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CONVERGENCE OF PENALTY AND REGULARIZATION METHOD FOR CONTACT PROBLEMS IN LOCKING MATERIALS WITH NORMAL COMPLIANCE

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Abstract. We study a static contact problem involving a locking body and an obstacle considering normal compliance and frictional effects. The mechanical behavior follows an elastic constitutive law for a locking material, which is modeled by using a subdifferential approach. The weak formulation is presented as a variational inequality for the displacement field. We establish the existence and uniqueness of the weak solution under suitable regularity conditions. Next, we introduce a penalty formulation expressed as a variational identity achieved by integrating the penalization of the terms in the constitutive law on one hand and, on the other hand, by regularizing the contact condition. Under the scaling relation $\epsilon = \frac{1}{n}$ between the penalty parameters, we establish the convergence of the penalized solutions as $n \to \infty$. Finally, we analyze error estimates for the finite element discretization of the penalty method, showing that for the discretization parameter $h = \frac{1}{n^2}$ the error terms yield estimates comparable to the constrained problem case. Our analysis demonstrates consistent convergence behavior between the penalized and constrained formulations.

Keywords: locking material, normal compliance, variational inequality, penalty method, finite element.

Mathematics Subject Classification: 74B15, 74G30, 74H15, 65L60

1. Introduction

Locked materials exhibit a mechanical response constrained by an impassable limit, typically in deformation or sliding. In elasticity, this locking effect induces a blockage of deformation at the threshold s, generating unilateral conditions and nonlinearities in contact laws. They are widely used in engineering, particularly in civil engineering, aeronautics, automotive applications, and biomedical devices.

The analysis of variational problems for such materials was first introduced by Prager [23], [25], [24] and later expanded upon by Demengel and Suquet in 1985 [9], [10]. They investigated the fundamental properties of elastic–locking materials, using a penalty method to establish the inf–sup condition and to analyze the problem of locking limit. Locking materials are characterized by a limited elastic range, within which the strain tensor is constrained to a subset of a convex set denoted by C. Additionally, it is assumed that a normality rule holds in the strain space.

Bourichi et al. proposed a penalty–based approach to model elastic–locking materials in a frictional unilateral contact problem [4]. The study of elastic locking materials incorporating memory effects was further explored mathematically in [29]. More recently, in [13], a novel

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framework was presented for analyzing contact problems involving electroelastic–locking materials interacting with a conductive foundation under frictional conditions.

In 2024, Bouallala [5] studied a viscoelastic contact problem featuring ideal locking behavior, governed by the Signorini condition and nonlocal Coulomb friction. The work presented the mathematical formulation, existence and uniqueness results were proved, and a finite element approximation along with its error estimates were analyzed.

In [2], [15], [20], [26] contact problems with normal compliance and friction were studied, both their mathematical formulation and numerical resolution. They examined the effects of normal compliance on the mechanical response of the system, particularly its impact on the formation and evolution of contact under the influence of friction. We also mention some more works [1], [6], [7], [14], [16], [17], [18], [21], which were devoted to investigating numerical analysis schemes and their error estimates for the dynamic and quasi–static contact problem, with or without friction.

Penalty methods have recently gained popularity for solving constrained problems in mechanics, such as unilateral contact problems and problems with Dirichlet boundary conditions. This approach enables the use of standard methods to solve the resulting nonlinear algebraic equations, as demonstrated by Chouly and Hild [8] and other works [3], [11], [12]. Inexact integration is required to obtain physically meaningful solutions, see, for instance, [19], [30].

In this paper we deal with a theoretical model for frictional contact problems incorporating normal compliance. The study involves an elastic body subjected to mechanical forces and surface tractions and interacting with a deformable foundation that permits penetration. The contact interaction is modeled using the normal compliance condition, while friction is described by Coulomb law. The analysis focuses on this fundamental contact framework with normal compliance.

The primary objective of this study is to investigate the convergence properties of the penalty method applied to a frictional contact problem governed by normal compliance. Subsequently, we derive the discrete formulation using finite elements and establish the convergence of the numerical solution through a mathematical proof.

The paper is organized as follows. First, we present the governing equations, notations, and assumptions for the contact problem, casting it as a variational inequality with proven existence/uniqueness of solutions followed by the derivation of apriori estimates. To address nonlinearities and nonsmoothness, we employ a penalty method, analyze its convergence, and subsequently discretize the problem using linear finite elements. The analysis ensures theoretical convergence across both continuous and discrete settings.

2. Mathematical model

We consider the problem of contact with normal compliance between a locking body and a deformable foundation. The physical setting is as follows. We consider a body made of a locking material that occupies the domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, in the reference configuration. The domain is assumed to be bounded with a smooth boundary $\partial\Omega = \Gamma$. This boundary is partitioned into three open disjoint parts: Γ_D, Γ_N, and Γ_C. Additionally, we consider a partition of $\Gamma_D \cup \Gamma_N$. We assume that the measure of Γ_D is positive, i.e., $\text{meas}(\Gamma_D) > 0$, and the body is subjected to volume forces with density f_0. The body is fixed on Γ_D, implying that the displacement field vanishes there. Surface tractions with density f_N act on Γ_N. The body is in contact with a deformable obstacle along Γ_C, and the contact is frictional with normal compliance.

Now we introduce some the notation and preliminary facts, which will be used in the next sections. We use standard notations $\mathbb{R}^d$, $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ and $\|\cdot\|$; by $\mathbb{S}^d$ we denote the symbol the space of second order symmetric tensors on $\mathbb{R}^d$, or, equivalently, the space of symmetric matrices of size $d \times d$. We shall also use scalar products $u \cdot v := \sum_i u_i v_i$, $u, v \in \mathbb{R}^d$ and $\sigma \cdot \tau := \sum_{i,j} \sigma_{ij}\tau_{ij}$, $\sigma, \tau \in \mathbb{S}^d$. Similarly, the normal and tangential components of the displacement u and the stress tensor σ, with respect to the outward unit normal vector ν on the boundary Γ, are respectively defined by

$$u_\nu = u \cdot \nu, \qquad u_\tau = u - u_\nu \nu, \qquad \sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

where ν and τ denote the outward unit normal vector and the tangential vector on the surface Γ, respectively.

The model for this process is as follows.

Problem (P): Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$ such that

$$\sigma(u) \in \mathscr{A}\mathcal{E}(u) + \partial\left(I_C(\mathcal{E}(u))\right) \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\operatorname{Div}\sigma(u) + f_0 = 0 \quad \text{in} \quad \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{2.3}$$

$$\sigma\nu = f_N \quad \text{on} \quad \Gamma_N. \tag{2.4}$$

The frictional contact on Γ_C with normal compliance condition reads

$$-\sigma_\nu(u) = \varpi_\nu\left(u_\nu - g\right) \quad \text{on} \quad \Gamma_C, \tag{2.5}$$

and

$$\begin{aligned} &\|\sigma_\tau(u)\| \leqslant \varpi_\tau\left(u_\nu - g\right), \qquad \|\sigma_\tau(u)\| < \varpi_\tau\left(u_\nu - g\right) \to u_\tau = 0 \quad \text{on} \quad \Gamma_C, \\ &\|\sigma_\tau(u)\| = \varpi_\tau\left(u_\nu - g\right) \to \sigma_\tau = -\lambda u_\tau, \quad \lambda > 0. \end{aligned} \tag{2.6}$$

The indication function I_C of the set $C = \left\{\phi \in \mathbb{S}^d : \ \|\phi\| \leqslant s\right\}$ is defined as

$$I_C(\phi) = 0 \qquad \text{if} \quad \phi \in C, \qquad I_C(\phi) = +\infty \qquad \text{otherwise.}$$

Equation (2.1) represents the constitutive law of the material. The equilibrium of the stress and displacement fields is described by equation (2.2). The boundary conditions for displacement and traction are specified by relations (2.3) and (2.4), respectively. Equation (2.5) represents the normal compliance law describes a constitutive relationship between the normal stress σ_ν and the displacement gap $(u_\nu - g)$ on the contact boundary Γ_C. It is commonly used in contact mechanics to model elastic or semi–rigid interfaces, such as in friction problems, biomechanics, or granular materials. Here

- $\sigma_\nu(u)$ is the normal stress (or contact pressure), acting perpendicular to the surface Γ_C.
- u_ν is the normal displacement of the body at the contact interface.
- g denotes the gap function (or reference distance), representing the initial clearance between the body and the obstacle. If $u_\nu \leqslant g$, no contact occurs ($\sigma_\nu = 0$).
- ϖ_ν is the *compliance coefficient* (or penalty parameter), which quantifies the stiffness of the interface. A higher ϖ_ν implies a more rigid contact (smaller penetration).

Finally, friction law (2.6) governs the tangential contact behavior on Γ_C, coupling shear stress σ_τ with normal displacement u_ν. It generalizes Coulomb friction by incorporating normal compliance $\varpi_\tau(u_\nu - g)$, where

- $\sigma_\tau(u)$: Shear stress (tangential to Γ_C).
- u_τ: Tangential displacement.
- $u_\nu - g$: Effective gap (drives frictional resistance).
- ϖ_τ: Friction coefficient (scales with normal compliance).

- λ: Slip multiplier (ensures σ_τ opposes slip direction).

The physical interpretation is as follows.

- First condition: The shear stress cannot exceed the frictional capacity $\varpi_\tau(u_\nu - g)$.
- Second condition: If $\|\sigma_\tau(u)\| < \varpi_\tau(u_\nu - g)$, no slip occurs ($u_\tau = 0$).
- Third condition: When $\|\sigma_\tau(u)\| = \varpi_\tau(u_\nu - g)$, slip initiates and $\sigma_\tau = -\lambda u_\tau$ with $\lambda > 0$.

To formulate the variational problem, we introduce the Hilbert spaces

$$H = L^2(\Omega)^d, \qquad H_1 = H^1(\Omega)^d,$$
$$\mathscr{H} = \left\{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \qquad \mathscr{H}_1 = \{\sigma \in \mathscr{H} \mid \operatorname{Div}\sigma \in H\},$$

equipped with the scalar products

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad (u,v)_{H_1} = (u,v)_H + (\mathcal{E}(u), \mathcal{E}(v))_{\mathscr{H}},$$

$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx, \quad (\sigma,\tau)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_{\mathscr{H}},$$

and the associated norms $\|\cdot\|_H, \|\cdot\|_{H_1}, \|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}_1}$, respectively.

To considering boundary condition (2.5), we define

$$V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_D\}.$$

Let V' be the dual of V, the for all $v \in V$, $f \in V'$ we denote $\langle f, v\rangle = f(v)$.

We consider the closed convex set

$$K = \{v \in V,\ \|\mathcal{E}(v)\| \leqslant s \text{ a.e. on } \Omega\}.$$

We know that meas $(\Gamma_D) > 0$ and by the Korn's inequality we conclude, see, e.g., [27],

$$\|\mathcal{E}(v)\|_{\mathscr{H}} \geqslant c_k \|v\|_{H_1},$$

where $c_k > 0$ depends only on Ω and Γ_D. We define the scalar product in V

$$(u,v)_V = (\mathcal{E}(u), \mathcal{E}(v))_{\mathscr{H}}, \qquad \|u\|_V = (u,u)_V^{\frac{1}{2}}, \tag{2.7}$$

and denote by $\|\cdot\|_V$ the norm in V associated with this scalar product.

By the Sobolev trace theorem, there exists $c_0 > 0$ such that

$$\|u\|_{L^2(\Gamma)^d} \leqslant c_0 \|u\|_V.$$

To treat the problem (2.1)-(2.6), we define the functions p_r with $r = \nu$ or $r = \tau$ such that

$$\begin{aligned}
&(a) \quad p_r : \Gamma_C \times \mathbb{R}^+ \to \mathbb{R}^+;\\
&(b) \quad \text{there exists } L_r > 0 \text{ such that for all } u, v \in \mathbb{R}^+ \text{ and a.e. } x \in \Gamma_C;\\
&\qquad |p_r(x,u) - p_r(x,v)| \leqslant L_r \|u - v\|;\\
&(c) \quad \text{the function } x \to p_r(x,u) \text{ is Lebesgue measurable on } \ \Gamma_C \text{ for all } u \in \mathbb{R}^+;\\
&(d) \quad p_r(x,s) = 0 \text{ for } s \leqslant 0 \text{ and a.e. } x \in \Gamma_C;\\
&(e) \quad p_r \text{ is bounded: there exists } \mu^* > 0 \ p_r(x,u) \leqslant \mu^* \text{ for all } u \in \mathbb{R}^+ \text{ and a.e. } x \in \Gamma_C.
\end{aligned} \tag{2.8}$$

We define the element $f \in V$ and the mapping and $j : V \times V \to \mathbb{R}$

$$(f,v)_V = \int_\Omega f_0 \cdot v dx + \int_{\Gamma_N} f_N \cdot v da \quad \text{for all} \quad v \in V, \tag{2.9}$$

and

$$\ell(u,v)=\int\limits_{\Gamma_C}\varpi_\nu\left(u_\nu-g\right)v_\nu da+\int\limits_{\Gamma_C}\varpi_\tau\left(u_\nu-g\right)\|v_\tau\|\,da\quad\text{for all}\quad v\in V.$$

By applying (2.1) and Green formula, we obtain

$$(\sigma(u),\mathcal{E}(v))_{\mathscr{H}}-\langle\sigma_\nu(u),v_\nu\rangle_{\Gamma_C}-\langle\sigma_\tau(u),v_\tau\rangle_{\Gamma_C}=(f,v)_V,$$

and

$$(\mathscr{A}\mathcal{E}(u),\mathcal{E}(v))_{\mathscr{H}}+(\mathscr{Z}(\mathcal{E}(u)),\mathcal{E}(v))_{\mathscr{H}}-\langle\sigma_\nu(u),v_\nu\rangle_{\Gamma_C}-\langle\sigma_\tau(u),v_\tau\rangle_{\Gamma_C}=(f,v)_V,$$

with

$$\mathscr{Z}(\mathcal{E}(u))\in\partial I_C(\mathcal{E}(u)).$$

We consider the bilinear form on $V\times V$

$$a(u,v)=(\mathscr{A}\mathcal{E}(u),\mathcal{E}(v))_{\mathscr{H}}.$$

Using the above assumptions, we see that $a(\,\cdot\,,\,\cdot\,)$ is bilinear, symmetric, V–elliptic and continuous on $V\times V$ imply that there exists $\alpha>0$ such that

$$a(v-w,v-w)\geqslant\alpha\|v-w\|_V^2.\tag{2.10}$$

The convexity of J_C implies

$$\langle\mathscr{Z}(\mathcal{E}(u)),\mathcal{E}(v-u)\rangle\leqslant I_C(\mathcal{E}(v))-J_C(\mathcal{E}(u))=0,\quad\text{for all}\quad v\in K.$$

Using (2.5), (2.7) and (2.10) to get the variational **Problem** (PV).

Problem (PV): Find a displacement field $u:\Omega\to\mathbb{R}^d$ such that

$$\begin{cases}u\in K,\\ a(u,v-u)+\ell(u,v)-\ell(u,u)\geqslant(f,v-u)_V\quad\text{for all}\quad v\in K.\end{cases}\tag{2.11}$$

Theorem 2.1. *Assume that* (2.8), (2.9), (2.10) *and*

$$c_0^2\left(L_\nu+L_\tau\right)<\alpha$$

hold. Then ***Problem*** (PV) *has a unique solution.*

Proof. In view of the closing convexity of K, by using the V-ellipticity, continuity and bilinearity of the mapping $a(\,\cdot\,,\,\cdot\,)$ we verify that for all $u\in V$, $j(u,\,\cdot\,)$ is convex and lower semi continuous on V, and for all u_1, $u_2\in V$ we have

$$\begin{aligned}&\left|\ell\left(u_1,u_2\right)-\ell\left(u_1,u_1\right)+\ell\left(u_2,u_1\right)-\ell\left(u_2,u_2\right)\right|\\ &\quad=\Bigg|\int\limits_{\Gamma_C}\left(\varpi_\nu\left(u_{1\nu}-g\right)-\varpi_\nu\left(u_{2\nu}-g\right)\right)\left(u_{2\nu}-u_{1\nu}\right)da\\ &\qquad+\int\limits_{\Gamma_C}\left(\varpi_\tau\left(u_{1\nu}-g\right)-\varpi_\tau\left(u_{2\nu}-g\right)\right)\left(\|u_{2\tau}\|-\|u_{1\tau}\|\right)da\Bigg|\\ &\quad\leqslant\int\limits_{\Gamma_C}L_\nu\left|u_{2\nu}-u_{1\nu}\right|^2da+\int\limits_{\Gamma_C}L_\tau\|u_{2\nu}-u_{1\nu}\|\|u_{2\tau}-u_{1\tau}\|da\\ &\quad\leqslant c_0^2v\left(L_\nu+L_\tau\right)\|u_2-u_1\|_V^2.\end{aligned}$$

Hence,

$$\left|\ell\left(u_1,u_2\right)-\ell\left(u_1,u_1\right)+\ell\left(u_2,u_1\right)-\ell\left(u_2,u_2\right)\right|\leqslant c_0^2\left(L_V+L_\tau\right)\|u_2-u_1\|_V^2.$$

Now, by using [21, Thm. 3.7] we conclude that problem (2.11) has a unique solution. The proof is complete. □

3. Penalty problem

In this section we focus on the penalization method. This approach involves, first, penalizing the terms of the constitutive law arising from the blocking property, and second, regularizing the contact conditions used in the modeling.

For each $\rho \in \mathbb{R}$, we define

$$\rho^+ = \rho \quad \text{if} \quad \rho \geqslant 0, \qquad \rho^+ = 0 \quad \text{if} \quad \rho \leqslant 0.$$

Hereinafter we shall extensively use the following properties:

$$\rho \leqslant \rho^+, \qquad \rho\rho^+ = \rho^{+2}, \quad \rho \in \mathbb{R}.$$

The penalty method consist in the regularization of the behavior law and contact condition, namely, for all $n \in \mathbb{N}^*$ and $\epsilon > 0$ we set $u^\epsilon_{n,\nu} = u^\epsilon_n \nu$. In the penalty problem, we replace $I_C\left(\mathcal{E}\left(u^\epsilon_n\right)\right)$ by $n\left[\left(\|\mathcal{E}\left(u^\epsilon_n\right)\| - s\right)^+\right]^2$ and ℓ by ℓ_ϵ such that

$$\ell_\epsilon(u,v) = \int\limits_{\Gamma_C} \varpi_\nu \left(u_\nu - g\right) v_\nu da + \int\limits_{\Gamma_C} \varpi_\tau \left(u_\nu - g\right) \Psi_\epsilon(v)\, da,$$

where

$$\Psi_\epsilon(v) = \sqrt{\|v\|^2 + \epsilon^2}, \quad v \in V.$$

We denote

$$\langle \ell'_\epsilon(u,v), w\rangle = \int\limits_{\Gamma_C} \varpi_\nu \left(u_\nu - g\right) w_v\, da + \int\limits_{\Gamma_C} \varpi_\tau \left(u_\nu - g\right) \frac{v_\tau w_\tau\, da}{\sqrt{\|v_\tau\|^2 + \epsilon^2}}, \quad v, w \in V, \tag{3.1}$$

and

$$\mathscr{R}_n(\phi) = 2n\left(\left(\|\phi\| - s\right)^+\right)\frac{\phi}{\|\phi\|}, \quad \phi \in \mathbb{S}^d. \tag{3.2}$$

By applying the Green formula, we get the following formulation:

Problem $\left(PV^{\mathcal{E}}_n\right)$: Find a displacement field $u^\epsilon_n \in V$ such that

$$a\left(u^\epsilon_n, v\right) + \mathscr{R}_n\left(\mathcal{E}\left(u^\epsilon_n\right), \mathcal{E}(v)\right)_{\mathscr{H}} + \left\langle \ell'_\epsilon\left(u^\epsilon_n, u^\epsilon_n\right), v\right\rangle = (f, v)_V.$$

3.1. Existence and uniqueness of penalized problem.

Theorem 3.1. *Let $n \in \mathbb{N}^*$ and $\epsilon > 0$. Then*

1. *The penalty **Problem** (PV^ϵ_n) admits at least one solution.*
2. *Suppose that $c_0^2\left(L_\nu + L_\tau\right) < \alpha$, then the **Problem** (PV^ϵ_n) has a unique solution u^ϵ_n.*

Before proceeding with the proof of this result, we make some preliminary preparations. By applying the Riesz representation theorem, we introduce a (nonlinear) operator B^ϵ_n

$$(B^\epsilon_n v, w)_V = a(v,w) + \mathscr{R}_n(\mathcal{E}(v), \mathcal{E}(w))_{\mathscr{H}}, \qquad v, w \in V.$$

Then the problem (PV^ϵ_n) is rewritten as

Problem PV^ϵ_n : Find the displacement $u^\epsilon_n : \Omega \to \mathbb{R}^d$ such that

$$(B^\epsilon_n u^\epsilon_n, v)_V + \left\langle \ell'_\epsilon\left(u^\epsilon_n, u^\epsilon_n\right), v\right\rangle = (f, v)_V, \quad v \in V.$$

We define

$$G=\left\{z\in L^2\left(\Gamma_C\right);\|z\|_{L^2(\Gamma_C)}\leqslant\kappa\right\},$$

κ is a real positive number. For each $z\in G$ we consider the mapping $\widetilde{\ell}_z$ on V

$$\widetilde{\ell}_z(v)=\int\limits_{\Gamma_C} z\cdot v\,da.$$

According to the Riesz representation theorem, there exists an element $f^z\in V$ such that

$$(f^z,v)_V=(f,v)_V-\widetilde{\ell}_z(v).$$

For each $z\in G$ we consider the problem

Problem $(PV_n^{z\epsilon})$: Find $u_n^{z\epsilon}:\Omega\to\mathbb{R}^d$ such that

$$a\left(u_n^{z\epsilon},v\right)+\mathscr{R}_n\left(\mathcal{E}\left(u_n^{h\epsilon}\right),\mathcal{E}(v)\right)_{\mathscr{H}}=\left(f^z,v\right)_V.$$

Let us show that the operator B_n^ϵ is both strongly monotone and Lipschitz continuous. Indeed, the function $\mathrm{H}:\mathbb{S}^d\to\mathbb{R}$, $\phi\longmapsto n\left(\left(\|\phi\|-s\right)^+\right)^2$ is continuously differentiable and convex, and for all $\phi,\psi\in\mathbb{S}^d$ we have

$$\langle\mathrm{H}'(\phi),\psi\rangle=\mathscr{R}_n(\phi,\psi)_{\mathscr{H}}.$$

This yields

$$\langle H'(\mathcal{E}(u)),\mathcal{E}(v)\rangle=\mathscr{R}_n(\mathcal{E}(u),\mathcal{E}(v))_{\mathscr{H}}.$$

By the convexity of H we see that H' is monotone and hence,

$$\langle\mathrm{H}'(\mathcal{E}(u))-\mathrm{H}'(\mathcal{E}(v)),\mathcal{E}(u)-\mathcal{E}(v)\rangle\geqslant 0.$$

for all $u,v\in V$, and therefore,

$$(B_n^\epsilon v-B_n^\epsilon u,v-u)_V\geqslant a(v-u,v-u)\geqslant\alpha\|v-u\|_V^2.$$

Thus, B_n^ϵ is strongly monotone.

We proceed to proving the Lipschitz property for B_n^ϵ. Let $u_1,u_2,w\in V$. We know that $a(\cdot,\cdot)$ is bilinear continuous and we let

$$\begin{aligned}\mathrm{R}&=\mathscr{R}_n\left(\mathcal{E}\left(u_1\right),\mathcal{E}(v)\right)_{\mathscr{H}}-\left(\mathscr{R}_n\left(\mathcal{E}\left(u_2\right)\right),\mathcal{E}(w)\right)_{\mathscr{H}}\\&=2n\int\limits_\Omega\left(\left[\|\mathcal{E}\left(u_1\right)\|-s\right]^+\frac{\mathcal{E}\left(u_1\right)}{\|\mathcal{E}\left(u_1\right)\|}-\left[\|\mathcal{E}\left(u_2\right)\|-s\right]^+\frac{\mathcal{E}\left(u_2\right)}{\|\mathcal{E}\left(u_2\right)\|}\right)\mathcal{E}(v)\,da\\&=2n\int\limits_\Omega\left(\frac{\left[\|\mathcal{E}\left(u_1\right)\|-s\right]^+\mathcal{E}\left(u_1\right)\|\mathcal{E}\left(u_2\right)\|-\left[\|\mathcal{E}\left(u_2\right)\|-s\right]^+\mathcal{E}\left(u_2\right)\|\mathcal{E}\left(u_1\right)\|}{\|\mathcal{E}\left(u_2\right)\|\,\|\mathcal{E}\left(u_1\right)\|}\right)\mathcal{E}(w)\,da.\end{aligned}$$

To simplify the arguing, we treat the following cases. If $\|\mathcal{E}\left(u_1\right)\|\leqslant s$, $\|\mathcal{E}\left(u_2\right)\|\leqslant s$, then $R=0$. If $\|\mathcal{E}\left(u_1\right)\|>s$, $\|\mathcal{E}\left(u_2\right)\|\leqslant s$, we have

$$\begin{aligned}|\mathrm{R}|=2n&\left|\int\limits_\Omega\left(\|\mathcal{E}\left(u_1\right)\|-s\right)\frac{\mathcal{E}\left(u_1\right)}{\|\mathcal{E}\left(u_1\right)\|}\mathcal{E}(w)\,da\right|\leqslant 2n\int\limits_\Omega\left(\|\mathcal{E}\left(u_1\right)\|-s\right)\frac{\|\mathcal{E}\left(u_1\right)\|}{\|\mathcal{E}\left(u_1\right)\|}\|\mathcal{E}(w)\|\,da\\&=2n\int\limits_\Omega\left(\|\mathcal{E}\left(u_1\right)\|-s\right)\|\mathcal{E}(v)\|\,da\leqslant 2n\int\limits_\Omega\left(\|\mathcal{E}\left(u_1\right)\|-\|\mathcal{E}\left(u_2\right)\|\right)\|\mathcal{E}(w)\|\,da\\&\leqslant 2n\int\limits_\Omega\|\mathcal{E}\left(u_1\right)-\mathcal{E}\left(u_2\right)\|\,\|\mathcal{E}(w)\|\,da\leqslant 2n\left\|u_1-u_2\right\|_V\|w\|_V.\end{aligned}$$

In the third case $\|\mathcal{E}(u_1)\| > s$ and $\|\mathcal{E}(u_2)\| > s$ we deduce

$$\begin{aligned}
|\mathrm{R}| &= 2n\left|\int_\Omega \left((\|\mathcal{E}(u_1)\| - s)\frac{\mathcal{E}(u_1)}{\|\mathcal{E}(u_1)\|} - (\|\mathcal{E}(u_2)\| - s)\frac{\mathcal{E}(u_2)}{\|\mathcal{E}(u_2)\|}\right)\mathcal{E}(w)\,da\right| \\
&= 2n\left|\int_\Omega (\mathcal{E}(u_1) - \mathcal{E}(u_2)) - s\left(\frac{\mathcal{E}(u_1)}{\|\mathcal{E}(u_1)\|} - \frac{\mathcal{E}(u_2)}{\|\mathcal{E}(u_2)\|}\right)(\mathcal{E}(w))\,da\right| \\
&\leqslant 2n\int_\Omega \|\mathcal{E}(u_1) - \mathcal{E}(u_2)\|\|\mathcal{E}(w)\|\,da + 2n\left|\int_\Omega s\left(\frac{\mathcal{E}(u_1)}{\|\mathcal{E}(u_1)\|} - \frac{\mathcal{E}(u_2)}{\|\mathcal{E}(u_2)\|}\right)\mathcal{E}(v)\right|da \\
&\leqslant 2n\|u_1 - u_2\|_V\|w\|_V + 2n\int_\Omega s\left|\frac{\mathcal{E}(u_1)}{\|\mathcal{E}(u_1)\|} - \frac{\mathcal{E}(u_2)}{\|\mathcal{E}(u_2)\|}\right|\|\mathcal{E}(w)\|\,da \\
&\leqslant 2n\|u_1 - u_2\|_V\|v\|_V + 2n\int_\Omega \|\mathcal{E}(u_1) - \mathcal{E}(u_2)\|\|\mathcal{E}(w)\|\,da \\
&\leqslant 4n\|u_1 - u_2\|_V\|w\|_V.
\end{aligned}$$

Then we finally obtain

$$|\mathrm{R}| \leqslant 4n\|u_1 - u_2\|_V\|w\|_V,$$

and hence, the operator B_n^ϵ is Lipschitz continuous. This allows us to conclude that $(PV_n^{z\epsilon})$ possesses a unique solution $u_n^{z\epsilon}$.

We shall also some technical lemmas.

Lemma 3.1. *For each $z \in G$, the problem $PV_n^{z\epsilon}$ has a unique solution $u_n^{z\epsilon}$. The function $z \in \mathrm{G} \to u_n^{z\epsilon}$ is Lipshitz. These exists $c > 0$ such that*

$$\|u_n^{z\epsilon}\|_V \leqslant c\|f^z\|_V. \tag{3.3}$$

Proof. We consider the mapping the mapping $z \to u_n^{z\epsilon}$ and for two arbitrary elements $z_1, z_2 \in G$ we have

$$\begin{aligned}
(B_n^\epsilon u_n^{z_1\epsilon}, v - u_n^{z_1\epsilon})_V &= (f^{z_1}, w - u_n^{z_1\epsilon}), \quad w \in V, \\
(B_n^\epsilon u_n^{z_2}, v - u_n^{z_2\epsilon})_V &= (f^{z_2}, w - u_n^{z_2\epsilon}), \quad w \in V.
\end{aligned}$$

We take $w = u_n^{z_2}$ in the first identity and $w = u_n^{z_1}$ in the second one to obtain

$$\begin{aligned}
(B_n^\epsilon u_n^{z_1\epsilon}, u_n^{z_2\epsilon} - u_n^{z_1\epsilon})_V &= (f_1^z, u_n^{z_2\epsilon} - u_n^{z_1\epsilon}), \\
(B_n^\epsilon u_n^{z_2\epsilon}, u_n^{z_1\epsilon} - u_n^{z_2\epsilon})_V &= (f_2^z, u_n^{z_1\epsilon} - u_n^{z_2\epsilon}).
\end{aligned}$$

Keeping the last equalities we find

$$(B_n^\epsilon u_n^{z_1} - B_n^\epsilon z_n^{z_2}, u_n^{z_1} - u_n^{z_2})_V = (f^{z_1} - f^{z_2}, u_n^{z_1\epsilon} - u_n^{z_2\epsilon})_V.$$

Using $(PV_n^{z_1\epsilon})$ and $(PV_n^{z_2\epsilon})$, we obtain:

$$\begin{aligned}
(B_n^\epsilon u_n^{z_1} - B_n^\epsilon u_n^{z_2}, u_n^{z_1} - u_n^{z_2})_V &= \int_{\Gamma_C} (z_2 - h_1)(u_{n\tau}^{z_1\epsilon} - u_{n\tau}^{z_2\epsilon})\,da \\
&\leqslant \|z_1 - z_2\|_{L^2(\Gamma_C)}\|u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\|_{L^2(\Gamma_C)} \\
&\leqslant c_0\|z_1 - z_2\|_{L^2(\Gamma_C)}\|u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\|_V.
\end{aligned}$$

By the V-ellipticity of $a(\cdot,\cdot)$ to obtain

$$\alpha \|u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\|_V^2 \leqslant a\left(u_n^{z_1}\mathcal{E} - u_n^{z_2}\mathcal{E}, u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\right) \leqslant \left(B_n^\epsilon u_n^{z_1\epsilon} - B_n^\epsilon u_n^{z_2\epsilon}, u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\right)_V.$$

Hence,

$$\|u_n^{z_1\epsilon} - u_n^{z_2\epsilon}\|_V \leqslant \frac{c_0}{\alpha}\|z_1 - z_2\|_{L^2(\Gamma_C)}.$$

We proceed to proving inequality (3.3). By the identity

$$\left(B_n^\epsilon u_n^{z\epsilon}, u_n^z\right)_V = \left(f^z, u_n^{z\epsilon}\right)$$

we obtain

$$\alpha \|u_n^{z\epsilon}\|_V^2 \leqslant a\left(u_n^{z\epsilon}, u_n^{z\epsilon}\right) \leqslant \left(B_n^\epsilon u_n^z, u_n^{z\epsilon}\right)_V \leqslant \|f^z\|_V \|u_n^{z\epsilon}\|_V.$$

This yields

$$\|u_n^{z\epsilon}\|_V \leqslant \frac{1}{\alpha}\|f^z\|_V.$$

We consider a sequence $(z_k)_k$ in $L^2(\Gamma_C)$ weakly converging in $L^2(\Gamma_C)$ to an element z, and since

$$\|u_n^{z_k\epsilon}\|_V \leqslant \frac{1}{\alpha}\|f^{z_k}\|_V \leqslant \frac{1}{\alpha}\|f\|_V + \frac{c_0}{\alpha}\|z_k\|_{L^2(\Gamma_C)},$$

we see that $(u_n^{z_k\epsilon})_k$ is bounded in V. Thus, there exists a subsequence $(u_n^{z_k\epsilon})_k$, which converges weakly to $\widetilde{u}_n^\epsilon$.

For all $w \in V$ we have

$$\left(f^{z_k}, v - u_n^{z_k\epsilon}\right) = \left(f, w - u_n^{z_k\epsilon}\right)_V - \widetilde{\ell}_{z_k}\left(v - u_n^{z_k\epsilon}\right),$$

$$\left|\widetilde{\ell}_{z_k}\left(u_n^{z_k\epsilon}\right) - \widetilde{\ell}_{z_k}\left(\widetilde{u}_n^\epsilon\right)\right| \leqslant \|z_k\|_{L^2(\Gamma_C)}\|u_n^{z_k\epsilon} - \widetilde{u}_n^\epsilon\|_{L^2(\Gamma_C)}.$$

Since the Sobolev trace $\gamma : V \to L^2(\Gamma_C)$ is compact, $(u_n^{z_k\epsilon})_k \to \widetilde{u}_n^\epsilon$ as $k \to \infty$ and

$$\left(f^{z_k}, w - u_n^{z_k\epsilon}\right)_k \to \left(f, w - \widetilde{u}_n^\epsilon\right). \tag{3.4}$$

For $v = B_n^\epsilon u_n^{h_k\mathcal{E}}$ in the identity

$$\left(B_n^\epsilon u_n^{h_k\mathcal{E}}, v\right)_V = \left(f^{h_k}, v\right)_V$$

we get

$$\left\|B_n^\epsilon u_n^{h_k\mathcal{E}}\right\|_V \leqslant \left\|f^{h_k}\right\|_V \leqslant \|f\|_V + c_0\|h_k\|_{L^2(C)}.$$

This implies the existence of a constant $\gamma > 0$ such that $\|B_n^\epsilon u_n^{z_k\epsilon}\|_V \leqslant \gamma$ for all $k \in \mathbb{N}$.

It follows from

$$\left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_n\epsilon} - \widetilde{u}_n\right)_V = \left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_k\epsilon} - v\right)_V + \left(B_n^\epsilon u_n^{z_k\epsilon}, v - \widetilde{u}_n\right)_V.$$

Hence,

$$\left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_k\epsilon} - \widetilde{u}_n^\epsilon\right)_V \leqslant \left(f^{z_k}, u_n^{z_k\epsilon} - w\right)_V + \gamma\|w - \widetilde{u}_n^\epsilon\|_V.$$

Therefore,

$$\limsup_{k\to+\infty}\left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_k\epsilon} - \widetilde{u}_n^\epsilon\right)_V \leqslant \left(f, \widetilde{u}_n^\epsilon - w\right)_V + \gamma\|w - \widetilde{u}_n^\epsilon\|_V.$$

For $w = \widetilde{u}_n^\epsilon$ we get

$$\limsup_{k\to\infty}\left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_k\epsilon} - \widetilde{u}_n^\epsilon\right)_V \leqslant 0.$$

Then

$$\left(B_n^\epsilon \widetilde{u}_n^\epsilon, \widetilde{u}_n^\epsilon - v\right)_V \leqslant \liminf_{k\to\infty}\left(B_n^\epsilon u_n^{z_k\epsilon}, u_n^{z_k\epsilon} - w\right)_V. \tag{3.5}$$

We combine the relationship (3.4) and (3.5) to obtain

$$\begin{cases} \widetilde{u}_n^\epsilon \in V \\ (B_n^\epsilon \widetilde{u}_n^\epsilon, \widetilde{u}_n^\epsilon - w)_V \leqslant (f, \widetilde{u}_n^\epsilon - w) \quad \forall w \in V. \end{cases} \tag{3.6}$$

We make the change $\widetilde{w} = \widetilde{u}_n^\epsilon + w$ and $\widetilde{w} = \widetilde{u}_n^\epsilon - w$ in (3.6) and we find that $\widetilde{u}_n^\epsilon$ is a solution of PV_n^ϵ. Hence, $\widetilde{u}_n^\epsilon = u_n^{h\epsilon}$ and $\left(u_n^{h_k \mathcal{E}}\right)_k \rightharpoonup u_n^{h\epsilon}$ as $k \to +\infty$. Finally, the mapping $z \to u_n^{z\epsilon}$ is weakly continuous on $L^2(\Gamma_C)$.

Now we are going to show that the mapping

$$\Lambda : z \longmapsto \varpi_\nu \left(u_{n,\nu}^{z\epsilon} - g\right) v + \varpi_\tau \left(u_{n,\nu}^{z\epsilon} - g\right) \frac{u_{n,\tau}^{z\epsilon}}{\sqrt{\left\|u_{n,\tau}^{z\epsilon}\right\|^2 + \epsilon^2}},$$

has a fixed point. Let $z \in G$. We have

$$\|\Lambda z\|_{L^2(\Gamma_C)} \leqslant 2\mu^* \left(\operatorname{meas}(\Gamma_C)\right)^{\frac{1}{2}}.$$

We choose $\kappa = 2\mu^* \left(\operatorname{meas}(\Gamma_C)\right)^{\frac{1}{2}}$ and see that Λ maps G into itself. Since G is a closed non–empty convex set of $L^2(\Gamma_C)$, we apply the fixed point theorem at Λ and this completes the proof. □

Proof of Theorem 2.1. There exists z^* such that $\Lambda z^* = z^*$ and for each $v \in V$

$$\int_{\Gamma_C} \varpi_\nu \left(u_{n,\nu}^{z^*\epsilon} - g\right) v_\nu da + \int_{\Gamma_C} \varpi_\tau \left(u_{n,\nu}^{z^*\epsilon} - g\right) \frac{u_{n,\tau}^{z^*\epsilon} v_\tau \, da}{\sqrt{\left\|u_{n,\tau}^{z^*\epsilon}\right\|^2 + \epsilon^2}} = \widetilde{\ell}_z(v), \quad v \in V.$$

Hence, $u_n^{z^*\epsilon}$ is a solution of PV_n^ϵ.

Let us establish the uniqueness. Let $u_{1,n}^\epsilon$ and $u_{2,n}^\epsilon$ be two solutions of PV_n^ϵ. For all $v \in V$ we have

$$\left(B_n^\epsilon u_{1,n}^\epsilon, v\right)_V + \left\langle \ell'_{\mathcal{E}}\left(u_{1,n}^\epsilon, u_{1,n}^\epsilon\right), v\right\rangle = (f, v)_V, \qquad \left(B_n^\epsilon u_{2,n}^\epsilon, v\right)_V + \left\langle \ell'_{\mathcal{E}}\left(u_{2,n}^\epsilon, u_{2,n}^\epsilon\right), v\right\rangle = (f, v)_V.$$

Then

$$\left(B_n^\epsilon u_{1,n}^\epsilon - B_n^\epsilon u_{2,n}^\epsilon, v\right)_V = \left\langle \ell'_\epsilon\left(u_{2,n}^\epsilon, u_{2,n}^\epsilon\right), v\right\rangle - \left\langle \ell'_\epsilon\left(u_{1,n}^\epsilon, u_{1,n}^\epsilon\right), v\right\rangle. \tag{3.7}$$

Taking $v = u_{1,n}^\epsilon - u_{2,n}^\epsilon$ in (3.7), we obtain

$$\left(B_n^\epsilon u_{1,n}^\epsilon - B_n^\epsilon u_{2,n}^\epsilon, u_{1,n}^\epsilon - u_{2,n}^\epsilon\right)_V = \left\langle \ell'_\epsilon\left(u_{2,n}^\epsilon, u_{2,n}^\epsilon\right), u_{1,n}^\epsilon - u_{2,n}^\epsilon\right\rangle - \left\langle \ell'_\epsilon\left(u_{1,n}^\epsilon, u_{1,n}^\epsilon\right), u_{1,n}^\epsilon - u_{2,n}^\epsilon\right\rangle.$$

By the convexity of $j_{\mathcal{E}}(u, \cdot)$ we get

$$\begin{aligned}
\left(B_n^\epsilon u_{1,n}^\epsilon - B_n^\epsilon u_{2,n}^\epsilon, u_{1,n}^\epsilon - u_{2,n}^\epsilon\right)_V \leqslant & \ell_\epsilon\left(u_{1,n}^\epsilon, u_{2,n}^\epsilon\right) - \ell_\epsilon\left(u_{1,n}^\epsilon, u_{1,n}^\epsilon\right) + j_{\mathcal{E}}\left(u_{2,n}^\epsilon, u_{1,n}^\epsilon\right) - \ell_n\left(u_{2,n}^\epsilon, u_{2,n}^\epsilon\right) \\
= & \int_{\Gamma_C} \left(\varpi_\nu\left(u_{1,n\nu}^\epsilon - g\right) - \varpi_\nu\left(u_{1,n\nu}^\epsilon - g\right)\right)\left(u_{2,n\nu}^\epsilon - u_{1,n\nu}^\epsilon\right) da \\
& + \int_{\Gamma_C} \left(\varpi_\tau\left(u_{1,n\nu}^\epsilon - g\right) - \varpi_\tau\left(u_{2,n\nu}^\epsilon - g\right)\right) \\
& \quad \cdot \left(\sqrt{\left\|u_{2,n}^\epsilon\right\|^2 + \epsilon^2} - \sqrt{\left\|u_{1,n\tau}^\epsilon\right\|^2 + \epsilon^2}\right) da \\
\leqslant & \left(L_\nu c_0^2 + L_\tau c_0^2\right) \left\|u_{1,n}^\epsilon - u_{2,n}^\epsilon\right\|_V^2.
\end{aligned}$$

Hence,

$$\alpha \left\|u_{1,n}^\epsilon - u_{2,n}^\epsilon\right\|_V^2 \leqslant c_0^2 \left(L_\nu + L_\tau\right) \left\|u_{1,n}^\epsilon - u_{2,n}^\epsilon\right\|_V^2.$$

Let $L^* = \frac{1}{c_0^2}$. If $\frac{L_\mu+\mu^*}{\alpha} < L^*$ then $u_{1,n}^\epsilon = u_{2,n}^\epsilon$. The proof is complete. □

3.2. Convergence of penalty method. In this section, we analyze the convergence of both continuous and discrete penalization methods. We first suppose $\epsilon = \frac{1}{n}$, $B_n^\epsilon = B_n$, and $\ell_\epsilon = \ell_n$. Then **Problem** PV_n^ϵ becomes.

Problem PV_n**:** Find the displacement $u_n : \Omega \to \mathbb{R}^d$ such that

$$(B_n u_n, v)_V + \langle \ell_n' (u_n, u_n), v\rangle = (f, v)_V.$$

We first prove the technical necessary lemma.

Lemma 3.2. *Let $n \in \mathbb{N}$, and $f_n : \Omega \to \mathbb{R}^+$ be such that $(f_n) \rightharpoonup f$ in $L^2(\Omega)$, and $\beta > 0$. The set $B_n = \{x \in \Omega : \ f_n(x) \geqslant \beta\}$ is Borel. Suppose that $(|B_n|)_n \to 0$, then $f \in L^\infty(\Omega)$ and f is bounded by β in L^∞.*

Proof. For a test function θ and $\epsilon > 0$ for sufficiently large n we have

$$\left| \int\limits_{\Omega\setminus B_n} \theta\,(f_n - f) \right| \leqslant \epsilon, \qquad \left| \int\limits_{\Omega\setminus B_n} \theta f \right| \leqslant \epsilon + \left| \int\limits_{\Omega\setminus B_n} \theta f_n \right| \leqslant \epsilon + \beta \int\limits_{\Omega\setminus B_n} |\theta|.$$

Since $(|B_n|) \to 0$, we have $\theta f \chi_{B_n} \to 0$ a.e in Ω as well as $|\theta f \chi_{B_n}| \leqslant |f\theta| \in L^1(\Omega)$. Applying the dominate convergence theorem, we find

$$\left| \int\limits_{A_n} \theta f \right| \to 0.$$

As $\mathcal{E}$ vanishes, we get

$$\left| \int\limits_{\Omega} \theta f \right| \leqslant \beta \int\limits_{\Omega} |\theta|.$$

The proof is complete. □

Theorem 3.2. *Let u_n the solution of PV_n. The sequence (u_n) converge strongly to $u \in V$.*

Proof. We first prove that u_n converges weakly to the solution u of the Problem as $n \to \infty$. Since u_n the solution of PV_n, we have

$$(B_n u_n, u_n)_V + \langle \ell_n' (u_n, u_n), u_n\rangle = (f, u_n)_V.$$

The inequality $\langle \ell_n' (u_n, u_n), u_n\rangle \geqslant 0$ yields

$$(B_n u_n, u_n)_V \leqslant \|f\|_V \|u_n\|_V.$$

Since B_n is elliptic, we obtain

$$\|u_n\|_V \leqslant \frac{1}{\alpha}\|f\|_V.$$

This implies the existence a subsequence, still denoted by (u_n), such that $(u_n) \rightharpoonup \widetilde{u} \in V$.

Using lemma 3.2 with $f_n = \mathcal{E}(u_n)$, we take

$$\beta = s + \gamma, \qquad 0 < \gamma < 1, \qquad B_n = \{x \in \Omega : \ |\mathcal{E}(u_n)| > s + \gamma\}.$$

We have

$$\gamma^2 \|B_n\| \leqslant \int\limits_{B_n} (\|\mathcal{E}(u_n)\| - s)^2 \leqslant \int\limits_{\Omega} \left((\|\mathcal{E}(u_n)\| - s)^+\right)^2 \leqslant \frac{C}{n} \to 0.$$

Using Lemma 3.2 and letting $\gamma \to 0$, we obtain $\|\mathcal{E}(\widetilde{u})\| \leqslant s$. Thus, $\widetilde{u} \in K$.

Let $n \in \mathbb{N}^*$ and $v \in K$, we are going to show that $\mathscr{R}_n(\mathcal{E}(u_n), \mathcal{E}(v - u_n))_{\mathscr{H}} \leqslant 0$.

If $\|\mathcal{E}(u_n)\| \leqslant s$, then then $\mathscr{R}_n(\mathcal{E}(u_n), \mathcal{E}(v - u_n))_{\mathscr{H}} = 0$.

If $\|\mathcal{E}(u_n)\| \geqslant s$, then by the Cauchy — Schwarz — Bunyakovsky inequality we deduce:

$$\begin{aligned}
\mathscr{R}_n(\mathcal{E}(u_n), \mathcal{E}(v-u_n))_{\mathscr{H}} &= 2n\int_\Omega \left((\|\mathcal{E}(u_n)\| - s)^+\right) \frac{\mathcal{E}(u_n)}{\|\mathcal{E}(u_n)\|} (\mathcal{E}(v) - \mathcal{E}(u_n)) \\
&= 2n\int_\Omega \frac{(\|\mathcal{E}(u_n)\| - s)}{\|\mathcal{E}(u_n)\|} \mathcal{E}(u_n)(\mathcal{E}(v) - \mathcal{E}(u_n)) \\
&= 2n\int_\Omega \frac{(\|\mathcal{E}(u_n)\| - s)}{\|\mathcal{E}(u_n)\|} \left(\mathcal{E}(u_n)\cdot\mathcal{E}(v) - \|\mathcal{E}(u_n)\|^2\right) \\
&\leqslant 2n\int_\Omega \frac{(\|\mathcal{E}(u_n)\| - s)}{\|\mathcal{E}(u_n)\|} \left(\|\mathcal{E}(u_n)\| \|\mathcal{E}(v)\| - \|\mathcal{E}(u_n)\|^2\right) \\
&= 2n\int_\Omega \frac{(\|\mathcal{E}(u_n)\| - s)}{\|\mathcal{E}(u_n)\|} \|\mathcal{E}(u_n)\| (\|\mathcal{E}(v)\| - \|\mathcal{E}(u_n)\|) \\
&\leqslant 2n\int_\Omega \frac{(\|\mathcal{E}(u_n)\| - s)}{\|\mathcal{E}(u_n)\|} \|\mathcal{E}(u_n)\| (s - \|\mathcal{E}(u_n)\|) \leqslant 0.
\end{aligned} \tag{3.8}$$

Hence, for all $v \in K$,

$$\mathscr{R}_n(\mathcal{E}(u_n), \mathcal{E}(v-u_n))_{\mathscr{H}} \leqslant 0. \tag{3.9}$$

Let us show that the sequence (u_n) converges weakly to u. By (3.9) we get

$$a(u_n, v-u_n) + \langle \ell'_n(u_n, u_n), v-u_n\rangle \geqslant (f, v-u_n)_V, \quad v \in K.$$

Using (3.1), we get

$$a(u_n, v-u_n) + \ell_n(u_n, v) - \ell_n(u_n, u_n) \geqslant (f, v-u_n)_V, \quad v \in K.$$

We pass to the limit in the above inequality and see that $\widetilde{u}$ is a weak solution of the problem. We conclude on $\widetilde{u} = u$ and the uniqueness of $\widetilde{u}$ for each subsequence in (u_n), which converges weakly to u.

Let us show that $(u_n) \to u$ as $n \to +\infty$. We have

$$\begin{aligned}
\alpha\|u_n - u\|_V^2 \leqslant & a(u_n - u, u_n - u) = a(u_n, u_n - u) - a(u, u_n - u) \\
=& -a(u_n, u - u_n) - a(u, u_n - u) \\
=& \mathscr{R}_n(\mathcal{E}(u_n), \mathcal{E}(u-u_n))_{\mathscr{H}} + (S_n u_n, u-u_n)_V + \langle \ell'_n(u_n, u_n), u-u_n\rangle \\
& -(f, u-u_n) - a(u, u_n - u) \\
\leqslant & \ell(u,u) - \ell(u, u_n) - a(u, u_n - u).
\end{aligned}$$

As $n \to +\infty$, we obtain $(u_n) \to u$ strongly in V. The proof is complete. □

4. Finite element approximation

Let $T = \{T_h\}$ be the set of triangulations of Ω, where $h > 0$ is the spatial discretization parameter with $h = \max_{T \in T_h} h_T$, and h_T being the diameter of T. Let $V^h \subset V$ be a family of finite–dimensional vector spaces indexed by h. We define a finite element space V^h composed of continuous, piecewise linear polynomials

$$V^h = \left\{v \in \mathscr{C}(\bar{\Omega})^d : \ v \ \text{ is an affine function on } T_h \text{ and } v = 0 \text{ on } \Gamma_D\right\}. \tag{4.1}$$

We let $\epsilon = \frac{1}{n}$ and $\ell_n = \ell_\epsilon$. The discrete problem of the penalty problem PV_n reads as follows.

Problem PV_n^h: Find the displacement $u_n^h : \Omega \to \mathbb{R}^d$ such that

$$a\left(u_n^h, v^h\right) + \left(\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right)\right), \mathcal{E}\left(v^h\right)\right)_{\mathscr{H}} + \left\langle \ell_n'\left(u_n^h, u_n^h\right), v^h\right\rangle = \left(f, v^h\right)_V, \quad v^h \in V^h.$$

By Theorem 2.1 we conclude that the above problem possesses a unique solution u_n^h in V^h.

Theorem 4.1. *Let $n \in \mathbb{N}^*$ we suppose $h = \frac{1}{n^2}$. Let u_n^h and u be the solutions respectively of PV_n^h and PV. Then:*

$$\left(u_n^h\right) \to u \quad as \quad h \to 0.$$

Proof. We let

$$\mathscr{D} = \left\{v \in \mathscr{C}^\infty(\Omega)^d : \ \Gamma_D \subset \Omega \setminus \operatorname{supp}(v)\right\}$$

and $R^h : \mathscr{D} \to V^h$ is such that for $v \in \mathscr{D}$, $R^h v$ is the linear interpolation of v. We recall that $\mathscr{D}$ is dense in V and for all $v \in \mathscr{D}$

$$\left\|R^h v - v\right\|_V \leqslant Ch\|v\|_{2,\Omega}, \qquad \left\|R^h v - v\right\|_{0,\Gamma} \leqslant Ch\|v\|_V, \qquad \left(R^h v\right)_h \to v \quad \text{strongly}.$$

Taking $v = u_n^h$ in (4.1) and applying the relations

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h\right)\right)_{\mathscr{H}} \geqslant 0, \qquad \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h\right\rangle \geqslant 0,$$

we find

$$a\left(u_n^h, u_n^h\right) \leqslant \left(f, u_n^h\right)_V.$$

Using the V-ellipticity of $a(\cdot\,,\cdot)$, we get

$$\left\|u_n^h\right\|_V \leqslant \frac{\|f\|_V}{\alpha}.$$

It follows from that

$$a\left(u_n^h, u_n^h\right) \geqslant 0, \qquad \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h\right\rangle \geqslant 0.$$

Hence,

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h\right)\right)_{\mathscr{H}} \leqslant \left(f, u_n^h\right)_V \leqslant \|f\|_V \left\|u_n^h\right\|_V \leqslant \frac{\|f\|_V^2}{\alpha},$$

and

$$2n \int_\Omega \left(\left\|\mathcal{E}\left(u_n^h\right)\right\| - s\right)^+ \left\|\mathcal{E}\left(u_n^h\right)\right\| \leqslant C,$$

with a positive constant C. This yields

$$2n \int_\Omega \left(\left\|\mathcal{E}\left(u_n^h\right)\right\| - s\right)^2 \leqslant C.$$

Hence,

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h\right)\right)_{\mathscr{H}} \leqslant C, \quad h > 0.$$

Since $a(\cdot\,,\cdot)$ is continuous, there exists $m_a > 0$ such that

$$\left|a\left(u_n^h, w\right)\right| \leqslant m_a \|w\|_V \left\|u_n^h\right\|_V, \quad w \in V, \tag{4.2}$$

and therefore,

$$a\left(u_n^h, w\right) \leqslant C\|w\|_V, \quad w \in V.$$

There exists a subsequence of $\left(u_n^h\right)$, still denoted by $\left(u_n^h\right)$, which converges weakly to u^*. Arguing similarly to (3.2), we find $\left(u_n^h\right) \to \tilde{u}$ on $L^2\left(\Gamma_C\right)$ and $\tilde{u} \in K$. Then for all $v \in \mathscr{D}$ we have

$$\begin{aligned} a\left(u_n^h, u_n^h - R^h v\right) + \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h - R^h v\right)\right)_{\mathscr{H}} \\ + \left(\ell_n'\left(u_n^h, u_n^h\right), u_n^h - R^h v\right)_V = \left(f, u_n^h - R^h v\right)_V, \end{aligned}$$

and

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h - R^h v\right)\right)_{\mathscr{H}} = \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h - v\right)\right)_{\mathscr{H}} + \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}}.$$

By (3.9) we have

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - u_n^h\right)\right)_{\mathscr{H}} \leqslant 0 \tag{4.3}$$

for all $v \in K \cap U$. Then

$$\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h - R^h v\right)\right)_{\mathscr{H}} \geqslant \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}}.$$

On one hand we have

$$a\left(u_n^h, u_n^h - \tilde{u}\right) = a\left(u_n^h, u_n^h - R^h v\right) + a\left(u_n^h, R^h v - \tilde{u}\right),$$

and then

$$\begin{aligned} a\left(u_n^h, u_n^h - \tilde{u}\right) =& - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h - R^h v\right)\right)_{\mathscr{H}} - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h - R^h v\right\rangle \\ &+ \left(f, u_n^h - R^h v\right)_V + a\left(u_n^h, R^h v - u^*\right) \\ \leqslant& - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}} - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h - R^h v\right\rangle \\ &+ \left(f, u_n^h - R^h v\right)_V + a\left(u_n^h, R^h v - u^*\right). \end{aligned}$$

Then

$$\begin{aligned} a\left(u_n^h, u_n^h - u^*\right) \leqslant& - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}} - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h - R^h v\right\rangle \\ &+ \left(f, u_n^h - R^h v\right)_V + a\left(u_n^h, R^h v - u^*\right). \end{aligned} \tag{4.4}$$

On the other hand, we use (4.2) to obtain

$$\limsup_{h\to 0} a\left(u_n^h, R^h v - \tilde{u}\right) \leqslant C \limsup_{h\to 0} \left\|R^h v - \tilde{u}\right\|_V = C\|v - \tilde{u}\|_V.$$

We have

$$\lim_{h\to 0} \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h - R^h v\right\rangle = \ell(\tilde{u}, \tilde{u}) - \ell\left(u^*, v\right), \qquad \lim_{h\to 0} \left(f, u_n^h - R^h v\right)_V = (f, \tilde{u} - v)_V$$

and

$$\left|\mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}}\right| \leqslant C\sqrt{n}\left\|v - R^h v\right\|_V \leqslant Ch\sqrt{n}\|v\|_{H^2(\Omega)^d}, \tag{4.5}$$

with a positive constant C. Since $h = \frac{1}{n^2}$ in (4.5), we get

$$\lim_{h\to 0} \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v - R^h v\right)\right)_{\mathscr{H}} = 0.$$

Using (4.4) and (4.5), for all $v \in K \cap \mathscr{D}$ we find

$$\limsup_{h\to 0} a\left(u_n^h, u_n^h - \tilde{u}\right) \leqslant \ell(\tilde{u}, v) - \ell(\tilde{u}, \tilde{u}) + (f, \tilde{u} - v)_V + C\|v - \tilde{u}\|_V. \tag{4.6}$$

Since $K \cap \mathscr{D}$ is dense in K, it follows that (4.6) also holds in K.

By letting $v = \tilde{u}$ in (4.6), we obtain

$$\limsup_{h\to 0} a\left(u_n^h, u_n^h - \tilde{u}\right) \leqslant 0,$$

and for all $v \in K$ we have

$$a\left(u_n^h, v - u_n^h\right) = a\left(u_n^h, v - \tilde{u}\right) + a\left(u_n^h, \tilde{u} - u_n^h\right)$$

$$= \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}(v-\tilde{u})\right)_{\mathscr{H}} + \left\langle \ell_n'\left(u_n^h, u_n^h\right), v-\tilde{u}\right\rangle + a\left(u_n^h, \tilde{u}-u_n^h\right).$$

We pass to the limit in the above equality and using (4.6) we find

$$a(\tilde{u}, v-\tilde{u}) \geqslant -\ell(\tilde{u}, v) + \ell(\tilde{u}, \tilde{u}) + (f, v-\tilde{u})_V \quad \forall v \in K.$$

Then $\tilde{u}$ is a solution of a variational problem, and hence, it is unique and $\tilde{u}=u$.

Now we are going to show that $\left(u_n^h\right)$ converges strongly to u. Let $v \in K \cap \mathscr{D}$, then

$$\begin{aligned}
\alpha\left\|u_n^h-u\right\|_V^2 \leqslant & a\left(u_n^h-u, u_n^h-u\right) = a\left(u_n^h, u_n^h-u\right) - a\left(u, u_n^h-u\right) \\
= & a\left(u_n^h, u_n^h-R^h v\right) + a\left(u_n^h, R^h v-u\right) - a\left(u, u_n^h-u\right) \\
= & \left(f, u_n^h-R^h v\right)_V - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(u_n^h-R^h v\right)\right)_{\mathscr{H}} \\
& - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h-R^h v\right\rangle - a\left(u, u_n^h-u\right) \\
= & \left(f, u_n^h-R^h v\right)_V + \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v-u_n^h\right)\right)_{\mathscr{H}} \\
& - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v-R^h v\right)\right)_{\mathscr{H}} \\
& - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h-R^h v\right\rangle - a\left(u, u_n^h-u\right).
\end{aligned}$$

Using (4.3), we obtain

$$\begin{aligned}
\alpha\left\|u_n^h-u\right\|_V^2 \leqslant & \left(f, u_n^h-R^h v\right)_V - \mathscr{R}_n\left(\mathcal{E}\left(u_n^h\right), \mathcal{E}\left(v-R^h v\right)\right)_{\mathscr{H}} \\
& - \left\langle \ell_n'\left(u_n^h, u_n^h\right), u_n^h-R^h v\right\rangle - a\left(u, u_n^h-u\right).
\end{aligned}$$

We now take the limit in the above identity and, by applying (4.5), we obtain

$$\lim_{h\to 0} \alpha\left\|u_n^h-u\right\|_V^2 \leqslant (f, u-v)_V - \ell(u,u) + \ell(u,v), \quad v \in K \cap U. \tag{4.7}$$

Since $K \cap \mathscr{D}$ is dense in K, we conclude that (4.7) is valid in K. Then we take $v=u$ to get the strong convergence of $\left(u_n^h\right)$ to u in V. The proof is complete. □

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