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ON VECTOR SCHWARZ — KdV EQUATION

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Abstract. A collection of miscellaneous continuous, semi-discrete, and discrete integrable systems can be associated with each integrable evolution equation of the KdV type. We provide them for the Schwarz — KdV equation and generalize to the vector case. The existence of these vector generalizations is a non-trivial found fact, no mathematical explanation of which is known yet.

Keywords: auto–Bäcklund transformation, Volterra type chain, discrete integrable systems.

Mathematics Subject Classification: 37K10, 35Q53

1. Introduction

Using the symmetry approach to classification of integrable equations, see [12], [14], [15], in [8], [16] there were listed all scalar integrable evolution equations of the form

$$u_t = u_{xxx} + f(u, u_x, u_{xx}) \tag{1.1}$$

that have infinite sequence of local conservation laws.

A generalization of this approach to the case of systems of the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\, \boldsymbol{u}_{xx} + f_1\, \boldsymbol{u}_x + f_0\, \boldsymbol{u} \tag{1.2}$$

where the coefficients f_i depend on scalar products of the vectors $\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}$, was proposed in [9]. Some lists of integrable equations (1.2) were obtained in [7], [10], [11].

One of the most beautiful examples of such integrable systems is the so-called vector Schwarz — KdV equation [17]

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} - 3\frac{(\boldsymbol{u}_x, \boldsymbol{u}_{xx})}{(\boldsymbol{u}_x, \boldsymbol{u}_x)}\boldsymbol{u}_{xx} + \frac{3}{2}\frac{(\boldsymbol{u}_{xx}, \boldsymbol{u}_{xx})}{(\boldsymbol{u}_x, \boldsymbol{u}_x)}\boldsymbol{u}_x, \tag{1.3}$$

where $\boldsymbol{u}(x,t)$ is a vector in the N-dimensional vector space V equipped with a scalar product $(\,\cdot\,,\,\cdot\,)$, and $\boldsymbol{u}_t = \partial \boldsymbol{u}/\partial t, \boldsymbol{u}_i = \partial^i \boldsymbol{u}/\partial x^i$. In the component form this is a system of N evolution equations invariant with respect to the orthogonal group.

Each integrable equation (1.1) generates several associated differential, difference–differential and fully discrete integrable equations [1], [2]. In Section 2 we describe these integrable models in the case of the scalar Schwarz — KdV equation

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}. \tag{1.4}$$

The main question we discus in this paper is whether there exists such a variety of associated integrable vector equations for each vector equation (1.2). The answer is not apriori obvious. We shall show that this is the case for vector equation (1.3). While in the scalar case the search for related integrable systems is reduced to rather simple algorithmizable calculations, in the vector case the calculations become complex and sometimes require non-trivial tricks.

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2. Scalar case

We provide the collection of associated integrable systems for scalar Schwarz — KdV equation (1.4).

2.1. Bäcklund transformation. We assume that each function $u_n(x,t)$, $n \in Z$, satisfies equation (1.4). Then the formula

$$(u_{n+1})_x = \alpha \frac{(u_{n+1}-u_n)^2}{(u_n)_x}, \qquad \alpha \in \mathbb{C} \tag{2.1}$$

defines an auto–Bäcklund transformation [3] for (1.4). This semi–discrete chain is an integrable model as well. In [18] such chains related to equations (1.1) were called *dressing*.

To obtain formula (2.1), we write

$$(u_{n+1})_x = Q\Big((u_n)_x, u_n, u_{n+1}\Big), \tag{2.2}$$

differentiate it by t, and eliminate all t-derivatives by using (1.4) and all x-derivatives of u_{n+1} in virtue of (2.2). The remaining variables $u_{n+1}, u_n, (u_n)_x, \dots, (u_n)_{xxxx}$ are treated as independent jet variables. Splitting with respect to $(u_n)_{xx}, (u_n)_{xxx}, (u_n)_{xxxx}$, we obtain an overdetermined system of non–linear PDEs for $Q\Big((u_n)_x, u_n, u_{n+1}\Big)$. This system can be solved quite easily. Chain (2.1) corresponds to the most interesting solution of this system. The existence of other solutions can be explained by the fact that equation (1.4) is invariant with respect to the Möbius transformations

$$\tilde{u} = \frac{au+b}{cu+d}, \qquad a,b,c,d \in \mathbb{C}.$$

2.2. Volterra type chain. The pair of equations (1.4) and (2.1) are compatible with the following chain of Volterra type[1]:

$$(u_n)_z = \beta \frac{(u_{n+1}-u_n)(u_{n-1}-u_n)}{u_{n+1}-u_{n-1}}, \qquad \beta \in \mathbb{C}. \tag{2.3}$$

Compatibility ensures the existence of solutions $u(n,x,t,z)$ satisfying (1.4).

Chain (2.1) can be regarded as a hyperbolic equation, where the variable x is continuous and n is discrete. Usual hyperbolic integrable equation $u_{xy} = F$ has integrable evolution symmetries with respect to both variables x and y. Equation (2.1) has x-symmetry (1.4) and n-symmetry (2.3).

2.3. Negative flow. Here we consider relation (2.1), its shift

$$(u_n)_x = \alpha \frac{(u_n-u_{n-1})^2}{(u_{n-1})_x},$$

and (2.3) together with its x-derivative. Expressing $(u_{n+1})_x$, $(u_{n-1})_x$, u_{n+1}, u_{n-1} in terms of u_n, $(u_n)_x$, $(u_n)_z$, $(u_n)_{zx}$ from these four equations and differentiating (2.3) in x twice, we obtain the equation

$$u_{zxx} = \frac{u_{zx}^2 - \beta^2 u_x^2}{2u_z} + \frac{u_{xx}u_{zx}}{u_x} + 2\alpha u_z, \tag{2.4}$$

where $u = u_n$. This equation is a negative flow for (1.4). Moreover, a recursion operator for (1.4) can be extracted from (2.4), see [1], [2].

[1] Such integrable chains were classified in [19].

2.4. System of NLS type. Equation (1.4) is an x-symmetry for (2.4). But equation (2.4) turns out to be really integrable and therefore has z-symmetries (cf. [4]) of the form

$$u_\tau = F_1(u, u_x, u_z, u_{zx}, u_{zz}, \ldots), \qquad (u_x)_\tau = F_2(u, u_x, u_z, u_{zx}, u_{zz}, \ldots).$$

Computation shows that the simplest z-symmetry reads

$$F_1 = -\frac{u_{zzx}u_z}{u_x} + \frac{u_{zz}u_{zx}}{u_x} + \frac{1}{2}\frac{u_z u_{zx}^2}{u_x^2}, \qquad F_2 = (F_1)_x.$$

Denoting u_x by v, we obtain the following second order system:

$$u_\tau = -\frac{v_{zz}u_z}{v} + \frac{v_z u_{zz}}{v} + \frac{1}{2}\frac{u_z v_z^2}{v^2}, \qquad v_\tau = -\frac{v_{zz}v_z}{v} + \frac{u_{zz}}{vu_z}\left(v_z^2 - \beta^2 v^2\right) + \frac{1}{2}\frac{v_z^3}{v^2} + \beta^2 v_z. \tag{2.5}$$

The differential substitution

$$\tau = \beta^{-1}y + \frac{1}{2}z, \qquad u_z = e^{\sqrt{\beta}u' - \sqrt{\beta}v'/2}\Big(\beta e^{\sqrt{\beta}v'} + (e^{\sqrt{\beta}v'})_z\Big), \qquad v = e^{\sqrt{\beta}v'}$$

reduces (2.5) to the potential derivative NS–system

$$u_y = u_{zz} + v_z u_z^2 - \frac{1}{4}\beta v_z, \qquad v_y = -v_{zz} + u_z v_z^2 - \beta u_z.$$

Note that it is integrable [13] (a special case of the system $w3$) for all constant coefficients of v_z and u_z.

It is possible to continue creating integrable systems generated by the Schwarz – KdV equation. In particular, for the system

$$u_y = u_{zz} + v_z u_z^2 + k_1 v_z, \qquad v_y = -v_{zz} + u_z v_z^2 + k_2 u_z. \tag{2.6}$$

one can find the following Bäcklund transformation:

$$U_z = f u_z - (1 + f^{-1}) f_v, \qquad V_z = f^{-1}(v_z - (1 + f^{-1}) f_u),$$

where the function $f(u, v, U, V)$ is defined by the compatible system of ODEs

$$\begin{aligned} f_U &= f_u, \qquad f_{uu} = -\frac{1}{2}\frac{k_2(f-1)f^2 - (f+3)f_u^2}{f(f+1)}, \\ f_V &= f_v, \qquad f_{vv} = -\frac{1}{2}\frac{k_1(f-1)f^2 - (f+3)f_v^2}{f(f+1)}, \end{aligned}$$

which is semi–discrete system for two fields u and v. However, in this paper we do not aim to describe in some sense a complete list integrable systems that can be obtained from the Schwarz – KdV equation.

2.5. Superposition formula. Now we assume that functions $u(x, t, z_1, \ldots, z_k, n_1, \ldots n_k)$ satisfy equation (1.4), equation (2.1) with different parameters α_i and equation (2.3) with different parameters β_i. By T_i we denote the shift $n_i \mapsto n_{i+1}$. Then

$$T_i(u_x) = \alpha_i \frac{\big(T_i(u) - u\big)^2}{u_x}.$$

Consider the relations

$$\begin{aligned} u_x T_i(u_x) &= \alpha_i\big(T_i(u) - u\big)^2, \qquad & T_j(u_x) T_i T_j(u_x) &= \alpha_i\big(T_i T_j(u) - T_j(u)\big)^2, \\ u_x T_j(u_x) &= \alpha_j\big(T_j(u) - u\big)^2, \qquad & T_i(u_x) T_j T_i(u_x) &= \alpha_j\big(T_j T_i(u) - T_i(u)\big)^2. \end{aligned}$$

Eliminating x-derivatives from these equations and using that $T_i T_j(u) = T_j T_i(u)$, we arrive at the relation

$$\alpha_j^2\Big(T_i T_j(u) - T_i(u)\Big)^2\Big(T_j(u) - u\Big)^2 = \alpha_i^2\Big(T_i T_j(u) - T_j(u)\Big)^2\Big(T_i(u) - u\Big)^2,$$

Factorizing this relation, we obtain two candidates for the superposition formula. It can be verified that only the relation

$$\alpha_j\Big(T_iT_j(u)-T_i(u)\Big)\Big(T_j(u)-u\Big)=\alpha_i\Big(T_iT_j(u)-T_j(u)\Big)\Big(T_i(u)-u\Big), \tag{2.7}$$

is invariant with respect to the x-derivation. This superposition formula for equation (1.4) can be represented by the commutative diagram

$$\begin{tikzcd} T_i(u) \arrow[r, "\alpha_j"] & T_iT_j(u) \\ u \arrow[u, "\alpha_i"] \arrow[r, "\alpha_j"'] & T_j(u) \arrow[u, "\alpha_i"'] \end{tikzcd}.$$

It is well–known [6] that this integrable discrete equation satisfies the $3d$-consistency condition.

2.6. 3-D consistent hyperbolic system [1]. We enlarge the system

$$u_{z_ixx}=\frac{u_{z_ix}^2-\beta_i^2u_x^2}{2u_{z_i}}+\frac{u_{xx}u_{z_ix}}{u_x}+2\alpha_iu_{z_i},\qquad i=1,\dots,k, \tag{2.8}$$

of equations (2.4) by

$$u_{z_iz_j}=\frac{\alpha_iu_{z_i}u_{xz_j}-\alpha_ju_{z_j}u_{xz_i}}{(\alpha_i-\alpha_j)u_x},\qquad i\neq j,\quad i,j=1,\dots,k. \tag{2.9}$$

One can verify that system (2.8), (2.9) is compatible

$$\Big(u_{z_ixx}\Big)_{z_j}=\Big(u_{z_jxx}\Big)_{z_i}=\Big(u_{z_jz_i}\Big)_{xx},\qquad \Big(u_{z_iz_j}\Big)_{z_k}=\Big(u_{z_jz_k}\Big)_{z_i}=\Big(u_{z_kz_i}\Big)_{z_j},$$

where the derivatives are calculated in virtue of (2.8), (2.9). This means that there exists a simultaneous solution $u(x,z_1,\dots,z_k)$ for this system. For given equation (2.8), equations (2.9) are found by the compatibility conditions via some non–trivial computation. It is interesting to note that (2.9) is compatible without using consequences of system (2.8).

Some of equations from Sections 2.1–2.6 are well known. The others were found by more or less simple calculations by the algorithms described in the above cited by V.A. Adler. In contrast with this, finding similar objects for a vector Schwarz — KdV equation is not yet algorithmizable, it requires various non-trivial tricks, and is sometimes extremely time and memory–consuming.

3. Around vector Schwarz — KdV equation

In this section, for aesthetic reasons, instead of a single variable $\boldsymbol{u}$ with subscripts, we use various bold characters. The transition to index notation in the vector formulas is easily performed by comparing with the corresponding formulas from Section 2, taking into consideration that scalar formulas should coincide with vector ones for $N=1$.

3.1. Bäcklund transformation. For vector equations (1.2) the Bäcklund transformations, which we use, reads

$$\boldsymbol{u}_x=K_1\boldsymbol{v}_x+K_2\boldsymbol{u}+K_3\boldsymbol{v},$$

were K_i are some functions of the six scalar products of vector variables $\boldsymbol{v}_x$, $\boldsymbol{u}$, $\boldsymbol{v}$. In [10], a Bäcklund transformation for the vector Schwarz — KdV equation (1.3) was found

$$\boldsymbol{u}_x=\frac{\mu}{\boldsymbol{v}_x^2}(\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v}_x)(\boldsymbol{u}-\boldsymbol{v})-\frac{\mu}{2\boldsymbol{v}_x^2}(\boldsymbol{u}-\boldsymbol{v})^2\boldsymbol{v}_x. \tag{3.1}$$

It is easy to verify that (3.1) coincides with (2.1) if $N=1$, $\boldsymbol{u}\to u_{n+1}$, $\boldsymbol{v}\to u_n$, $\mu=2\alpha$.

At first glance, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are not involved in formula (3.1) symmetrically. However, multiplying (3.1) by $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{v}_x$ and $\boldsymbol{u}_x$ and expressing scalar products $(\boldsymbol{u},\boldsymbol{v}_x)$, $(\boldsymbol{v},\boldsymbol{v}_x)$, $(\boldsymbol{u}_x,\boldsymbol{v}_x)$ and $\boldsymbol{v}_x^2$ in terms of the remaining ones, it is not difficult to check that

$$\boldsymbol{v}_x=\frac{\mu}{\boldsymbol{u}_x^2}(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{u}_x)(\boldsymbol{v}-\boldsymbol{u})-\frac{\mu}{2\boldsymbol{u}_x^2}(\boldsymbol{v}-\boldsymbol{u})^2\boldsymbol{u}_x.$$

3.2. Volterra type chain.

The vector Volterra type chain has the form

$$\boldsymbol{v}_z=Q_1\boldsymbol{u}+Q_2\boldsymbol{v}+Q_3\boldsymbol{w},$$

where the coefficients Q_i are functions of all scalar products of the vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$. This chain has to be compatible with equation (1.3) for $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ modulo equation (3.1) taken together with

$$\boldsymbol{w}_x=\frac{\mu}{\boldsymbol{v}_x^2}(\boldsymbol{w}-\boldsymbol{v},\boldsymbol{v}_x)(\boldsymbol{w}-\boldsymbol{v})-\frac{\mu}{2\boldsymbol{v}_x^2}(\boldsymbol{w}-\boldsymbol{v})^2\boldsymbol{v}_x. \tag{3.2}$$

The compatibility condition

$$(Q_1\boldsymbol{u}+Q_2\boldsymbol{v}+Q_3\boldsymbol{w})_t=\left(\boldsymbol{v}_{xxx}-3\frac{(\boldsymbol{v}_x,\boldsymbol{v}_{xx})}{(\boldsymbol{v}_x,\boldsymbol{v}_x)}\boldsymbol{v}_{xx}+\frac{3}{2}\frac{(\boldsymbol{v}_{xx},\boldsymbol{v}_{xx})}{(\boldsymbol{v}_x,\boldsymbol{v}_x)}\boldsymbol{v}_x\right)_z \tag{3.3}$$

involves vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, the x-derivatives of which are related by (3.1) and (3.2). It is easy to see that the vector $\boldsymbol{v}_{xxx}$ in relation (3.3) cancels out, and we can take

$$\boldsymbol{u},\quad \boldsymbol{v},\quad \boldsymbol{w},\quad \boldsymbol{v}_x,\quad \boldsymbol{v}_{xx} \tag{3.4}$$

for independent vector jet variables. Finding the scalar products of relations (3.1) and (3.2) with independent vectors, we obtain a set of algebraic relations for scalar products of vectors (3.4). Each of these scalar products can be expressed in terms of products of vectors (3.4). We regard the latter products as independent ones.

Splitting (3.3) with respect to the independent vectors, we obtain a system of five PDEs for functions Q_i. Each of these PDEs can be split with respect to the independent scalar products that depend on x-derivatives. Solving the resulting system of linear PDEs for the functions Q_i, each of which depends on six variables, we arrive at the vector Volterra type chain

$$\boldsymbol{v}_z=\frac{(\boldsymbol{w}-\boldsymbol{v})^2(\boldsymbol{u}-\boldsymbol{v})-(\boldsymbol{u}-\boldsymbol{v})^2(\boldsymbol{w}-\boldsymbol{v})}{(\boldsymbol{u}-\boldsymbol{w})^2}. \tag{3.5}$$

It can be verified that (3.5) is compatible with (3.1). If $N=1$, then (3.5) coincides with (2.3) after the replacement $\{\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}\}\to\{u_n,u_{n-1},u_{n+1}\}$. Under additional assumption $|\boldsymbol{v}|=|\boldsymbol{u}|=|\boldsymbol{w}|=1$ this integrable vector Volterra type chain was found in [5] (see formula V_3) .

3.3. Negative flow.

Suppose that equations (3.5), (3.1), and (3.2) hold. We are looking for a relation of the form

$$\boldsymbol{v}_{zxx}=a_1\boldsymbol{v}_{zx}+a_2\boldsymbol{v}_{xx}+a_3\boldsymbol{v}_z+a_4\boldsymbol{v}_x+a_5\boldsymbol{v}, \tag{3.6}$$

where the coefficients depend on the scalar products of vectors

$$\boldsymbol{v}_{zx},\quad \boldsymbol{v}_{xx},\quad \boldsymbol{v}_z,\quad \boldsymbol{v}_x,\quad \boldsymbol{v}. \tag{3.7}$$

It is clear that $\boldsymbol{v}_z,\boldsymbol{v}_{zx},\boldsymbol{v}_{zxx}$ are linear combinations of

$$\boldsymbol{u},\quad \boldsymbol{v},\quad \boldsymbol{w},\quad \boldsymbol{v}_x,\quad \boldsymbol{v}_{xx}. \tag{3.8}$$

Using these relations, we can express $\boldsymbol{u}$, $\boldsymbol{w}$ in terms of $\boldsymbol{v}_{zx}$, $\boldsymbol{v}_{xx}$, $\boldsymbol{v}_z$, $\boldsymbol{v}_x$, $\boldsymbol{v}$, and obtain a relation of form (3.6). The main problem is that the coefficients in this relation depend on the scalar products of vectors (3.8) and we have to express these scalar products via the scalar products of vectors (3.7). These two collections of scalar products are related by a huge system of non–linear algebraic equations, which can be explicitly solved. As a result of such a computation we obtain

$$a_1 = \Big(\ln(\boldsymbol{v}_z, \boldsymbol{v}_x) \Big)_x, \qquad a_2 = \frac{1}{2}\Big(\ln \boldsymbol{v}_x^2 \Big)_z,$$

$$a_3 = \mu - \frac{(\boldsymbol{v}_{zx}, \boldsymbol{v}_{xx})}{(\boldsymbol{v}_z, \boldsymbol{v}_x)} - \frac{(\boldsymbol{v}_{zx}, \boldsymbol{v}_x)^2}{2(\boldsymbol{v}_z, \boldsymbol{v}_x)^2} - \frac{(\boldsymbol{v}_{zx}, \boldsymbol{v}_z)^2 \boldsymbol{v}_x^2}{2(\boldsymbol{v}_z, \boldsymbol{v}_x)^2 \boldsymbol{v}_z^2} + \frac{\boldsymbol{v}_{zx}^2 \boldsymbol{v}_x^2}{2(\boldsymbol{v}_z, \boldsymbol{v}_x)^2} + \frac{(\boldsymbol{v}_{zx}, \boldsymbol{v}_x)(\boldsymbol{v}_{xx}, \boldsymbol{v}_x)}{(\boldsymbol{v}_z, \boldsymbol{v}_x)\boldsymbol{v}_x^2} + \frac{\boldsymbol{v}_x^2}{2\boldsymbol{v}_z^2},$$

$$a_4 = \frac{(\boldsymbol{v}_{zx}, \boldsymbol{v}_z)^2}{(\boldsymbol{v}_z, \boldsymbol{v}_x)\boldsymbol{v}_z^2} - \frac{(\boldsymbol{v}_{xx}, \boldsymbol{v}_z)(\boldsymbol{v}_{zx}, \boldsymbol{v}_x)}{(\boldsymbol{v}_z, \boldsymbol{v}_x)\boldsymbol{v}_x^2} - \frac{\boldsymbol{v}_{zx}^2}{(\boldsymbol{v}_z, \boldsymbol{v}_x)} - \frac{(\boldsymbol{v}_z, \boldsymbol{v}_x)}{\boldsymbol{v}_z^2}, \qquad a_5 = 0. \tag{3.9}$$

In the scalar limit $N = 1$ formula (3.6) gives us (2.4) after the replacement $\boldsymbol{v} \to u$, $\mu = 2\alpha$ and $\beta = 1$.

3.4. System of NLS type. The simplest z-symmetry of (3.6), (3.9) reads

$$\begin{aligned} \boldsymbol{u}_\tau = & -\frac{\boldsymbol{u}_z^2}{(\boldsymbol{u}_z, \boldsymbol{u}_x)}\boldsymbol{u}_{zzx} + \frac{(\boldsymbol{u}_z, \boldsymbol{u}_{zx})}{(\boldsymbol{u}_z, \boldsymbol{u}_x)}\boldsymbol{u}_{zz} - \boldsymbol{u}_z^2 \Big(\frac{1}{(\boldsymbol{u}_z, \boldsymbol{u}_x)} \Big)_z \boldsymbol{u}_{zx} + \\ & + \left(\frac{\boldsymbol{u}_z^2 (\boldsymbol{u}_x, \boldsymbol{u}_{zx})_z}{\boldsymbol{u}_x^2 (\boldsymbol{u}_z, \boldsymbol{u}_x)} - \frac{\boldsymbol{u}_z^2 (\boldsymbol{u}_x, \boldsymbol{u}_{zx})(\boldsymbol{u}_x, \boldsymbol{u}_z)_z}{\boldsymbol{u}_x^2 (\boldsymbol{u}_z, \boldsymbol{u}_x)^2} - \frac{\boldsymbol{u}_z^2 (\boldsymbol{u}_x, \boldsymbol{u}_{zx})^2}{\boldsymbol{u}_x^4 (\boldsymbol{u}_z, \boldsymbol{u}_x)} \right) \boldsymbol{u}_x + \\ & + \left(\frac{(\boldsymbol{u}_z, \boldsymbol{u}_{zzx}) - (\boldsymbol{u}_{zx}, \boldsymbol{u}_{zz})}{(\boldsymbol{u}_z, \boldsymbol{u}_x)} - 2\frac{(\boldsymbol{u}_x, \boldsymbol{u}_{zx})_z}{\boldsymbol{u}_x^2} + 2\frac{(\boldsymbol{u}_x, \boldsymbol{u}_{zx})^2}{\boldsymbol{u}_x^4} - \frac{3}{2}\frac{(\boldsymbol{u}_z, \boldsymbol{u}_{zx})^2}{(\boldsymbol{u}_z, \boldsymbol{u}_x)^2} - \right. \\ & \left. - \frac{(\boldsymbol{u}_z, \boldsymbol{u}_{zx})(\boldsymbol{u}_x, \boldsymbol{u}_{zz})}{(\boldsymbol{u}_z, \boldsymbol{u}_x)^2} + \frac{1}{2}\frac{\boldsymbol{u}_z^2 \boldsymbol{u}_{zx}^2}{(\boldsymbol{u}_z, \boldsymbol{u}_x)^2} + 2\frac{(\boldsymbol{u}_x, \boldsymbol{u}_{zx})(\boldsymbol{u}_z, \boldsymbol{u}_x)_z}{\boldsymbol{u}_x^2 (\boldsymbol{u}_z, \boldsymbol{u}_x)} - \frac{1}{2}\frac{\boldsymbol{u}_z^2 (\boldsymbol{u}_x, \boldsymbol{u}_{zx})^2}{\boldsymbol{u}_x^2 (\boldsymbol{u}_z, \boldsymbol{u}_x)^2} \right) \boldsymbol{u}_z. \end{aligned} \tag{3.10}$$

If $N = 1$, this symmetry coincides with the symmetry from Section 2.4. As in the scalar case, we can rewrite (3.10) as the system of equations

$$\boldsymbol{u}_\tau = F_1(\boldsymbol{u}_z, \boldsymbol{u}_{zz}, \boldsymbol{v}, \boldsymbol{v}_z, \boldsymbol{v}_{zz}), \qquad \boldsymbol{v}_\tau = F_2(\boldsymbol{u}_z, \boldsymbol{u}_{zz}, \boldsymbol{v}, \boldsymbol{v}_z, \boldsymbol{v}_{zz}),$$

where $\boldsymbol{v} = \boldsymbol{u}_x$. After the transfomation

$$\boldsymbol{u} = \frac{\boldsymbol{v}'_z}{\xi} + \frac{(\boldsymbol{v}', \boldsymbol{v}'_z)\boldsymbol{u}'}{\xi f g} + \frac{1}{2\xi}\Big(f - (f\boldsymbol{u}' + g\boldsymbol{v}', \boldsymbol{v}'_z)\xi^{-2} \Big)\Big(\frac{\boldsymbol{u}'}{g} + \frac{\boldsymbol{v}'}{f} \Big), \qquad \boldsymbol{v} = \boldsymbol{v}',$$

where $\xi^2 = (\boldsymbol{u}', \boldsymbol{v}') + fg$, $f = |\boldsymbol{v}'|$, $g = |\boldsymbol{u}'|$, we obtain a constant separant system

$$\boldsymbol{u}'_\tau = \boldsymbol{u}'_{zz} + \Phi_1(\boldsymbol{u}', \boldsymbol{u}'_z, \boldsymbol{v}', \boldsymbol{v}'_z), \qquad \boldsymbol{v}'_\tau = -\boldsymbol{v}'_{zz} + \Phi_2(\boldsymbol{u}', \boldsymbol{u}'_z, \boldsymbol{v}', \boldsymbol{v}'_z). \tag{3.11}$$

In the scalar limit this system is reduced to

$$u_\tau = u_{zz} + \frac{(u^2 - u_z^2)v_z}{2uv} - \frac{u_z^2}{u} + \frac{u_z}{2}, \qquad v_\tau = -v_{zz} + \frac{(v^2 - v_z^2)u_z}{2uv} + \frac{v_z^2}{v} + \frac{v_z}{2}.$$

The terms $\frac{1}{2}u_z$ and $\frac{1}{2}v_z$ in the right hand sides can be eliminated by the Galilean transformation, and using then the transformation $u = e^{au'}$, $v = e^{-2v'/a}$, we obtain system (2.6) with $k_1 = -a^{-2}$, $k_2 = -a^2/4$ for (u', v') .

The explicit form of system (3.11) is very unpresentable. After restriction $\boldsymbol{u}'^2 = \boldsymbol{v}'^2 = 1$ to the sphere it becomes more compact

$$\begin{aligned}
\boldsymbol{u}_\tau =& \boldsymbol{u}_{zz} + 4\left(\frac{(\boldsymbol{u},\boldsymbol{v}_z)(\boldsymbol{u}_z,\boldsymbol{v})}{\boldsymbol{w}^4} - \frac{(\boldsymbol{u}_z,\boldsymbol{v}_z)}{\boldsymbol{w}^2} - \frac{1}{4}(\ln \boldsymbol{w}^2)_z\right)\boldsymbol{u}_z +, \\
&+ 2\left(\frac{\boldsymbol{u}_z^2}{\boldsymbol{w}^2} - \frac{(\boldsymbol{w}+\boldsymbol{u}_z,\boldsymbol{v})^2}{\boldsymbol{w}^4}\right)\boldsymbol{v}_z + 4\left(\frac{(\boldsymbol{u},\boldsymbol{v}_z)(\boldsymbol{w}+\boldsymbol{u}_z,\boldsymbol{v})^2}{\boldsymbol{w}^6} + \frac{\boldsymbol{u}_z^2(\boldsymbol{w}-\boldsymbol{v}_z,\boldsymbol{u})}{\boldsymbol{w}^4}\right)\boldsymbol{w}, \\
\boldsymbol{v}_\tau =& - \boldsymbol{v}_{zz} + 4\left(\frac{(\boldsymbol{u},\boldsymbol{v}_z)(\boldsymbol{u}_z,\boldsymbol{v})}{\boldsymbol{w}^4} - \frac{(\boldsymbol{u}_z,\boldsymbol{v}_z)}{\boldsymbol{w}^2} + \frac{1}{4}(\ln \boldsymbol{w}^2)_z\right)\boldsymbol{v}_z + \\
&+ 2\left(\frac{\boldsymbol{v}_z^2}{\boldsymbol{w}^2} - \frac{(\boldsymbol{w}-\boldsymbol{v}_z,\boldsymbol{u})^2}{\boldsymbol{w}^4}\right)\boldsymbol{u}_z + 4\left(\frac{(\boldsymbol{u}_z,\boldsymbol{v})(\boldsymbol{w}-\boldsymbol{v}_z,\boldsymbol{u})^2}{\boldsymbol{w}^6} - \frac{\boldsymbol{v}_z^2(\boldsymbol{w}+\boldsymbol{u}_z,\boldsymbol{v})}{\boldsymbol{w}^4}\right)\boldsymbol{w},
\end{aligned}$$

where $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ and the terms $\frac{1}{2}\boldsymbol{u}_z$ and $\frac{1}{2}\boldsymbol{v}_z$ are omitted as trivial.

3.5. $3D$ **consistent hyperbolic system.** The vector analog of equations (2.8), which satisfy the $3D$ consistency condition has the form

$$\boldsymbol{u}_{z_1 z_2} = b_1\boldsymbol{u}_{xz_1} + b_2\boldsymbol{u}_{xz_2} + b_3\boldsymbol{u}_{z_1} + b_4\boldsymbol{u}_{z_2} + b_5\boldsymbol{u}_x,$$

where

$$\begin{aligned}
b_1 =& \frac{\alpha_2}{(\alpha_2-\alpha_1)}\frac{(\boldsymbol{u}_{z_1},\boldsymbol{u}_{z_2})}{(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)}, \qquad b_2 = \frac{\alpha_1}{(\alpha_1-\alpha_2)}\frac{(\boldsymbol{u}_{z_1},\boldsymbol{u}_{z_2})}{(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)}, \\
b_3 =& \frac{(\boldsymbol{u}_{xz_2},\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)\alpha_1 + \big((\boldsymbol{u}_x,\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_{xz_1}) - (\boldsymbol{u}_{xz_1},\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)\big)\alpha_2}{(\boldsymbol{u}_x,\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)(\alpha_1-\alpha_2)}, \\
b_4 =& \frac{(\boldsymbol{u}_{xz_1},\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)\alpha_2 + \big((\boldsymbol{u}_x,\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_{xz_2}) - (\boldsymbol{u}_{xz_2},\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)\big)\alpha_1}{(\boldsymbol{u}_x,\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)(\alpha_2-\alpha_1)}, \\
b_5 =& \frac{(\boldsymbol{u}_{z_1},\boldsymbol{u}_{z_2})\big((\boldsymbol{u}_{xz_2},\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)\alpha_1 - (\boldsymbol{u}_{xz_1},\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)\alpha_2\big)}{(\boldsymbol{u}_x,\boldsymbol{u}_x)(\boldsymbol{u}_{z_1},\boldsymbol{u}_x)(\boldsymbol{u}_{z_2},\boldsymbol{u}_x)(\alpha_2-\alpha_1)}.
\end{aligned}$$

The calculations for finding these coefficients are extremely difficult, and the intermediate formulas contain several hundred thousand terms. The found equations are compatible by virtue of equations (3.6), (3.9). Unlike the scalar case, they are incompatible without the differential consequences of (3.6), (3.9).

3.6. Vector superposition formula. Assume that the following identities hold:

$$\begin{aligned}
\boldsymbol{u}_x &= \Phi(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}_x,\mu), \qquad & \boldsymbol{w}_x &= \Phi(\boldsymbol{w},\boldsymbol{v},\boldsymbol{v}_x,\nu), \\
\boldsymbol{s}_x &= \Phi(\boldsymbol{s},\boldsymbol{u},\boldsymbol{u}_x,\nu), \qquad & \boldsymbol{s}_x &= \Phi(\boldsymbol{s},\boldsymbol{w},\boldsymbol{w}_x,\mu),
\end{aligned} \tag{3.12}$$

where $\Phi(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}_x,\mu)$ denotes the right hand side of Bäcklund transformation (3.1).

By the vector formula of superposition we mean a relation of the form

$$\boldsymbol{s} = k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w},$$

where the coefficients k_i depend on the scalar products of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Eliminating the vectors $\boldsymbol{u}_x$, $\boldsymbol{w}_x$ and $\boldsymbol{s}_x$ from system (3.12), we obtain the relation

$$(\boldsymbol{u}-\boldsymbol{s},\boldsymbol{v}_x)\boldsymbol{A} + (\boldsymbol{w}-\boldsymbol{s},\boldsymbol{v}_x)\boldsymbol{B} + (\boldsymbol{v}-\boldsymbol{s},\boldsymbol{v}_x)\boldsymbol{C} + a\boldsymbol{v}_x = 0,$$

where the vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and the scalar a are independent of $\boldsymbol{v}_x$.

It is natural to assume[1] that this relation is an identity with respect to $\boldsymbol{v}_x$. This means that $a = 0$ and that, replacing $\boldsymbol{v}_x$ by $\boldsymbol{u}-\boldsymbol{s}$, $\boldsymbol{w}-\boldsymbol{s}$, and $\boldsymbol{v}-\boldsymbol{s}$, we obtain three vector relations, which do not involve the derivative $\boldsymbol{v}_x$. Their scalar products with the vector variables, together with

[1]This was suggested by V. Adler

the relation $a = 0$, give us an algebraic system for the scalar products $(\boldsymbol{u}, \boldsymbol{s})$, $(\boldsymbol{v}, \boldsymbol{s})$, $(\boldsymbol{w}, \boldsymbol{s})$, $(\boldsymbol{s}, \boldsymbol{s})$.

Non–trivial calculations show that this overdetermined system has two solutions. One of them leads to superposition formula [10]

$$\boldsymbol{s} = \boldsymbol{v} + (\mu - \nu)\frac{\nu(\boldsymbol{v} - \boldsymbol{w})^2(\boldsymbol{v} - \boldsymbol{u}) - \mu(\boldsymbol{v} - \boldsymbol{u})^2(\boldsymbol{v} - \boldsymbol{w})}{\big(\mu(\boldsymbol{v} - \boldsymbol{u}) - \nu(\boldsymbol{v} - \boldsymbol{w})\big)^2}. \tag{3.13}$$

The second solution corresponds to the replacement $\mu \mapsto -\mu$.

The corresponding Bianchi diagram

$$\begin{tikzcd} \boldsymbol{u} \arrow[r, "\nu"] & \boldsymbol{s} \\ \boldsymbol{v} \arrow[u, "\mu"] \arrow[r, "\nu"'] & \boldsymbol{w} \arrow[u, "\mu"'] \end{tikzcd}$$

admits the symmetry group generated by the reflection $\boldsymbol{u} \leftrightarrow \boldsymbol{w}$ and $\pi/2$ rotation. The symmetry of formula (3.13) with respect to the reflection (with $\mu \leftrightarrow \nu$) is obvious. Calculations similar to those described at the end of Section 3.1 show that formula (3.13) is symmetric with respect to the rotation.

Notice that (3.13) can be rewritten as

$$\mu\frac{\boldsymbol{w} - \boldsymbol{v}}{(\boldsymbol{w} - \boldsymbol{v})^2} - \nu\frac{\boldsymbol{u} - \boldsymbol{v}}{(\boldsymbol{u} - \boldsymbol{v})^2} = (\mu - \nu)\frac{\boldsymbol{s} - \boldsymbol{v}}{(\boldsymbol{s} - \boldsymbol{v})^2}.$$

In this form it is a vector analog of the cross–ratio model [6]. In the scalar limit the latter formula reduces to (2.7).

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