb{Z}_+)$ $(E^{1\times p}(\mathbb{Z}_-))$.

If $E^{p\times 1}(\mathbb{Z}_+) = l_1^{p\times 1}(\mathbb{Z}_+)$, then $T_k^*$ coincides with the operator $U_k$ acting in $l_\infty^{1\times p}(\mathbb{Z}_+)$. Thus, in this case $\alpha_k^* = \beta_k < \infty$, i.e., the operator $T_k$ is Noetherian and $\operatorname{ind} T_k = \alpha_k - \beta_k$. In the same we prove that the operator $U_k$ is Noetherian in the space $l_1^{1\times p}(\mathbb{Z}_-)$ and $\operatorname{ind} U_k = \beta_k - \alpha_k$.

We consider the operator $T_k$ in the space $E^{p\times 1}(\mathbb{Z}_+) = l_\infty^{p\times 1}(\mathbb{Z}_+)$. By Proposition 2.1 it is adjoint to the Noetherian operator $U_k$ acting in the space $l_1^{1\times p}(\mathbb{Z}_-)$. Then, by a well–known theorem from the functional analysis (see, e.g., [8, Thm. 18]), $T_k$ is Noetherian in $l_\infty^{p\times 1}(\mathbb{Z}_+)$. Moreover, since the indices of an operator and of its adjoint differ only by sign, $\operatorname{ind} T_k = -\operatorname{ind} U_k = \alpha_k - \beta_k$, the operator $T_k$ is Noetherian in all spaces $E^{p\times 1}(\mathbb{Z}_+)$ and its index is independent on the choice of the space. The same statement holds for the operator $U_a$. The proof is complete. □

**Remark 3.1.** *The results of this section hold in all Krein spaces $E^{p\times 1}(\mathbb{Z}_+) = l_s^{p\times 1}(\mathbb{Z}_+)$ $(1 \leqslant s \leqslant \infty)$, $c^{p\times 1}(\mathbb{Z}_+)$, $c_0^{p\times 1}(\mathbb{Z}_+)$ and $E^{1\times p}(\mathbb{Z}_-) = l_s^{1\times p}(\mathbb{Z}_-)$, $c^{1\times p}(\mathbb{Z}_-)$, $c_0^{1\times p}(\mathbb{Z}_-)$. In order not to overload the work, we have considered only those pairs of spaces that will be needed for the new proof of the existence of a solution to the Wiener — Hopf factorization problem.*

## 4. Indices and essential functions of infinite sequence of complex matrices

Now we can proceed to the definition of indices and essential functions for an infinite sequence of matrices. We shall follow the lines of the case of a finite sequence of matrices [1], [13].

In the spaces $E^{p\times 1}(\mathbb{Z}_+)$ and $E^{1\times p}(\mathbb{Z}_-)$ we consider the sequence of subspaces $\ker T_k$ and $\ker U_k$ for $k\in\mathbb{Z}$:

$$\ker T_k = \Big\{x_+ \in E^{p\times 1}(\mathbb{Z}_+) \ : \ \sum_{j=0}^{\infty} a_{i-j}x_j = 0,\ i\geqslant k\Big\},$$

$$\ker U_k = \Big\{f_- \in E^{1\times p}(\mathbb{Z}_-) \ : \ \sum_{j=-\infty}^{0} f_j a_{i-j} = 0,\ i\leqslant k\Big\}.$$

These formulas show that

$$\ker T_k \subseteq \ker T_{k+1}, \qquad \ker U_k \supseteq \ker U_{k+1}.$$

We note that the sequence of spaces $\ker U_k$ is the analogue of the sequence $\ker_L T_k$ in the finite–dimensional case.

Let us study the structure of the kernels $\{\ker T_k\}_{k\in\mathbb{Z}}$ ($\{\ker U_k\}_{k\in\mathbb{Z}}$).

By Theorem 3.2 we can restrict ourselves to studying the spaces $\ker T_k$, ($\ker U_k$) only for $l_1^{p\times 1}(\mathbb{Z}_+)$ ($l_1^{1\times p}(\mathbb{Z}_-)$). In this case it is more convenient to pass from the spaces of sequences to the corresponding spaces of generating vector functions, i.e., to $W_+^{p\times 1}$ ($W_-^{1\times p}$).

For the matrix function $a(t)=\sum\limits_{j=-\infty}^{\infty} a_j t^j \in W^{p\times p}$ we define two linear bounded operators

$$\sigma_R : W^{p\times 1} \mapsto \mathbb{C}^{p\times 1}, \qquad \sigma_L : W^{1\times p} \mapsto \mathbb{C}^{1\times p},$$

acting by the formulas

$$\sigma_R\{R(t)\} = \sum_{j=-\infty}^{\infty} a_{-j}r_j, \qquad \sigma_L\{L(t)\} = \sum_{j=-\infty}^{\infty} l_j a_{-j}, \tag{4.1}$$

for

$$R(t) = \sum_{j=-\infty}^{\infty} r_j t^j \in W^{p\times 1}, \qquad L(t) = \sum_{j=-\infty}^{\infty} l_j t^j \in W^{1\times p}.$$

We denote by $\mathcal{N}_k^R$, $\mathcal{N}_k^L$, the space of generating vector functions (i.e., the space of Fourier transforms) of elements in $\ker T_k$, $\ker U_k$, respectively. It is obvious that

$$\begin{aligned}\mathcal{N}_k^R &= \Big\{R(t)\in W_+^{p\times 1} : \sigma_R\{t^{-i}R(t)\} = 0,\ i\geqslant k\Big\},\\ \mathcal{N}_k^L &= \Big\{L(t)\in W_-^{1\times p} : \sigma_L\{t^{-i}L(t)\} = 0,\ i\leqslant k\Big\}.\end{aligned} \tag{4.2}$$

We recall that these spaces are finite–dimensional.

We denote

$$d_k^R = \dim\mathcal{N}_k^R, \qquad d_k^L = \dim\mathcal{N}_k^L, \qquad \Delta_k^R = d_k^R - d_{k-1}^R, \qquad \Delta_k^L = d_k^L - d_{k+1}^L.$$

**Theorem 4.1.** *For each matrix function $a(t)\in GW^{p\times p}$ there exists an integer $n\geqslant 0$ such that*

$$\begin{aligned}0 = \ldots = \Delta_{-n-1}^R = \Delta_{-n}^R \leqslant \Delta_{-n+1}^R \leqslant \ldots \leqslant \Delta_n^R \leqslant \Delta_{n+1}^R = \Delta_{n+2}^R = \ldots = p,\\ p = \ldots = \Delta_{-n-2}^L = \Delta_{-n-1}^L \geqslant \Delta_{-n}^L \geqslant \ldots \geqslant \Delta_{n-1}^L \geqslant \Delta_n^L = \Delta_{n+1}^L = \ldots = 0.\end{aligned} \tag{4.3}$$

*Proof.* Definition (4.2) implies immediately that $\mathcal{N}_k^R$ and $t\mathcal{N}_k^R$ are subspaces of $\mathcal{N}_{k+1}^R$. It is also easy to observe that $\mathcal{N}_k^R \cap t\mathcal{N}_k^R = t\mathcal{N}_{k-1}^R$. Therefore, by the Grassmann formula,

$$\dim\,[\mathcal{N}_k^R + t\mathcal{N}_k^R] = 2d_k^R - d_{k-1}^R.$$

If $h^R_{k+1}$ is the dimension of the complement of $\mathcal{N}^R_k + t\mathcal{N}^R_k$ to the entire space $\mathcal{N}^R_{k+1}$, then $h^R_{k+1} = \Delta^R_{k+1} - \Delta^R_k$. This means that $\Delta^R_{k+1} \geqslant \Delta^R_k$ for each $k \in \mathbb{Z}$. Moreover, by Theorem 3.2 there exists $n \geqslant 0$ such that $d^R_k = 0$ for $k \leqslant -n$. Therefore, $\Delta^R_k = 0$ for $k \leqslant -n$.

Let us show that $\Delta^R_k = p$, starting from $k = n+1$. If $x(t)$ and $y(t)$ belong to $\mathcal{N}^R_{k+1}$ and $x(t) - y(t) \in \mathcal{N}^R_k$, then $\sigma_R\{t^{-k}x(t)\} = \sigma_R\{t^{-k}y(t)\}$. Therefore, the mapping $\pi : \mathcal{N}^R_{k+1}/\mathcal{N}^R_k \mapsto \mathbb{C}^{p\times 1}$ is well–defined by the rule

$$\pi(X) = \sigma_R\{t^{-k}x(t)\}.$$

Here $x(t) \in \mathcal{N}^R_{k+1}$ is any representative of the class $X$. This mapping is the embedding of $\mathcal{N}^R_{k+1}/\mathcal{N}^R_k$ into $\mathbb{C}^{p\times 1}$ and hence, $\Delta^R_{k+1} = \dim \mathcal{N}^R_{k+1}/\mathcal{N}^R_k \leqslant p$ for any $k \in \mathbb{Z}$.

Let us prove the reverse inequality for $k \geqslant n$. By the same Theorem 3.2, for $k \geqslant n$ we have $\operatorname{Im} T_k = l_1^{p\times 1}(\mathbb{Z}_+)$. Hence, for such $k$ there exist solutions $x^{(l)} = \{x^{(l)}_j\}_{j=0}^{\infty}$, $l = 1, \ldots, p$, to the systems of equations

$$\sum_{j=0}^{\infty} a_{k-j}x^{(l)}_j = [I_p]^l, \tag{4.4}$$

$$\sum_{j=0}^{\infty} a_{i-j}x^{(l)}_j = 0, \qquad i \geqslant k+1 \tag{4.5}$$

in the space $l_1^{p\times 1}(\mathbb{Z}_+)$. Here $[I_p]^l$ is the $l$-th column of the identity matrix $I_p$. Equations (4.5) mean that the generating functions $x^{(l)}(t)$, $l = 1, \ldots, p$, belong to the space $\mathcal{N}^R_{k+1}$, and condition (4.4) is equivalent to the fact that $x^{(1)}(t), \ldots, x^{(p)}(t)$ are linearly independent modulo the space $\mathcal{N}^R_k$. Hence, $\Delta^R_{k+1} \geqslant p$ and finally $\Delta^R_{k+1} = p$ for $k \geqslant n$.

The part of the theorem on the sequence $\Delta^L_k$ can be proved in the same way. The proof is complete. □

Inequalities (4.3) mean that there exist integers $-n \leqslant \mu_1 \leqslant \ldots \leqslant \mu_p \leqslant n$ such that

$$\begin{aligned} \ldots =&\Delta^R_{-n} = \ldots = \Delta^R_{\mu_1} = 0,\\ &\ldots\ldots\\ &\Delta^R_{\mu_i+1} = \ldots = \Delta^R_{\mu_{i+1}} = i,\\ &\ldots\ldots\\ &\Delta^R_{\mu_p+1} = \ldots = \Delta^R_{n+1} = p = \ldots. \end{aligned} \tag{4.6}$$

If the $i$-th row is absent in these relations, we let $\mu_i = \mu_{i+1}$.

Similarly, the monotonicity of the sequence $\Delta^L_k$ yields that there exist integers $\nu_1 \leqslant \ldots \leqslant \nu_p$ such that

$$\begin{aligned} \ldots =&\Delta^L_{-n-1} = \ldots = \Delta^L_{\nu_1-1} = p,\\ &\ldots\ldots\\ &\Delta^L_{\nu_i} = \ldots = \Delta^L_{\nu_{i+1}-1} = p - i,\\ &\ldots\ldots\\ &\Delta^L_{\nu_p} = \ldots = \Delta^L_{n} = 0 = \ldots. \end{aligned} \tag{4.7}$$

Similar to the finite-dimensional case, in the next theorem we prove that the numbers $\mu_i$ and $\nu_i$ coincide.

**Theorem 4.2.** *For each matrix function $a(t) \in GW^{p\times p}$ the numbers $\mu_1, \ldots, \mu_p$ coincide with $\nu_1, \ldots, \nu_p$. Moreover,*

$$\sum_{j=1}^{p} \mu_j = -\operatorname{ind} T_a. \tag{4.8}$$

*Proof.* By the definition of the index of the operator $T_k$ we obtain

$$d_k^R = \operatorname{ind} T_k + \dim \ker T_k^*.$$

Since the adjoint operator to $T_k$ in the space $l_1^{p\times 1}(\mathbb{Z}_+)$ coincides with the operator $U_k$ in the space $l_\infty^{1\times p}(\mathbb{Z}_-)$ and the dimension of the space $\ker U_k$ is one and the same in all spaces $E^{1\times p}(\mathbb{Z}_-)$, then $\dim \ker T_k^* = d_k^L$. The definition $T_k = T_{t^{-k}a}$ and partial multiplicativity imply that $T_{t^{-k}a}$ is the product of Noetherian operators $T_{t^{-k}I_p}$ and $T_a$. Then

$$\operatorname{ind} T_k = \operatorname{ind} T_{t^{-k}I_p} + \operatorname{ind} T_a = kp + \operatorname{ind} T_a$$

for all $k \in \mathbb{Z}$. Since

$$d_k^R = kp + \operatorname{ind} T_a + d_k^L, \qquad d_{k+1}^R = (k+1)p + \operatorname{ind} T_a + d_{k+1}^L,$$

we find $\Delta_{k+1}^R = p - \Delta_k^L$. Relations (4.6), (4.7) hence imply that the numbers $\mu_i$ and $\nu_i$ coincide for $i = 1, \dots, p$.

We proceed to proving the second statement of the theorem. Since $d_{n+1}^R = \sum\limits_{j=-n}^{n+1} \Delta_j^R$, by (4.6) we obtain $d_{n+1}^R = (n+1)p - \sum\limits_{j=1}^{p} \mu_j$. On the other hand, by Corollary 3.2 we have $\operatorname{Im} T_{n+1} = l_1^{p\times 1}(\mathbb{Z}_+)$, i.e., $\operatorname{ind} T_{n+1} = \dim \ker T_{n+1}$ and thus

$$d_{n+1}^R = \operatorname{ind} T_{n+1} = (n+1)p + \operatorname{ind} T_a.$$

Comparing these two formulas, we obtain (4.8). The proof is complete. □

**Definition 4.1.** *The integers $\mu_1, \dots, \mu_p$ determined by relations (4.6) are be called the essential indices (or, briefly, indices) of the invertible matrix function $a(t)$ and of the corresponding sequence $\{a_n\}_{n=-\infty}^{\infty}$.*

Since the dimension $h_{k+1}^R$ of the complement $\mathcal{H}_{k+1}^R$ of the subspace $\mathcal{N}_k^R + t\mathcal{N}_k^R$ to the entire space $\mathcal{N}_{k+1}^R$ is equal to $\Delta_{k+1}^R - \Delta_k^R$, it follows from (4.6) that $h_{k+1}^R \neq 0$ only for $k = \mu_i$. In this case $h_{k+1}^R$ coincides with the multiplicity $r_i$ of the index $\mu_i$. Therefore,

$$\mathcal{N}_{k+1}^R = \mathcal{N}_k^R + t\mathcal{N}_k^R \tag{4.9}$$

for $k \neq \mu_i$ and

$$\mathcal{N}_{k+1}^R = (\mathcal{N}_k^R + t\mathcal{N}_k^R) \oplus \mathcal{H}_{k+1}^R \tag{4.10}$$

for $k = \mu_i$.

**Definition 4.2.** *Each vector functions $R_i(t), \dots, R_{i+r_i-1}(t)$ from $W_+^{p\times 1}$ forming a basis of the complement $\mathcal{H}_{\mu_i+1}^R$, are called the right essential vector functions of the matrix function $a(t)$ corresponding to the index $\mu_i$ of multiplicity $r_i$, $i = 1, \dots, p$. Thus, we obtain the vector functions $R_1(t), \dots, R_p(t)$ from the space $W_+^{p\times 1}$, which are called the right essential vector functions of $a(t)$ and of the corresponding sequence $\{a_n\}_{n=-\infty}^{\infty}$.*

Similarly, for $k \neq \mu_i$

$$\mathcal{N}_{k-1}^L = \mathcal{N}_k^L + t^{-1}\mathcal{N}_k^L$$

and for $k = \mu_i$

$$\mathcal{N}_{k-1}^L = (\mathcal{N}_k^L + t^{-1}\mathcal{N}_k^L) \oplus \mathcal{H}_{k-1}^L.$$

Choosing the bases in the spaces $\mathcal{H}_{\mu_i-1}^L$ $(i = 1, \dots, p)$, we obtain the set $L_1(t), \dots, L_p(t)$ of vector functions in $W_-^{1\times p}$, which are called the *left essential vector functions* of the matrix function $a(t)$.

In terms of indices and essential functions we can now describe the structure of the spaces $\ker T_k$, $\ker U_k$.

**Theorem 4.3.** *Let $\mu_1, \dots, \mu_p$ be the essential indices of $a(t) \in GW^{p\times p}$ and $R_1(t), \dots, R_p(t)$; $L_1(t), \dots, L_p(t)$ the right and left essential functions of $a(t)$, respectively. Then in any space $E^{p\times 1}(\mathbb{Z}_+)$ for $k \in (-\infty, \mu_1]$ we have $\ker T_k = 0$. For $k \in (\mu_i, \mu_{i+1}]$ $(i = 1, \dots, p-1)$ or $k \in (\mu_p, +\infty)$ the vector functions*

$$\{R_j(t), tR_j(t), \dots, t^{k-\mu_j-1}R_j(t)\}_{j=1,2,\dots,i} \tag{4.11}$$

*from $W_+^{p\times 1}$ are generating functions for the elements of a basis of the kernel of the operator $T_k$ in any space $E^{p\times 1}(\mathbb{Z}_+)$.*

*Similarly, $\ker U_k = 0$ for $k \in [\mu_p, +\infty)$, and for $k \in [\mu_{i-1}, \mu_i)$ $(i = 2, \dots, p)$ or for $k \in (-\infty, \mu_1)$ the vector functions*

$$\{L_j(t), t^{-1}L_j(t), \dots, t^{-(\mu_j-k-1)}L_j(t)\}_{j=i,i+1,\dots,p}$$

*from $W_-^{1\times p}$ are generating functions for the elements of a basis of the kernel of the operator $U_k$ in any space $E^{1\times p}(\mathbb{Z}_-)$.*

*Proof.* The first statement of the theorem is obvious. Now let $k \in (\mu_i, \mu_{i+1}]$, $(i = 1, \dots, p-1)$ or $k \in (\mu_p, +\infty)$. Relations (4.9) and (4.10) imply that functions (4.11) generate the space $\mathcal{N}_k^R$. Since $d_k^R = \sum_{j=-n}^{k} \Delta_j^R$, by (4.6) we obtain

$$d_k^R = ki - \sum_{j=1}^{i} \mu_j.$$

Thus, $d_k^R$ coincides with the number of vector functions in (4.11). Therefore, they form a basis of the space $\mathcal{N}_k^R$. The second part of the theorem can be proved in the same way. The proof is complete. □

## 5. Essentiality criterion

For an invertible matrix function $a(t)$ there exists a criterion of essentiality similar to the corresponding criterion in the finite–dimensional case [13].

**Theorem 5.1.** *Let $a(t) \in GW^{p\times p}$, $\kappa_1 \leqslant \dots \leqslant \kappa_p$ be integers satisfying the identity*

$$\sum_{j=1}^{p} \kappa_j = -\operatorname{ind} T_a$$

*and $R_1(t), \dots, R_p(t)$; $L_1(t), \dots, L_p(t)$ be vector functions such that $R_j(t) \in \mathcal{N}_{\kappa_j+1}^R$, $L_j(t) \in \mathcal{N}_{\kappa_j-1}^L$.*

*The numbers $\kappa_1, \dots, \kappa_p$ are the essential indices of $a(t)$, and $R_1(t), \dots, R_p(t)$ are its right essential functions if and only if the matrix*

$$\Lambda_R = \big(\sigma_R\{t^{-\kappa_1}R_1(t)\} \dots \sigma_R\{t^{-\kappa_p}R_p(t)\}\big) \text{ or } \hat{\Lambda}_R = \big(R_1(0) \dots R_p(0)\big).$$

*is invertible.*

*Similarly, $\kappa_1, \dots, \kappa_p$ are the essential indices, and $L_1(t), \dots, L_p(t)$ are the left essential functions of $a(t)$ if and only if the matrix*

$$\Lambda_L = \begin{pmatrix} \sigma_L\{t^{-\kappa_1}L_1(t)\} \\ \vdots \\ \sigma_L\{t^{-\kappa_p}L_p(t)\} \end{pmatrix} \quad \text{or} \quad \hat{\Lambda}_L = \begin{pmatrix} L_1(\infty) \\ \vdots \\ L_p(\infty) \end{pmatrix}.$$

*is invertible.*

*Proof. Necessity.* Let $\kappa_1, \dots, \kappa_p$ be the essential indices, and $R_1(t), \dots, R_p(t)$ be the right essential functions of $a(t)$. Suppose that the matrix $\Lambda_R$ is not invertible. Then there exist numbers $\alpha_1, \dots, \alpha_p$, not all being zero, such that

$$\alpha_1 \sigma_R\{t^{-\kappa_1} R_1(t)\} + \dots + \alpha_p \sigma_R\{t^{-\kappa_p} R_p(t)\} = 0. \tag{5.1}$$

Let the last index $\kappa_p$ have multiplicity $r$, i.e.,

$$\kappa_{p-r} < \kappa_{p-r+1} = \dots = \kappa_p.$$

We introduce the vector function $Q(t) \in W_+^{p\times 1}(t)$:

$$\begin{aligned} Q(t) =& \alpha_1 t^{\kappa_p - \kappa_1} R_1(t) + \dots + \alpha_{p-r} t^{\kappa_p - \kappa_{p-r}} R_{p-r}(t) \\ &+ \alpha_{p-r+1} R_{p-r+1}(t) + \dots + \alpha_p R_p(t). \end{aligned}$$

It is obviously that $Q(t)$ belongs to the space $\mathcal{N}^R_{\kappa_p+1}$.

Condition (5.1) means that $\sigma_R\{t^{-\kappa_p} Q(t)\} = 0$, i.e., $Q(t) \in \mathcal{N}^R_{\kappa_p}$. But then

$$\alpha_{p-r+1} R_{p-r+1}(t) + \dots + \alpha_p R_p(t) \in \mathcal{N}^R_{\kappa_p} + t\mathcal{N}^R_{\kappa_p}.$$

Since $R_{p-r+1}(t), \dots, R_p(t)$ are the right essential functions corresponding to the index $\kappa_p$, this is possible only if $\alpha_{p-r+1} = \dots = \alpha_p = 0$.

Repeating these arguments successively for all the remaining indices, we obtain $\alpha_1 = \dots = \alpha_p = 0$. The contradiction shows that $\Lambda_R$ is invertible. In the same way we prove the invertibility of $\hat{\Lambda}_R$, $\Lambda_L$ and $\hat{\Lambda}_L$.

*Sufficiency.* Suppose that for the numbers $\kappa_1, \dots, \kappa_p$ and the vector functions $R_1(t), \dots, R_p(t)$, the matrix $\Lambda_R$ is invertible. We denote by $\mu_1, \dots, \mu_p$ the essential indices of $a(t)$. Let $\kappa_j \in [\mu_{s_j}, \mu_{s_j+1})$, $j = 1, \dots, p$. Here $0 \leqslant s_1 \leqslant \dots \leqslant s_p \leqslant p$, and these numbers are chosen so that, if $\kappa_j = \kappa_{j+1}$, then $s_j = s_{j+1}$ and $\mu_{s_j} < \mu_{s_j+1}$ for $\kappa_j < \kappa_{j+1}$.

Let us estimate the numbers $s_j$. In order to do this, by the first $j$ columns of the invertible matrix $\Lambda_R$ we form the matrix

$$\Lambda_R^j = \left(\sigma\{t^{-\kappa_1} R_1(t)\} \dots \sigma\{t^{-\kappa_j} R_j(t)\}\right),$$

$j = 1, \dots, p$. It is obvious that the rank of $\Lambda_R^j$ is equal to $j$.

We are going to estimate this rank. We consider an arbitrary $l$-th column of this matrix: $\sigma\{t^{-\kappa_l} R_l(t)$. By assumption, $R_l(t) \in \mathcal{N}^R_{\kappa_l+1}$ and $\kappa_l + 1 \in (\mu_{s_l}, \mu_{s_l+1}]$. By Theorem 4.3 a basis in the space $\mathcal{N}^R_{\kappa_l+1}$ consists of the functions

$$\left\{R_i(t), tR_i(t), \dots, t^{\kappa_l - \mu_i} R_i(t)\right\}_{i=1,2,\dots,s_l}.$$

We expand $R_l(t)$ over this basis selecting explicitly only $\{t^{\kappa_l - \mu_i} R_i(t)\}_{i=1,2,\dots,s_l}$ :

$$R_l(t) = \sum_{i=1}^{s_l} \alpha_i^{(l)} t^{\kappa_l - \mu_i} R_i(t) + u_l(t).$$

Here $u_l(t)$ is a vector function, which includes all the remaining vector functions in the basis $\mathcal{N}^R_{\kappa_l+1}$. It is obvious that $\sigma_R\{t^{-\kappa_l} u_l(t)\} = 0$. We denote $\sigma_i = \sigma_R\{t^{-\mu_i} R_i(t)\}$, $\sigma_i \in \mathbb{C}^{p\times 1}$.

Then the column vector $\sigma_R\{t^{-\kappa_l} R_l(t)\}$ is represented as the following sum of column vectors:

$$\sigma_R\{t^{-\kappa_l} R_l(t)\} = \sum_{i=1}^{s_l} \alpha_i^{(l)} \sigma_i = \sum_{i=1}^{s_j} \alpha_i^{(l)} \sigma_i, \quad 1 \leqslant l \leqslant j.$$

Here we have extended the coefficients $\alpha_i^{(l)}$ by letting $\alpha_i^{(l)} = 0$ for $s_l < i \leqslant s_j$. Thus, we have obtained the following representation of the matrix $\Lambda_R^j$:

$$\Lambda_R^j = \sum_{i=1}^{s_j} \left(\alpha_i^{(1)}\sigma_i, \ldots, \alpha_i^{(j)}\sigma_i\right) = \sum_{i=1}^{s_j} \sigma_i\big(\alpha_i^{(1)}, \ldots, \alpha_i^{(j)}\big).$$

This means that we have represented the matrix $\Lambda_R^j$ as a sum of rank one matrices. Therefore, the rank of the matrix $\Lambda_R^j$ does not exceed $s_j$ (see, e.g., [3, Subsect. 4.44]). We have obtained the estimate $s_j \geqslant j$ for $j = 1, \ldots, p$. Therefore, $\kappa_j \geqslant \mu_{s_j} \geqslant \mu_j$. Since

$$\sum_{j=1}^{p} \kappa_j = -\operatorname{ind} T_a = \sum_{j=1}^{p} \mu_j,$$

this is possible only if $\kappa_j = \mu_j$, $j = 1, \ldots, p$.

Now we are going to prove that the vector functions $R_1(t), \ldots, R_p(t)$ can serve as the right essential functions. Let $\mu_1$ have multiplicity $r_1$: $\mu_1 = \ldots = \mu_{r_1} < \mu_{r_1+1}$. If the functions $R_1(t), \ldots, R_{r_1}(t)$ from the $r_1$-dimensional space $\mathcal{N}_{\mu_1+1}^R$ were linearly dependent, then the first $r_1$ columns of the matrix $\Lambda_R$ would also be connected by a linear dependence, which is impossible due to the invertibility of $\Lambda_R$. Thus, $R_1(t), \ldots, R_{r_1}(t)$ can serve as the right essential functions corresponding to the index $\mu_1$.

Suppose that the vector functions $R_1(t), \ldots, R_j(t)$ are the essential functions corresponding to the indices $\mu_1, \ldots, \mu_j$. Let us prove that the functions $R_{j+1}(t), \ldots, R_{j+r_{j+1}}(t)$ from the space $\mathcal{N}_{\mu_{j+1}+1}^R$ can serve as the right essential functions corresponding to the index $\mu_{j+1}$. Here $r_{j+1}$ is the multiplicity of the index $\mu_{j+1}$:

$$\mu_j < \mu_{j+1} = \ldots = \mu_{j+r_{j+1}} < \mu_{j+r_{j+1}+1}.$$

Suppose that there exist numbers $\alpha_{j+1}, \ldots, \alpha_{j+r_{j+1}}$ such that

$$\alpha_{j+1}R_{j+1}(t) + \ldots + \alpha_{j+r_{j+1}}R_{j+r_{j+1}}(t) \in \mathcal{N}_{\mu_{j+1}}^R + t\mathcal{N}_{\mu_{j+1}}^R. \tag{5.2}$$

The definition of the right essential polynomials and Theorem 4.3 imply that a basis of the space $\mathcal{N}_{\mu_{j+1}}^R + t\mathcal{N}_{\mu_{j+1}}^R$ consists of the vector functions

$$\left\{R_i(t), tR_i(t), \ldots, t^{\mu_{j+1}-\mu_i}R_i(t)\right\}_{i=1,2,\ldots,j}.$$

Therefore, it follows from (5.2) that there exist numbers $\alpha_1, \ldots, \alpha_j$ such that

$$\begin{aligned}\alpha_{j+1}\sigma_R\left\{t^{-\mu_{j+1}}R_{j+1}(t)\right\} + \ldots + \alpha_{j+r_{j+1}}\sigma_R\left\{t^{-\mu_{j+r_{j+1}}}R_{j+r_{j+1}}(t)\right\}\\ = \alpha_1\sigma_R\left\{t^{-\mu_1}R_1(t)\right\} + \ldots + \alpha_j\sigma_R\left\{t^{-\mu_j}R_j(t)\right\}.\end{aligned}$$

But by the invertibility of the matrix $\Lambda_R$ this is possible only if $\alpha_1 = \ldots = \alpha_{j+r_{j+1}} = 0$. This means that the vector functions $R_{j+1}(t), \ldots, R_{j+r_{j+1}}(t)$ are linearly independent modulo the space $\mathcal{N}_{\mu_{j+1}}^R + t\mathcal{N}_{\mu_{j+1}}^R$. Therefore, they can serve as the right essential functions corresponding to the index $\mu_{j+1}$.

The statement of the theorem on the matrices $\hat{\Lambda}_R$, $\Lambda_L$ and $\hat{\Lambda}_L$ can be proved in the same way. The proof is complete. □

## 6. Essential functions and Wiener — Hopf factorization

Now we can find out how the indices and essential functions of an invertible matrix function $a(t)$ are related to its Wiener — Hopf factorization.

By the essential vector functions $R_1(t), \dots, R_p(t)$ and $L_1(t), \dots, L_p(t)$ we form the matrix functions

$$\mathcal{R}(t) = \big(R_1(t) \dots R_p(t)\big), \qquad \mathcal{L}(t) = \begin{pmatrix} L_1(t) \\ \vdots \\ L_p(t) \end{pmatrix}.$$

It is clear that $\mathcal{R}(t) \in W_+^{p\times p}$, $\mathcal{L}(t) \in W_-^{p\times p}$. Moreover, we introduce

$$d(t) = \operatorname{diag}[t^{\mu_1}, \dots, t^{\mu_p}].$$

In the next theorem we construct the right Wiener — Hopf factorization of $a(t)$ in terms of its indices and essential functions. In this way we obtain a new proof for the existence of the Wiener — Hopf factorization. We note that this proof explicitly uses only the tools of functional analysis, and from the complex analysis, only the maximum modulus principle is required.

**Theorem 6.1.** *Each matrix function $a(t) \in GW^{p\times p}$ admits the representations*

$$a(t) = a_-(t)d(t)\mathcal{R}^{-1}(t) = \mathcal{L}^{-1}(t)d(t)a_+(t),$$

*which are its right Wiener — Hopf factorizations.*

*Thus, the essential indices of $a(t)$ coincide with the right factorization indices of $a(t)$, and $\mathcal{R}(t)$ $\big(\mathcal{L}(t)\big)$ is the inverse matrix function to one of the factors $a_+(t)$ $\big(a_-(t)\big)$.*

*Proof.* The definition (4.1) of the operator $\sigma_R$ implies that

$$\sigma_R\{t^{-i}R(t)\} = \sum_{k=0}^{\infty} a_{i-k}R_k = (a * R)_i,$$

i.e., it coincides with the $i$-th Fourier coefficient of the vector function $a(t)R(t)$. Therefore, the condition $R(t) \in \mathcal{N}_k^R$ is equivalent to the fact that $a(t)R(t)$ has zero Fourier coefficients with indices $i \geqslant k$. This yields that the columns of the matrix function $a(t)\mathcal{R}(t)d^{-1}(t)$ belong to the space $W_-^{p\times 1}$, i.e.,

$$a(t)\mathcal{R}(t)d^{-1}(t) = a_-(t), \tag{6.1}$$

for some matrix function $a_-(t) \in W_-^{p\times p}$. By the essentiality criterion (Theorem 5.1) we obtain the invertibility of the matrix

$$a_-(\infty) = \big(\sigma_R\{t^{-\mu_1}R_1(t)\} \dots \sigma_R\{t^{-\mu_p}R_p(t)\}\big) = \Lambda_R.$$

Similarly,

$$d^{-1}(t)\mathcal{L}(t)a(t) = a_+(t), \tag{6.2}$$

for some $a_+(t) \in W_+^{p\times p}$, and $a_+(0) = \Lambda_L$ is invertible.

By (6.1), (6.2) we have

$$\mathcal{L}(t)a(t)\mathcal{R}(t) = \mathcal{L}(t)a_-(t)d(t) = d(t)a_+(t)\mathcal{R}(t).$$

Passing to the determinants in this identity, we find

$$\det a_+(t) \det \mathcal{R}(t) = \det a_-(t) \det \mathcal{L}(t).$$

Since the left hand side of this identity belongs to the algebra $W_+$, and the right hand side does to the algebra $W_-$, on the unit circle we have $\det a_+(t) \det \mathcal{R}(t) = \lambda$, $\lambda \in \mathbb{C}$. The maximum modulus principle then implies that $\det a_+(z) \det \mathcal{R}(z) = \lambda$ everywhere in the closed unit disk. Since $\det a_+(0) = \det \Lambda_L \neq 0$ and $\det \mathcal{R}(0) = \det \hat{\Lambda}_R \neq 0$, we obtain $\lambda \neq 0$. Thus,

$$\det a_+(z) \det \mathcal{R}(z) = \lambda \neq 0$$

everywhere in the closed unit disk. This means that the matrix functions $a_+(t)$, $\mathcal{R}(t)$ are invertible in the algebra $W_+^{p\times p}$. In the same way we prove the invertibility of $a_-(t)$, $\mathcal{L}(t)$ in the algebra $W_-^{p\times p}$.

Therefore, by (6.1) and (6.2) we obtain the right Wiener — Hopf factorizations of the matrix function $a(t)$ in terms of the right or left essential functions:

$$a(t) = a_-(t)d(t)\mathcal{R}^{-1}(t) = \mathcal{L}^{-1}(t)d(t)a_+(t).$$

The proof is complete. □

**Remark 6.1.** *We note that the obtained results imply that the sum of the factorization indices of $a(t)$, taken with the opposite sign, coincides with the index of the operator $T_a$.*

Now we are going to prove that knowing any right factorization of $a(t)$, we can find its essential indices and essential functions. Given a matrix $A$, we adopt the notation $[A]^j$ for its $j$-th column, and $[A]_j$ for its $j$-th row.

**Theorem 6.2.** *If*

$$a(t) = a_-(t)d(t)a_+(t) \tag{6.3}$$

*is a right Wiener — Hopf factorization of $a(t)$, then the right factorization indices of $a(t)$ are the essential indices of $a(t)$, the vector functions $[a_+^{-1}(t)]^1, \dots, [a_+^{-1}(t)]^p$ are the right essential functions, and $[a_-^{-1}(t)]_1, \dots, [a_-^{-1}(t)]_p$ are the left essential functions of $a(t)$.*

*Proof.* If $\mu_1, \dots, \mu_p$ are the right factorization indices of $a(t)$, then it follows from (6.3) that

$$\sum_{j=1}^{p} \mu_j = -\operatorname{ind} T_a.$$

Moreover, by the factorization (6.3) we also have

$$t^{-\mu_j} a(t)[a_+^{-1}(t)]^j = [a_-(t)]^j, \quad j = 1, \dots, p.$$

The Fourier coefficients with positive indices of the vector function $t^{-\mu_j} a(t)[a_+^{-1}(t)]^j$ are equal to zero. Therefore, $[a_+^{-1}(t)]^j \in \mathcal{N}^R_{\mu_j+1}$. Since the matrix

$$\Lambda_R = \left(\sigma_R\left\{t^{-\mu_1}[a_+^{-1}(t)]^1\right\} \dots \sigma_R\left\{t^{-\mu_p}[a_+^{-1}(t)]^p\right\}\right) = a_-(\infty),$$

is invertible. The essentiality criterion (Theorem 5.1) implies that the numbers $\mu_1, \dots, \mu_p$ are the essential indices, and the vector functions $[a_+^{-1}(t)]^1, \dots, [a_+^{-1}(t)]^p$ are the right essential functions of $a(t)$.

The second part of the theorem can be proved in the same way. The proof is complete. □

Thus, in the case of an invertible matrix function the method of essential indices and functions is completely equivalent to the method of Wiener — Hopf factorization.

**Remark 6.2.** *Let us compare the proof of the existence of factorization proposed in this work with the classical proof by I.Ts. Gokhberg, now presented in many monographs and surveys on this topic (see, e.g., [5], [7]). The base of both methods is the lemma on canonical factorization of matrix functions sufficiently close to the identity matrix. Then in both methods a representation of form (3.2) is used (see [7]). The further ways of the proof diverge. In the method by I.Ts. Gokhberg, the existence of a factorization of a matrix function analytic inside the domain is proved. This is the most technically complicated part of the proof, and it is a pure existence theorem. Explicit identification of the partial indices and factorization factors of an analytic matrix function within this method is hardly possible. The general case in the method of I.Ts. Gokhberg is reduced to the analytic one by means of representation (3.2).*

*In the method proposed in this work, representation (3.2) is used to study the chain of kernels of the operators $T_k$, $U_k$, which ultimately leads to the construction of the factorization. This is a longer path, but it has its advantages, which we shall demonstrate in the next section and in subsequent works.*

**Remark 6.3.** *If $a(t) = a_-(t)d(t)a_+(t)$ is a Wiener — Hopf factorization of $a(t)$, then for the Toeplitz operator $T_a$ the operator $T_a^\dagger := T_{a_+^{-1}} T_{d^{-1}} T_{a_-^{-1}}$ is its generalized inverse. Therefore, the kernel $T_a$ and the generalized inverse $T_a^\dagger = T_{\mathcal{R}} T_{d^{-1}} T_{\mathcal{L}}$ are described in terms of the indices and essential functions of $a(t)$. This fact is the analogue of the corresponding result for finite Toeplitz matrices. Thus, there exists a unified approach to the study of infinite–dimensional and finite–dimensional Toeplitz operators: this is the method of indices and essential functions (polynomials).*

**Remark 6.4.** *In this work we consider only the right Wiener — Hopf factorization. A similar theory can be constructed for left factorization (1.1) if one replaces assumption A by*

B. *The sequence $a := \{a_j\}_{j=-\infty}^{\infty}$ is the sequence of matrix Fourier coefficients for the matrix function $a^{-1}(t) \in W^{p\times p}$.*

## 7. Effective criterion for canonical factorization of strictly nonsingular matrix functions

In [12], we studied the problem on finding the partial indices of a strictly nonsingular matrix function

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \in GW^{2\times 2}, \quad a_{11}(t) \in GW.$$

The factorization of such matrix functions often appears in various problems of mathematical physics and mechanics, in particular, in the soliton theory [10] and in the diffraction theory [16]. Under the condition $\operatorname{ind} a_{11} = 2 \operatorname{ind} \det A$ the factorization problem for $A(t)$ was explicitly reduced to the factorization of the matrix function

$$a(t) = \begin{pmatrix} 1 & \beta_-(t) \\ \alpha_+(t) & 1 + \beta_-(t)\alpha_+(t) \end{pmatrix}, \tag{7.1}$$

where $\beta_-(t) \in W_-$, $\alpha_+(t) \in W_+$. In the particular case when these functions are Laurent polynomials, a criterion was obtained for the existence of a canonical factorization; however, the method of [12] did not allow one to construct this factorization.

In this section we shall show that the new factorization method based on indices and essential functions makes it possible to construct explicitly the canonical factorization of the Laurent matrix polynomial

$$a_N(t) = \begin{pmatrix} 1 & \beta_-^{(N)}(t) \\ \alpha_+^{(N)}(t) & 1 + \beta_-^{(N)}(t)\alpha_+^{(N)}(t) \end{pmatrix}, \tag{7.2}$$

where $\beta_-^{(N)}(t)$, $\alpha_+^{(N)}(t)$ are Laurent polynomials of degree at most $N$ for $\beta_-(t)$, $\alpha_+(t)$, respectively:

$$\beta_-^{(N)}(t) = \beta_0 + \beta_{-1}t^{-1} + \ldots + \beta_{-N}t^{-N}, \qquad \alpha_+^{(N)}(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_N t^N.$$

The factorization of $a_N(t)$ will be constructed in terms of the coefficients of $\beta_-^{(N)}(t)$, $\alpha_+^{(N)}(t)$.

Having found explicitly the canonical factorization of $a_N(t)$, we shall be able to obtain an effective criterion for the existence of the canonical factorization of $a(t)$. The method of obtaining such a criterion was proposed in [15]. In this work, for the practical implementation of the method, the ExactMPF package developed by the authors was used, which allows one to construct, by means of exact (error–free) computations, the factorization of matrix polynomials. Such an approach is suitable for obtaining the factorization of a specific matrix function $a(t)$ and is not suitable for the theoretical investigation of the problem. In this section we shall eliminate this shortcoming.

**7.1. Canonical factorization for $a_N(t)$.** Let us introduce the data necessary for what follows. We define the numerical matrix $D_N = I_N + K_N$, where $I_N$ is the identity matrix of order $N$ and $K_N$ is the product of Toeplitz matrices:

$$K_N = \begin{pmatrix} \alpha_N & \dots & \alpha_1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_N \end{pmatrix} \begin{pmatrix} \beta_{-N} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \beta_{-1} & \dots & \beta_{-N} \end{pmatrix}.$$

If the matrix $D_N$ is invertible and $H_N = \big(h_{i,j}^{(N)}\big)_{i,j=0}^{N-1}$ is its inverse matrix, then by $H_k^{(N)}$, $0 \leqslant k \leqslant N-1$, we denote the $k$-th column of $H_N$ and $H_k^{(N)}(t) = \sum\limits_{j=0}^{N-1} h_{jk}^{(N)} t^j$ stands for the generating polynomial of this column. We also introduce $H_N^{(N)}(t) := t^N$ and $H_\alpha^{(N)}(t) := \sum\limits_{k=0}^{N} \alpha_k H_k^{(N)}(t)$. We recall that $\mathcal{P}_+$ is the projection from $W$ into $W_+$ parallel to $(W_-)^0$, acting by the rule

$$\mathcal{P}_+\left( \sum_{k=-\infty}^{\infty} a_k t^k \right) = \sum_{k=0}^{\infty} a_k t^k,$$

and $\mathcal{P}_- = \mathcal{I} - \mathcal{P}_+$ is the complementary projection.

**Theorem 7.1.** *The Laurent matrix polynomial $a_N(t)$ admits a canonical factorization if and only if the matrix $D_N$ is invertible.*

*If this condition is satisfied, then the canonical factorization $a_N(t) = a_N^-(t) a_N^+(t)$, normalized by the condition $a_N^-(\infty) = I_2$, is constructed by the formulas*

$$\begin{aligned} a_N^-(t) &= \begin{pmatrix} 1 - \mathcal{P}_-\big(\beta_-^{(N)}(t) H_\alpha^{(N)}(t)\big) & \mathcal{P}_-\big(\beta_-^{(N)}(t) H_0^{(N)}(t)\big) \\ -\mathcal{P}_-\big(\beta_-^{(N)}(t)\alpha_+^{(N)}(t) H_\alpha^{(N)}(t)\big) & 1 + \mathcal{P}_-\big(\beta_-^{(N)}(t)\alpha_+^{(N)}(t) H_0^{(N)}(t)\big) \end{pmatrix}, \\ a_N^+(t) &= \begin{pmatrix} H_0^{(N)}(t) & \mathcal{P}_+\big(\beta_-^{(N)}(t) H_0^{(N)}(t)\big) \\ H_\alpha^{(N)}(t) & 1 + \mathcal{P}_+\big(\beta_-^{(N)}(t) H_\alpha^{(N)}(t)\big) \end{pmatrix}. \end{aligned} \tag{7.3}$$

*Proof.* By Theorem 6.1, the essential indices for the matrix function $a_N(t)$ coincide with the factorization indices of $a_N(t)$. The essential indices are equal to zero if and only if the kernel of the Toeplitz operator $T_0 := T_{a_N}$ is trivial. Indeed, if the essential indices are equal to zero, then $a_N(t)$ admits a canonical factorization, the operator $T_0$ is invertible and its kernel is trivial.

On the other hand, if $\ker T_0 = \{0\}$, then for the essential indices $\mu_1$, $\mu_2$ we have $\mu_2 \geqslant \mu_1 \geqslant 0$. If we suppose that $\mu_1 > 0$, then $0 = \operatorname{ind} \det a_N(t) = \mu_1 + \mu_2 > 0$, which is impossible. Therefore, $\mu_1 = \mu_2 = 0$. Thus, the triviality of the kernel of the operator $T_0$ is a criterion for the existence of a canonical factorization of $a_N(t)$.

The vector function $x_+(t) = \big(x_1(t), x_2(t)\big)^T$, $x_1(t), x_2(t) \in W_+$, belongs to the kernel of the Toeplitz operator $T_0$ if and only if this vector is a solution of the operator system of equations

$$\begin{cases} x_1(t) + T_{\beta_-^{(N)}} x_2(t) = 0, \\ T_{\alpha_+^{(N)}} x_1(t) + (\mathcal{I} + T_{\beta_-^{(N)}\alpha_+^{(N)}}) x_2(t) = 0. \end{cases}$$

This implies that the kernel of the operator $T_0$ consists of vector functions $x_+(t) = \big(-T_{\beta_-^{(N)}} x_2(t), x_2(t)\big)^T$, where $x_2(t)$ is an arbitrary function from the kernel of the operator $\mathcal{D} = \mathcal{I} + \big(T_{\beta_-^{(N)}} T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}} T_{\beta_-^{(N)}}\big)$. This means that the kernel of $T_0$ is trivial if and only if the kernel of $\mathcal{D}$ is trivial.

To find the kernel of the operator $\mathcal{D}$, we decompose the space $W_+$ into the direct sum $W_+ = P_N W_+ \oplus Q_N W_+$, where $P_N$ is the projection acting by the rule

$$P_N\left(\sum_{k=0}^{\infty} x_k t^k\right) = \sum_{k=0}^{N-1} x_k t^k,$$

and $Q_N = \mathcal{I} - P_N$ is the complementary projection. It is easy to see that the formula $Q_N = T_{t^N} T_{t^{-N}}$ holds.

Let us represent the operator $\mathcal{D}$ as an operator matrix corresponding to this decomposition of the space $W_+$:

$$\mathcal{D} = \begin{pmatrix} P_N\mathcal{D}P_N|\operatorname{Im} P_N & P_N\mathcal{D}Q_N|\operatorname{Im} Q_N \\ Q_N\mathcal{D}P_N|\operatorname{Im} P_N & Q_N\mathcal{D}Q_N|\operatorname{Im} Q_N \end{pmatrix}.$$

Let us find the elements of this operator matrix, taking into consideration that $\beta_-^{(N)}(t)$, $\alpha_+^{(N)}(t)$ are polynomials in $t^{-1}$, $t$, respectively, of degree at most $N$:

$$\begin{aligned} \mathcal{D}Q_N &= Q_N + \Big(T_{\beta_-^{(N)}} T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}} T_{\beta_-^{(N)}}\Big) T_{t^N} T_{t^{-N}} \\ &= Q_N + \Big(T_{\beta_-^{(N)} \alpha_+^{(N)} t^N} - T_{\alpha_+^{(N)}} T_{\beta_-^{(N)} t^N}\Big) T_{t^{-N}} \\ &= Q_N + \Big(T_{\beta_-^{(N)} \alpha_+^{(N)} t^N} - T_{\alpha_+^{(N)} \beta_-^{(N)} t^N}\Big) T_{t^{-N}} = Q_N. \end{aligned}$$

In the same we prove that $Q_N\mathcal{D} = Q_N$. This yields

$$P_N\mathcal{D}Q_N = 0, \qquad Q_N\mathcal{D}P_N = 0, \qquad Q_N\mathcal{D}Q_N = Q_N.$$

It remains to find $P_N\mathcal{D}P_N$

$$\begin{aligned} P_N\mathcal{D}P_N &= P_N + P_N\Big(T_{\beta_-^{(N)}} T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}} T_{\beta_-^{(N)}}\Big) P_N \\ &= P_N + P_N\Big(T_{t^{-N}\alpha_+^{(N)}} T_{t^N \beta_-^{(N)}} - T_{t^{-N}\alpha_+^{(N)}} T_{t^N} T_{t^{-N}} T_{t^N \beta_-^{(N)}}\Big) P_N \\ &= P_N + P_N T_{t^{-N}\alpha_+^{(N)}} (\mathcal{I} - Q_N) T_{t^N \beta_-^{(N)}} P_N = P_N + P_N T_{t^{-N}\alpha_+^{(N)}} P_N T_{t^N \beta_-^{(N)}} P_N. \end{aligned}$$

Finally,

$$\mathcal{D} = \begin{pmatrix} (P_N + \mathcal{K}_N)|\operatorname{Im} P_N & 0 \\ 0 & Q_N|\operatorname{Im} Q_N \end{pmatrix}, \tag{7.4}$$

where $\mathcal{K}_N = P_N T_{t^{-N}\alpha_+^{(N)}} P_N T_{t^N \beta_-^{(N)}} P_N$. Thus, $x_2(t) = P_N x_2(t) + Q_N x_2(t)$ belongs to the kernel of the operator $\mathcal{D}$ if and only if

$$(P_N + \mathcal{K}_N) P_N x_2(t) = 0, \qquad Q_N x_2(t) = 0.$$

In the standard basis the operator $(P_N + \mathcal{K}_N)|\operatorname{Im} P_N$ has the matrix $D_N = I_N + K_N$. We hence find that $x_2(t) = \sum\limits_{k=0}^{\infty} x_k t^k$ belongs to the kernel of $\mathcal{D}$ only in the case when the vector $(x_0, \ldots, x_{N-1})^T$ belongs to the kernel of the matrix $D_N = I_N + K_N$ and $x_N = x_{N+1} = \ldots = 0$.

This means that the kernel of $\mathcal{D}$ is trivial if and only if the matrix $D_N = I_N + K_N$ is invertible. The criterion for the canonical factorization of $a_N(t)$ is obtained.

Let the matrix $D_N$ be invertible; then the essential indices are equal to zero, $\mu_1 = \mu_2 = 0$. This means that the essential functions $R_1(t)$, $R_2(t)$ lie in the two–dimensional space $\ker T_1 := \ker T_{t^{-1} a_N}$. Let us find a basis of this space.

If $R(t) = \big(r_1(t), r_2(t)\big)^T \in \ker T_1$, then $T_{a_N} R(t) \in \ker T_{t^{-1} I_2}$. The latter kernel consists of constant vector functions. Thus,

$$\begin{cases} r_1(t) + T_{\beta_-^{(N)}} r_2(t) = u_1, \\ T_{\alpha_+^{(N)}} r_1(t) + (\mathcal{I} + T_{\beta_-^{(N)} \alpha_+^{(N)}}) r_2(t) = u_2, \end{cases}$$

where $u_1, u_2 \in \mathbb{C}$. Then $R(t) = \big(u_1 - T_{\beta_-^{(N)}} r_2(t), r_2(t)\big)^T$, where $r_2(t)$ is a solution of the operator equation $\mathcal{D} r_2(t) = u_2 - T_{\alpha_+^{(N)}} u_1$, i.e.,

$$\big[\mathcal{I} + \big(T_{\beta_-^{(N)}} T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}} T_{\beta_-^{(N)}}\big)\big] r_2(t) = u_2 - u_1 \alpha_+^{(N)}(t).$$

Taking into consideration formula (7.4), we obtain

$$(P_N + \mathcal{K}_N) P_N r_2(t) = u_2 - u_1 P_N \alpha_+^{(N)}(t), \qquad Q_N r_2(t) = Q_N(u_2 - u_1 \alpha_+^{(N)}(t)) = -u_1 \alpha_N t^N.$$

Thus, to find the Fourier coefficients $r_k^{(2)}$ of the function $r_2(t)$, we obtain the system

$$\begin{cases} D_N \begin{pmatrix} r_0^{(2)} \\ \vdots \\ r_{N-1}^{(2)} \end{pmatrix} = u_2 \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} - u_1 \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{pmatrix}, \\ r_N^{(2)} = -u_1 \alpha_N, \\ r_k^{(2)} = 0, k > N. \end{cases}$$

If $H_N$ is the matrix inverse to $D_N$, then

$$\begin{pmatrix} r_0^{(2)} \\ \vdots \\ r_{N-1}^{(2)} \end{pmatrix} = u_2 H_N \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} - u_1 H_N \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{pmatrix},$$

and we obtain

$$\begin{cases} P_N r_2(t) = u_2 H_0^{(N)}(t) - u_1 \sum\limits_{k=0}^{N-1} \alpha_k H_k^{(N)}(t), \\ Q_N r_2(t) = -u_1 \alpha_N t^N. \end{cases}$$

Recall that here $H_k^{(N)}(t)$ is the generating polynomial of the $k$-th column of the matrix $H_N$. Finally, we arrive at the expression for $r_2(t)$

$$r_2(t) = P_N r_2(t) + Q_N r_2(t) = u_2 H_0^{(N)}(t) - u_1 H_\alpha^{(N)}(t),$$

where $H_\alpha^{(N)}(t) = \sum\limits_{k=0}^{N} \alpha_k H_k^{(N)}(t)$. Hence,

$$r_1(t) = u_1\big(1 + T_{\beta_-^{(N)}} H_\alpha^{(N)}(t)\big) - u_2 T_{\beta_-^{(N)}} H_0^{(N)}(t).$$

Thus, an arbitrary vector function from $\ker T_1$ is represented as

$$R(t) = u_1 \begin{pmatrix} 1 + T_{\beta_-^{(N)}} H_\alpha^{(N)}(t) \\ -H_\alpha^{(N)}(t) \end{pmatrix} + u_2 \begin{pmatrix} -T_{\beta_-^{(N)}} H_0^{(N)}(t) \\ H_0^{(N)}(t) \end{pmatrix}.$$

The vector functions

$$R_1(t) = \begin{pmatrix} 1 + \mathcal{P}_+ \beta_-^{(N)} H_\alpha^{(N)}(t) \\ -H_\alpha^{(N)}(t) \end{pmatrix}, \qquad R_2(t) = \begin{pmatrix} -\mathcal{P}_+ \beta_-^{(N)} H_0^{(N)}(t) \\ H_0^{(N)}(t) \end{pmatrix}$$

form a basis of the space $\ker T_1$, i.e., they are the essential vector functions of $a_N(t)$.

By Theorem 6.1 we then obtain

$$\big(a_N^+(t)\big)^{-1} = \begin{pmatrix} 1 + \mathcal{P}_+ \beta_-^{(N)}(t) H_\alpha^{(N)}(t) & -\mathcal{P}_+ \beta_-^{(N)}(t) H_0^{(N)}(t) \\ -H_\alpha^{(N)}(t) & H_0^{(N)}(t) \end{pmatrix}.$$

The factor

$$a_N^-(t) = \begin{pmatrix} a_{11}^-(t) & a_{12}^-(t) \\ a_{21}^-(t) & a_{22}^-(t) \end{pmatrix}$$

can be found by the identity $a_N^-(t) = a_N(t)\big(a_N^+(t)\big)^{-1}$. Then

$$\begin{aligned}
a_{11}^-(t) &= 1 + \mathcal{P}_+\beta_-^{(N)}(t)H_\alpha^{(N)}(t) - \beta_-^{(N)}(t)H_\alpha^{(N)}(t)\\
&= 1 - \mathcal{P}_-\beta_-^{(N)}(t)H_\alpha^{(N)}(t) \in W_-, \qquad a_{11}^-(\infty) = 1,\\
a_{12}^-(t) &= \beta_-^{(N)}(t)H_0^{(N)}(t) - \mathcal{P}_+\beta_-^{(N)}(t)H_0^{(N)}(t) = \mathcal{P}_-\beta_-^{(N)}(t)H_0^{(N)}(t) \in W_-, \qquad a_{12}^-(\infty) = 0,\\
a_{21}^-(t) &= (\mathcal{P}_+ + \mathcal{P}_-)\big[\alpha_+^{(N)}(t) + \alpha_+^{(N)}(t)T_{\beta_-^{(N)}}H_\alpha^{(N)}(t) - H_\alpha^{(N)}(t) - \alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_\alpha^{(N)}(t)\big]\\
&= \alpha_+^{(N)}(t) - \big[\mathcal{I} + T_{\beta_-^{(N)}}T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}}T_{\beta_-^{(N)}}\big]H_\alpha^{(N)}(t) - \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_\alpha^{(N)}(t)\\
&= \alpha_+^{(N)}(t) - \mathcal{D}H_\alpha^{(N)}(t) - \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_\alpha^{(N)}(t)\\
&= -\mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_\alpha^{(N)}(t) \in W_-, \qquad a_{21}^-(\infty) = 0.
\end{aligned}$$

Here we have taken into consideration that $\mathcal{D}H_\alpha^{(N)}(t) = \sum\limits_{k=0}^{N}\alpha_k\mathcal{D}H_k^{(N)}(t)$, and since formula (7.4) implies immediately $\mathcal{D}H_k^{(N)}(t) = t^k$, we have $\mathcal{D}H_\alpha^{(N)}(t) = \alpha_+^{(N)}(t)$. Similar computations are made for the element $a_{22}^-(t)$:

$$\begin{aligned}
a_{22}^-(t) &= (\mathcal{P}_+ + \mathcal{P}_-)\big[H_0^{(N)}(t) + \alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_0^{(N)}(t) - \alpha_+^{(N)}(t)T_{\beta_-^{(N)}}H_0^{(N)}(t)\big]\\
&= \big[\mathcal{I} + T_{\beta_-^{(N)}}T_{\alpha_+^{(N)}} - T_{\alpha_+^{(N)}}T_{\beta_-^{(N)}}\big]H_0^{(N)}(t) + \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_0^{(N)}(t)\\
&= \mathcal{D}H_0^{(N)}(t) + \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_0^{(N)}(t) = 1 + \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_0^{(N)}(t) \in W_-,\\
a_{22}^-(\infty) &= 1.
\end{aligned}$$

This leads us to the following result:

$$a_N^-(t) = \begin{pmatrix} 1 - \mathcal{P}_-\beta_-^{(N)}(t)H_\alpha^{(N)}(t) & \mathcal{P}_-\beta_-^{(N)}(t)H_0^{(N)}(t)\\ -\mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_\alpha^{(N)}(t) & 1 + \mathcal{P}_-\alpha_+^{(N)}(t)\beta_-^{(N)}(t)H_0^{(N)}(t)\end{pmatrix} \in W_-^{2\times 2},$$

and moreover, $\det a_N^-(\infty) = 1$. Since $\det a_N^-(t) = \text{const}$, then $\det a_N^-(t) = 1$, and hence also $\det a_N^+(t) = 1$. Then

$$a_N^+(t) = \begin{pmatrix} H_0^{(N)}(t) & \mathcal{P}_+\beta_-^{(N)}(t)H_0^{(N)}(t)\\ H_\alpha^{(N)}(t) & 1 + \mathcal{P}_+\beta_-^{(N)}(t)H_\alpha^{(N)}(t)\end{pmatrix}.$$

The proof is complete. □

We are going to provide an example of the application of this theorem.

**Example 7.1.** *We consider the Laurent matrix polynomial*

$$a_3(t) = \begin{pmatrix} 1 & \beta_-^{(3)}(t)\\ \alpha_+^{(3)}(t) & 1 + \beta_-^{(3)}(t)\alpha_+^{(3)}(t)\end{pmatrix},$$

*where* $N = 3$ *and*

$$\alpha_+^{(3)}(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}, \qquad \beta_-^{(3)}(t) = 1 + \frac{1}{2t} + \frac{1}{2^2 2! t^2} + \frac{1}{2^3 3! t^3}.$$

*Here the Laurent polynomials of the functions* $\alpha_+(t) = e^t$, $\beta_-(t) = e^{1/2t}$ *have been taken as* $\alpha_+^{(3)}(t)$, $\beta_-^{(3)}(t)$, *respectively. All further computations were made in exact arithmetic by using the CAS Maple.*

*Since the matrix*

$$D_3 = I_3 + K_3 = \begin{pmatrix} \frac{451}{288} & \frac{13}{96} & \frac{1}{48} \\ \frac{13}{48} & \frac{307}{288} & \frac{1}{96} \\ \frac{1}{12} & \frac{1}{48} & \frac{289}{288} \end{pmatrix}$$

*is invertible, then $a_3(t)$ admits a canonical factorization. In order not to overload the exposition, we restrict ourselves to finding only the factor $a_3^+(t)$.*

*Let us find $H_3 = D_3^{-1}$ and write out the generating polynomials for the columns of this matrix*

$$H_0^{(3)}(t) = \frac{5109408}{7817645} - \frac{1294272}{7817645}t - \frac{79488}{1563529}t^2,$$
$$H_1^{(3)}(t) = -\frac{647136}{7817645} + \frac{7499232}{7817645}t - \frac{101952}{7817645}t^2,$$
$$H_2^{(3)}(t) = -\frac{19872}{1563529} - \frac{50976}{7817645}t + \frac{7799904}{7817645}t^2.$$

*By definition,*

$$H_3^{(3)}(t) = t^3, \qquad H_\alpha^{(3)}(t) = H_0^{(3)}(t) + H_1^{(3)}(t) + \frac{1}{2}H_2^{(3)}(t) + \frac{1}{6}H_3^{(3)}(t).$$

*We find $H_\alpha^{(3)}(t)$*

$$H_\alpha^{(3)}(t) = \frac{4412592}{7817645} + \frac{6179472}{7817645}t + \frac{680112}{1563529}t^2 + \frac{1}{6}t^3.$$

*We calculate $\beta_-^{(3)}(t)H_0^{(3)}(t)$, $\beta_-^{(3)}(t)H_\alpha^{(3)}(t)$, and in the obtained Laurent polynomials we select only the terms with nonnegative powers of $t$*

$$\mathcal{P}_+\big(\beta_-^{(3)}(t)H_0^{(3)}(t)\big) = \frac{4412592}{7817645} - \frac{1492992}{7817645}t - \frac{79488}{1563529}t^2,$$
$$1 + \mathcal{P}_+\big(\beta_-^{(3)}(t)H_\alpha^{(3)}(t)\big) = \frac{4542390029}{2251481760} + \frac{386045741}{375246960}t + \frac{9724873}{18762348}t^2 + \frac{1}{6}t^3.$$

*We have found all elements of the matrix polynomial $a_3^+(t)$: $H_0^{(3)}(t)$, $\mathcal{P}_+\big(\beta_-^{(3)}(t)H_0^{(3)}(t)\big)$, $H_\alpha^{(3)}(t)$, $1 + \mathcal{P}_+\big(\beta_-^{(3)}(t)H_\alpha^{(3)}(t)\big)$.*

**Remark 7.1.** *The factorization of $a_3(t)$ has been verified by means of the* ExactMPF *package* [14]. *After normalizing the factorization returned by* ExactMPF, *we obtain the same result as in Example 7.1.*

**Remark 7.2.** *It makes sense to compare the factorizations of $a_N(t)$ that are constructed by means of the* ExactMPF *package and by explicit formulas (7.3). The* ExactMPF *package is universal; it can construct the factorization of matrix polynomials not only of the form $a_N(t)$. However, the algorithm by which the package works is based on the notions of indices and essential polynomials of a certain finite sequence of numerical matrices* [1], [13]. *Therefore, it can be applied only for constructing the factorization of a matrix function $a_N(t)$ in which the polynomials $\alpha_+(t)$, $\beta_-(t)$ have numerical coefficients. The method of indices and essential functions proposed in this work allows one to construct the factorization for the case, when $\alpha_+(t)$, $\beta_-(t)$ have symbolic coefficients. This is very important for carrying out the theoretical investigation of the problem. Moreover, this method is economical; the factorization is constructed in terms of the inverse of one explicitly given matrix.*

**7.2. Asymptotics for norms $a_{\pm}^{(N)}(t)$.** In order to obtain an effective condition for the canonical factorization of $a(t)$, we shall need estimates for the norms $a_{\pm}^{(N)}(t)$.

The norm of matrix function $A(t) = (a_{ij}) \in W^{p\times p}$ obeys the estimate

$$\|A(t)\|_W \leqslant \sum_{i,j} \|a_{ij}\|_W.$$

The results obtained while using this estimate will be valid both for the norm $\|\cdot\|_{W,1}$ and for $\|\cdot\|_{W,\infty}$. We shall also need the obvious estimates

$$\|\beta_-^{(N)}(t)\|_W \leqslant \|\beta_-(t)\|_W, \qquad \|\alpha_+^{(N)}(t)\|_W \leqslant \|\alpha_+(t)\|_W.$$

In what follows $\|\cdot\|_W$ denotes $\|\cdot\|_{W,1}$.

Since $a_N^-(\infty) = I_2$, we have

$$\|a_N^-(t)\|_W = 1 + \|a_N^-(t) - I_2\|_W.$$

Formula (7.3) implies

$$\begin{aligned}\|a_N^-(t) - I_2\|_W \leqslant& \|\beta_-(t)\|_W\big(\|H_\alpha^{(N)}(t)\|_W + \|H_0^{(N)}(t)\|_W\big)\\ &+ \|\beta_-(t)\|_W\|\alpha_+(t)\|_W\big(\|H_\alpha^{(N)}(t)\|_W + \|H_0^{(N)}(t)\|_W\big)\\ =&\|\beta_-(t)\|_W\big(1 + \|\alpha_+(t)\|_W\big)\big(\|H_\alpha^{(N)}(t)\|_W + \|H_0^{(N)}(t)\|_W\big).\end{aligned}$$

By $\|\cdot\|_1$ we denote the Hölder norm for a vector and the maximum column norm for matrices. Then

$$\|H_0^{(N)}(t)\|_W = \|H_0^{(N)}\|_1 \leqslant \|H_N\|_1$$

and

$$\begin{aligned}\|H_\alpha^{(N)}(t)\|_W &\leqslant \sum_{k=0}^{N-1} |\alpha_k|\|H_k^{(N)}\|_1 + |\alpha_N|\\ &\leqslant \Big(\sum_{k=0}^{N-1} |\alpha_k|\Big)\|H_N\|_1 + |\alpha_N| \leqslant \|\alpha_+(t)||_W \max\{\|H_N\|_1, 1\}.\end{aligned}$$

Thus,

$$\|H_\alpha^{(N)}(t)\|_W + \|H_0^{(N)}(t)\|_W \leqslant \big(1 + \|\alpha_+(t)\|\big)_W \max\{\|H_N\|_1, 1\}$$

and, finally,

$$\|a_N^-(t)\|_W \leqslant 1 + \|\beta_-(t)\|_W\big(1 + \|\alpha_+(t)\|_W\big)^2 \max\{\|H_N\|_1, 1\}. \tag{7.5}$$

In the same way we obtain the estimate

$$\|a_N^+(t)\|_W \leqslant 1 + \big(1 + \|\beta_-(t)\|_W\big)\big(1 + \|\alpha_+(t)\|_W\big) \max\{\|H_N\|_1, 1\}. \tag{7.6}$$

Thus, we have obtained the asymptotics $\|a_N^{\pm}(t)\|_W = \mathrm{O}(\max\{\|H_N\|_1, 1\})$ as $N \to \infty$.

**7.3. An effective criterion for the canonical factorization of $a(t)$.** Now, relying on the results in [15], we can obtain an effective criterion for the canonical factorization of $a(t)$.

Let us approximate the matrix function $a(t)$ by the Laurent matrix polynomial $a_N(t)$. We denote $\delta_N = \|a(t) - a_N(t)\|_W$. Effective estimates of the quantity $\delta_N$ are given in [15].

**Theorem 7.2.** *If there exists a number $N$ such that the conditions hold*

1. *the matrix $D_N$ is invertible,*
2. *the inequality holds*

$$\begin{aligned}q_N :=&\big[1 + \|\beta_-(t)\|_W\big(1 + \|\alpha_+(t)\|_W\big)^2 \max\{\|H_N\|_1, 1\}\big]\\ &\cdot\big[1 + \big(1 + \|\beta_-(t)\|_W\big)\big(1 + \|\alpha_+(t)\|_W\big) \max\{\|H_N\|_1, 1\}\big]\delta_N < 1,\end{aligned}$$

*then $a(t)$ admits a canonical factorization. Here $H_N = D_N^{-1}$.*

*Proof.* We apply Theorem 4.2 in [15]. Condition 1) implies that a canonical factorization $a_N(t) = a_N^-(t)a_N^+(t)$ exists, i.e., the first condition of Theorem 4.2 is satisfied. To verify the second condition of this theorem, we shall need estimates of the norms $\|(a_N^\pm)^{-1}(t)\|_{W,1}$. Since $\det a_N^\pm(t) = 1$, these norms coincide with

$$\|(a_N^\pm)^\#(t)\|_{W,1} = \|a_N^\pm(t)\|_{W,\infty}.$$

Here $(a_N^\pm)^\#(t)$ is the matrix function adjoint to $a_N^\pm(t)$. As noted, estimates (7.5), (7.6) are valid both for $\|\cdot\|_{W,1}$ and for $\|\cdot\|_{W,\infty}$. Therefore, if condition 2) of the present theorem is satisfied, then the second condition of Theorem 4.2 is also satisfied. From this theorem it follows that $a(t)$ admits a canonical factorization. The proof is complete. □

Note that $a(t) \approx a_N^-(t)a_N^+(t)$ is an approximate factorization of $a(t)$. In [15] effective error estimates for this factorization are given.

The above criterion can be formulated without employing the factorization of $a_N(t)$, but using only the coefficients of the functions $\alpha_+(t)$, $\beta$.

**Corollary 7.1.** *If there exists a number $N_0$ such that*

1. *the matrix $D_N$ is invertible for $N \geqslant N_0$,*
2. $\sup\limits_{N\geqslant N_0} \|D_N^{-1}\| < \infty$,

*then $a(t)$ admits a canonical factorization. Here $\|\cdot\|$ is any matrix norm.*

*Proof.* Since $\delta_N \to 0$ as $N \to \infty$, and the sequence $\{\|H_N\|\}_{N\geqslant N_0}$ is bounded, the quantity $q_N$ can be made smaller than any prescribed number. Condition 2) of the preceding theorem is satisfied, and $a(t)$ admits a canonical factorization. The proof is complete. □

In conclusion, we consider an example.

**Example 7.2.** *Let*

$$a(t) = \begin{pmatrix} 1 & e^{\frac{1}{2t}} \\ e^t & 1 + e^{t+\frac{1}{2}t} \end{pmatrix}.$$

*In this example*

$$\|\alpha_+\|_W = \|e^t\|_W = e, \qquad \|\beta_-\|_W = \|e^{\frac{1}{2}t}\|_W = e^{\frac{1}{2}}.$$

*We estimate $\|a - a_N\|_W$*

$$\begin{aligned}\|a - a_N\|_W &\leqslant \|\alpha_+ - \alpha_+^{(N)}\|_W + \|\beta_- - \beta_-^{(N)}\|_W + \|\alpha_+\beta_- - \alpha_+^{(N)}\beta_-^{(N)}\|_W \\ &\leqslant (1 + \|\alpha_+\|_W)\|\beta_- - \beta_-^{(N)}\|_W + (1 + \|\beta_-\|_W)\|\alpha_+ - \alpha_+^{(N)}\|_W.\end{aligned}$$

*It remains to estimate $\|\alpha_+ - \alpha_+^{(N)}\|_W$, $\|\beta_- - \beta_-^{(N)}\|_W$:*

$$\|\alpha_+ - \alpha_+^{(N)}\|_W = \sum_{k=N+1}^{\infty} \frac{1}{k!} \leqslant \frac{1}{(N+1)!} \sum_{k=0}^{\infty} \frac{1}{(N+2)^k} \leqslant \frac{N+2}{(N+1)(N+1)!}.$$

*In the same way,*

$$\|\beta_- - \beta_-^{(N)}\|_W \leqslant \frac{N+2}{2^N(2N+3)(N+1)!}.$$

*Finally we obtain the required estimate*

$$\delta_N := \|a - a_N\|_W \leqslant (1+e)\frac{N+2}{2^N(2N+3)(N+1)!} + (1+e^{\frac{1}{2}})\frac{N+2}{(N+1)(N+1)!}.$$

*For $q_N$ we now have the expression*

$$q_N := \left[1 + e^{\frac{1}{2}}(1+e)^2 \max\{\|H_N\|_1, 1\}\right]\left[1 + (1+e^{\frac{1}{2}})(1+e)\max\{\|H_N\|_1, 1\}\right]\delta_N.$$

*The further computations are made by using the CAS Maple, and the results are summarized in the next table.*

Table 1.

| $N$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $\delta_N$ | 0.67713 | 0.14871 | 0.027543 | 0.0043788 | 0.00060676 | 0.000074285 | 0.0000081312 |
| $\det D_N$ | 1.6289 | 1.6363 | 1.6364 | 1.6364 | 1.6364 | 1.6364 | 1.6364 |
| $\|H_N\|$ | 1.0359 | 1.0550 | 1.0590 | 1.0597 | 1.0598 | 1.0598 | 1.0598 |
| $\|a_N^-\|_W$ | 24.614 | 25.050 | 25.139 | 25.156 | 25.159 | 25.159 | 25.159 |
| $\|a_N^+\|_W$ | 11.202 | 11.391 | 11.429 | 11.437 | 11.438 | 11.438 | 11.438 |
| $q_N$ | 186.72 | 42.436 | 7.9144 | 1.2598 | 0.17461 | 0.021378 | 0.0023401 |

*The estimates for the parameters $\delta_N$, $\|a_N^{\pm}\|_W$ are given in the table. All values are indicated with 5 significant digits.*

*All the quantities given in the table show a tendency to stabilize. For all $N = 2, \ldots, 8$ we have $\det D_N \neq 0$, which, in accordance with the criterion of canonical factorization, means that $a_N(t)$, $N = 2, \ldots, 8$, admits a canonical factorization. The parameter $q_N$ becomes less than 1 starting from $N = 6$. This means that the matrix function $a(t)$ admits a canonical factorization.*

## BIBLIOGRAPHY


1. V.M. Adukov. *Factorization of analytic matrix–valued functions* // Theor. Math. Phys. **118**:3, 255–263 (1999). https://doi.org/10.1007/BF02557319
2. N.P. Vekua. *Systems of singular integral equations*. Nauka, Moscow (1970); *English translation:* P. Noordhoff, Ltd., Groningen, The Netherlands (1967).
3. V.V. Voevodin, Yu.A. Kuznetsov. *Matrices and computations*. Nauka, Moscow (1984). (in Russian).
4. I. Gohberg, N. Krupnik. *Introduction into Theory of One–Dimensional Singular Integral Operators*. Shtiintsa, Kishinev (1973); *English translation: One–Dimensional Linear Singular Integral Equations. I. Introduction*. Birkhäuser Verlag, Basel (1992) & *One–Dimensional Linear Singular Integral Equations. Vol. II: General Theory and Applications*. Birkhäuser Verlag, Basel (1992).
5. I.Ts. Gokhberg. *A factorization problem in normed rings, functions of isometric and symmetric operators and singular integral equations* // Russ. Math. Surv. **19**:1, 63–114 (1964). https://doi.org/10.1070/rm1964v019n01ABEH004137
6. I.Ts. Gokhberg, M.G. Krein. *Systems of integral equations on the half–line with kernels depending on the difference of the arguments* // Usp. Matem. Nauk. **13**:2(80), 3–72 (1958). (in Russian). https://www.mathnet.ru/eng/rm7414
7. I.C. Gohberg, I.A. Fel'dman. *Convolution equations and projection methods for their solution*. Amer. Math. Soc. Providence, R.I. (1974).
8. A.A. Kirillov, A.D. Gvishiani. *Theorems and problems in functional analysis*. Nauka, Moscow (1979); *English translation:* Springer–Verlag, New York (1982).
9. N.I. Muskhelishvili. *Singular integral equations*. Nauka, Moscow (1968); *English translation:* Wolters–Noordhoff Publishing, Groningen (1967).
10. L.D. Faddeev, L.A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons*. Nauka, Moscow (1986); *English translation:* Springer, Berlin (2007). https://doi.org/10.1007/978-3-540-69969-9.
11. I.D. Abrahams. *On the application of the Wiener — Hopf technique to problems in dynamic elasticity* // Wave Motion **36**:4, 311–333 (2002). https://doi.org/10.1016/S0165-2125(02)00027-6

12. V.M. Adukov. *On factorization indices of strictly nonsingular* $2 \times 2$ *matrix function* // Integral Equations Oper. Theory **21**:1, 1–11 (1995). https://doi.org/10.1007/BF01262988
13. V.M. Adukov. *Generalized inversion of block Toeplitz matrices* // Linear Algebra Appl. **274**, 85–124 (1998). https://doi.org/10.1016/S0024-3795(97)00304-2
14. V.M. Adukov, N.V. Adukova, G. Mishuris. *An explicit Wiener — Hopf factorisation algorithm for matrix polynomials and its exact realizations within ExactMPF package* // Proc. R. Soc. Lond., A, Math. Phys. Eng. Sci. **478**:2263, 20210941 (2022). https://doi.org/10.1098/rspa.2021.0941
15. V.M. Adukov, N.V. Adukova, G. Mishuris. *An effective criterion for a stable factorisation of a strictly nonsingular* $2 \times 2$ *matrix functions: use of the ExactMPF package* // Proc. R. Soc. Lond., A, Math. Phys. Eng. Sci. **480**:2299, 20249116 (2024). https://doi.org/10.1098/rspa.2024.0116
16. V.G. Daniele, R.S. Zich. *The Wiener — Hopf Method in Electromagnetics.* SciTech Publishing, Edison, NJ (2014).
17. V.G. Daniele, G. Lombardi. *Scattering and diffraction by wedges 1: the Wiener — Hopf solution. Theory.* ISTE, London; John Wiley & Sons, Hoboken, New York (2020). https://doi.org/10.1002/9781119476733
18. I.C. Gohberg, M.A. Kaashoek, I.M. Spitkovsky. *An overview of matrix factorization theory and operator applications* // in "Factorization and integrable systems", Birkhäuser, Basel **141**, 1–102 (2003).
19. A. Kisil, G. Mishuris. *Special issue: advances in Wiener — Hopf techniques: theory and applications* // Proc. R. Soc. Lond., A, Math. Phys. Eng. Sci. **479**:2273, 20230210 (2023). https://doi.org/10.1098/rspa.2023.0210
20. G.S. Litvinchuk, I.M. Spitkovsky. *Factorization of measurable matrix functions.* Birkhäuser, Basel–Boston (1987).

Viktor Mikhailovich Adukov,
Institute of Natural Science and Mathematics,
South Ural State University,
Lenin av. 76,
454080, Chelyabinsk, Russia
E-mail: adukovvm@susu.ru