ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 3 (2026). P. 69-84.

doi:10.13108/2026-18-3-69

PERTURBED LINEAR EQUATIONS WITH HILFER FRACTIONAL DERIVATIVE

V.E. FEDOROV, A.A. BYKOV

Abstract. We study issues on unique solvability for the Cauchy–type problem for linear equations in a Banach space with the Hilfer fractional derivative, with an operator generating a strongly continuous resolving family, and with a bounded perturbation operator, either constant or time–dependent. It is shown that the addition of a constant bounded operator does not move us outside the class of generators of strongly continuous resolving families of operators. Based on the technique of the proof of this result, we show the existence of a unique solution to the Cauchy–type problem for the equation with a nonautonomous perturbation operator. We employ abstract results in the study of the unique solvability of an initial boundary value problem with periodic conditions in the spatial variable for the equation with the Hilfer fractional derivative in time and with the Laplace operator perturbed by a nonautonomous linear integral operator.

Keywords: Hilfer fractional derivative, resolving family of operators, perturbation theorem, nonautonomous perturbation, initial boundary value problem.

Mathematics Subject Classification: 34G10, 35R11.

1. Introduction

Differential equations with fractional derivatives have been the subject of active theoretical research in recent decades [9], [8] and are often used in mathematical modelling in applied problems [2], [10], [11], [12]. The interest in studying equations with the Hilfer fractional derivative $D^{\alpha,\beta}z = J^{\beta(m-\alpha)}D^m J^{(1-\beta)(m-\alpha)}z$ [7] is due to the fact that this class of derivatives includes the Gerasimov — Caputo fractional derivatives (for $\beta = 1$) and the Riemann — Liouville fractional derivatives (for $\beta = 0$), which are most commonly used in mathematical models. Previously, for equations in Banach spaces

$$D^{\alpha,\beta}z(t) = Az(t), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

necessary and sufficient conditions were obtained for the existence of strongly continuous [4] and analytic [5] resolving families of operators in terms of the resolvent of a linear closed densely defined operator A in a Banach space $\mathcal{Z}$. In [5], a perturbation theorem was established, which provided sufficient conditions on the perturbing linear operator B ensuring that once A generates an analytic resolving family of operators, then $A + B$ also generates such a family. Any bounded operator B satisfies this perturbation theorem.

In the present work, we study the solvability of the equation

$$D^{\alpha,\beta}z(t) = Az(t) + Bz(t), \qquad t \in \mathbb{R}_+,$$

V.E. Fedorov, A.A. Bykov, Perturbed linear equations with Hilfer fractional derivative.

The research is supported by the Russian Science Foundation, project no. 24-11-20002, https://rscf.ru/project/24-11-20002/ and is supported by the Goverment of Chelyabinsk Region.

Submitted February 28, 2026.

with an operator A generating a strongly continuous resolving family of operators. First, we show that adding a bounded operator B to such an operator A yields an operator $A+B$ which also generates a strongly continuous resolving family. Then the case of an operator $B=B(t)$ depending on t and bounded for each t is considered, and results on the solvability of homogeneous and inhomogeneous equations

$$D^{\alpha,\beta}z(t)=Az(t)+B(t)z(t) \qquad D^{\alpha,\beta}z(t)=Az(t)+B(t)z(t)+f(t),$$

are obtained.

In Section 2 we introduce preliminary information and define the class of operators $\mathcal{C}_{\alpha,\beta}$. We present a theorem on the necessity and sufficiency of the inclusion $A\in\mathcal{C}_{\alpha,\beta}$ for the existence of a strongly continuous resolving family of operators for equation (1.1). In Section 3 we obtain a theorem on the solvability of the linear inhomogeneous equation $D^{\alpha,\beta}z(t)=Az(t)+f(t)$ in the case $A\in\mathcal{C}_{\alpha,\beta}$ under more general conditions on the function f than in [4], allowing for a power–type singularity at zero. In Section 4 we obtain a perturbation theorem for the operator $A\in\mathcal{C}_{\alpha,\beta}$: we prove that the inclusions $A\in\mathcal{C}_{\alpha,\beta}$, $B\in\mathcal{L}(\mathcal{Z})$ imply $A+B\in\mathcal{C}_{\alpha,\beta}$. Section 5 provides a theorem on the solvability of the equation with the Hilfer derivative and a nonautonomous perturbation operator $B=B(t)$. In the last section, the obtained results are used to prove the unique solvability of an initial boundary value problem for a partial differential equation with a fractional derivative in time.

2. Strongly continuous and analytic resolving families of operators

Let $\mathcal{Z}$ be a Banach space, $h\in L_{1,\mathrm{loc}}(\mathbb{R}_+;\mathcal{Z})$. The Riemann — Liouville fractional integral of order $\beta>0$ for the function h is defined as

$$J^\beta h(t):=\frac{1}{\Gamma(\beta)}\int\limits_0^t (t-s)^{\beta-1}h(s)\,ds, \qquad t>0.$$

The Riemann — Liouville fractional derivative of order α reads $D^\alpha h(t):=D^mJ^{m-\alpha}h(t)$, where $m-1<\alpha\leqslant m\in\mathbb{N}$, D^m is the operator of differentiation of integer order m. For $\alpha>0$ we shall use the notation $J^\alpha h(t)=D^{-\alpha}h(t)$.

We denote

$$D^\gamma h(0):=\lim_{t\to 0+}D^\gamma h(t), \qquad \gamma\in\mathbb{R}.$$

The Hilfer derivative of order $\alpha\in(m-1,m]$, $m\in\mathbb{Z}$, and type $\beta\in[0,1]$ is defined as

$$\begin{aligned} D^{\alpha,\beta}h(t)&=D^{m-\beta(m-\alpha)}\left(J^{(1-\beta)(m-\alpha)}h(t)-\sum_{k=0}^{m-1}D^{k-(1-\beta)(m-\alpha)}h(0)\frac{t^k}{k!}\right)\\ &=D^m\left(J^{m-\alpha}h(t)-\sum_{k=0}^{m-1}\frac{D^{k-(1-\beta)(m-\alpha)}h(0)t^{k+\beta(m-\alpha)}}{\Gamma(k+\beta(m-\alpha)+1)}\right). \end{aligned} \tag{2.1}$$

For sufficiently smooth h, identity (2.1) implies the standard form of the Hilfer derivative

$$D^{\alpha,\beta}h(t)=J^{\beta(m-\alpha)}D^mJ^{(1-\beta)(m-\alpha)}h(t).$$

Remark 2.1. *It is clear that for $\beta=0$ the Hilfer fractional derivative coincides with the Riemann — Liouville fractional derivative, and for $\beta=1$ does with the Gerasimov — Caputo fractional derivative.*

Remark 2.2. *For $\alpha=m$ we have $D^{\alpha,\beta}=D^{m,\beta}=D^m$ for each $\beta\in[0,1]$.*

By $\mathfrak{L}[h]$ we denote the Laplace transform for the function $h : \mathbb{R}_+ \to \mathcal{Z}$. Hereinafter we shall use the notation $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$ and the principal branch of the power function of a complex variable.

Lemma 2.1 ([5]). *Let $m-1 < \alpha \leqslant m \in \mathbb{N}$, $\beta \in [0,1]$, and for $h : \mathbb{R}_+ \to \mathcal{Z}$ there exist $\mathfrak{L}[h]$, $D^{\alpha,\beta}h$, $\mathfrak{L}[D^{\alpha,\beta}h]$. Then*

$$\mathfrak{L}[D^{\alpha,\beta}h](\lambda) = \lambda^\alpha \mathfrak{L}[h](\lambda) - \sum_{k=0}^{m-1} D^{k-(1-\beta)(m-\alpha)}h(0)\lambda^{m-1-k-\beta(m-\alpha)}.$$

By the symbol $\mathcal{L}(\mathcal{Z})$ we denote the Banach space of all linear bounded operators on the space $\mathcal{Z}$, and by $\mathcal{C}l(\mathcal{Z})$ the set of all linear closed operators densely defined in the space $\mathcal{Z}$ and acting in this space. We equip the domain D_A of the operator $A \in \mathcal{C}l(\mathcal{Z})$ with its graph norm $\|\cdot\|_{D_A} := \|\cdot\|_\mathcal{Z} + \|A\cdot\|_\mathcal{Z}$ and thereby obtain the Banach space D_A.

By $AC^m(\mathbb{R}_+;\mathcal{Z})$ we denote the set of all functions $h \in C^{m-1}(\overline{\mathbb{R}}_+;\mathcal{Z})$ having absolutely continuous on each interval $[t_0,T] \subset \mathbb{R}_+$ derivative of order $m-1$.

Consider the Cauchy–type problem

$$D^{k-(1-\beta)(m-\alpha)}z(0) = z_k, \qquad k = 0,1,\dots,m-1, \tag{2.2}$$

for the linear equation

$$D^{\alpha,\beta}z(t) = Az(t), \qquad t \in \mathbb{R}_+, \tag{2.3}$$

where $m-1 < \alpha \leqslant m \in \mathbb{N}$, $A \in \mathcal{C}l(\mathcal{Z})$. A solution to problem (2.2), (2.3) is a function $z \in C(\mathbb{R}_+;D_A) \cap L_{1,\text{loc}}(\mathbb{R}_+;\mathcal{Z})$ such that $J^{(1-\beta)(m-\alpha)}z \in AC^m(\mathbb{R}_+;\mathcal{Z})$, $D^{\alpha,\beta}z \in C(\mathbb{R}_+;\mathcal{Z})$, conditions (2.2) hold, and identity (2.3) holds for $t \in \mathbb{R}_+$.

Let $m-1 < \alpha \leqslant m \in \mathbb{N}$, $\beta \in [0,1]$. A family of operators $\{S(t) \in \mathcal{L}(\mathcal{Z}) : t \in \mathbb{R}_+\}$ is called a resolving family for equation (2.3) if the following conditions hold:

(i) there exist $\omega \geqslant 0$ and $K > 0$ such that
$$\|S(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant Kt^{-(1-\beta)(m-\alpha)}e^{\omega t}$$
for all $t \in \mathbb{R}_+$;

(ii) for all $z_0 \in \mathcal{Z}$ $S(t)z_0 \in C(\mathbb{R}_+;\mathcal{Z})$,
$$\operatorname*{s-lim}_{t\to 0+} J^{(1-\beta)(m-\alpha)}S(t) = I;$$

(iii) $S(t)[D_A] \subset D_A$, $S(t)Az_0 = AS(t)z_0$ for all $z_0 \in D_A$, $t \in \mathbb{R}_+$;

(iv) for each $z_0 \in D_A$ the function $S(t)z_0$ is a solution of the Cauchy–type problem
$$D^{-(1-\beta)(m-\alpha)}z(0) = z_0, \qquad D^{k-(1-\beta)(m-\alpha)}z(0) = 0, \quad k = 1,2,\dots,m-1,$$
for the equation (2.3).

It is easy to show that for all $z_0, z_1, \dots, z_{m-1} \in D_A$, the sum $\sum\limits_{k=0}^{m-1} J^k S(t)z_0$ solves problem (2.2), (2.3).

Remark 2.3. *For $A \in \mathcal{L}(\mathcal{Z})$ the resolving family of operators for equation (2.3) reads (see, e.g., [1])*

$$S(t) = \sum_{l=0}^{\infty} \frac{t^{\alpha l+(1-\beta)(\alpha-m)}A^l}{\Gamma(\alpha l + (1-\beta)(\alpha-m)+1)} = t^{(1-\beta)(\alpha-m)}E_{\alpha,(1-\beta)(\alpha-m)+1}(t^\alpha A), \qquad t \in \mathbb{R}_+,$$

where

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$$

is the Mittag–Leffler function.

For $m-1<\alpha\leqslant m\in\mathbb{N}$, $\beta\in[0,1]$ an operator $A\in\mathcal{C}l(\mathcal{Z})$ is called an operator of class $\mathcal{C}_{\alpha,\beta}(K,\omega)$, where $K>0$, $\omega\geqslant 0$, if the following two conditions hold:

(i) if $\operatorname{Re}\lambda>\omega$, then $\lambda^\alpha\in\rho(A):=\{\mu\in\mathbb{C}:(\mu-A)^{-1}\in\mathcal{L}(\mathcal{Z})\}$;

(ii) for all $\operatorname{Re}\lambda>\omega$ and $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$

$$\left\|\frac{d^n}{d\lambda^n}\left(\lambda^{m-1-\beta(m-\alpha)}(\lambda^\alpha-A)^{-1}\right)\right\|_{\mathcal{L}(\mathcal{Z})}\leqslant\frac{K\Gamma((1-\beta)(\alpha-m)+n+1)}{(\operatorname{Re}\lambda-\omega)^{(1-\beta)(\alpha-m)+n+1}}.$$

We introduce the notations

$$\mathcal{C}_{\alpha,\beta}(\omega):=\bigcup_{K>0}\mathcal{C}_{\alpha,\beta}(K,\omega),\qquad \mathcal{C}_{\alpha,\beta}:=\bigcup_{\omega\geqslant 0}\mathcal{C}_{\alpha,\beta}(\omega).$$

Theorem 2.1 ([4]). *Let $\beta\in[0,1]$, $\alpha>2$. Then $C_{\alpha,\beta}\subset\mathcal{L}(\mathcal{Z})$.*

Let $A\in\mathcal{C}_{\alpha,\beta}(\omega)$ for some $\omega\geqslant 0$; for $\operatorname{Re}\lambda>\omega$,

$$H_\beta(\lambda):=\lambda^{m-1-\beta(m-\alpha)}(\lambda^\alpha-A)^{-1},$$

and $n\in\mathbb{N}$, $t\in\mathbb{R}_+$ we define the operators

$$\mathcal{S}_n(t):=e^{-nt}\sum_{k=0}^{\infty}\frac{(-1)^k(n(n+\omega)t)^{k+1}}{k!(k+1)!}H_\beta^{(k)}(n+\omega).$$

Theorem 2.2 ([4]). *Let $m-1<\alpha\leqslant m\in\mathbb{N}$, $\beta\in[0,1]$. Then a resolving family of operators for equation (2.3) exists if and only if $A\in\mathcal{C}_{\alpha,\beta}$. Moreover, $S(t)=\operatorname*{s-lim}_{n\to\infty}\mathcal{S}_n(t)$ for all $t>0$.*

3. Linear inhomogeneous equation with Hilfer derivative

We consider Cauchy–type problem (2.2) for the linear inhomogeneous equation

$$D^{\alpha,\beta}z(t)=Az(t)+f(t),\qquad t\in(0,T],\tag{3.1}$$

where $m-1<\alpha\leqslant m\in\mathbb{N}$, $\beta\in[0,1]$, $A\in\mathcal{C}l(\mathcal{Z})$, $f:[0,T]\to\mathcal{Z}$. A solution to problem (2.2), (3.1) is a function $z\in C((0,T];D_A)\cap L_1(0,T;\mathcal{Z})$ such that

$$J^{(1-\beta)(m-\alpha)}z\in C^{m-1}([0,T];\mathcal{Z})\cap AC^m((0,T];\mathcal{Z}),\qquad D^{\alpha,\beta}z\in C((0,T];\mathcal{Z}),$$

and it satisfies conditions (2.2) and equation (2.3) for $t\in(0,T]$.

The space of functions $v\in C((0,T];\mathcal{Z})$ such that $t^\gamma v(t)\in C([0,T];\mathcal{Z})$ for some $\gamma\in\mathbb{R}$ is denoted by $C_\gamma((0,T];\mathcal{Z})$ and equipped with the norm

$$\|v\|_{C_\gamma((0,T];\mathcal{Z})}:=\sup_{t\in[0,T]}\|t^\gamma v(t)\|_{\mathcal{Z}}.$$

Under this notation $C_0((0,T];\mathcal{Z})=C([0,T];\mathcal{Z})$. For $\gamma,\delta\in\mathbb{R}$, $\gamma<\delta$ we have $C_\gamma((0,T];\mathcal{Z})\subset C_\delta((0,T];\mathcal{Z})$.

For each $\gamma\in\mathbb{R}$ and $v\in C_\gamma((0,T];\mathcal{Z})$ we have

$$\|v(t)\|_{\mathcal{Z}}\leqslant t^{-\gamma}\|v\|_{C_\gamma((0,T];\mathcal{Z})},\qquad t\in(0,T].$$

This is why as $\gamma<0$, we have $v(0)=0$ for $v\in C_\gamma((0,T];\mathcal{Z})$.

Theorem 3.1. *Let $\alpha\in(1,2]$, $\beta\in[0,1]$, $A\in\mathcal{C}_{\alpha,\beta}$, $f\in C_\gamma((0,T];\mathcal{Z})$, $\gamma<1-\beta(2-\alpha)$, $f(t)\in D_A$ for $t>0$. Then the Cauchy–type problem*

$$D^{-(1-\beta)(2-\alpha)}z(0)=z_0,\qquad D^{1-(1-\beta)(2-\alpha)}z(0)=z_1$$

for equation (3.1) *for all* $z_0, z_1 \in D_A$ *has a unique solution. Moreover, it has the form*

$$z(t) = S(t)z_0 + J^1 S(t)z_1 + \int_0^t J^{1-\beta(2-\alpha)} S(t-s)f(s)\,ds.$$

Proof. The uniqueness of the solution of the problem follows from the uniqueness of the solution $z(t) = S(t)z_0 + J^1 S(t)z_1$ of the Cauchy–type problem for the homogeneous equation $D^{\alpha,\beta}z(t) = Az(t)$, which was proved in [4, Cor. 5]. It sufficient to prove that the function

$$z_f(t) = \int_0^t J^{1-\beta(2-\alpha)} S(t-s)f(s)\,ds$$

solves the problem

$$D^{-(1-\beta)(2-\alpha)}z(0) = 0, \qquad D^{1-(1-\beta)(2-\alpha)}z(0) = 0$$

for equation (3.1).

For $t > 0$, in view of the definition of the resolving family of operators,

$$\|J^{1-\beta(2-\alpha)}S(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant K \int_0^t \frac{(t-s)^{-\beta(2-\alpha)}s^{-(1-\beta)(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))} e^{\omega s}\,ds \leqslant C_1 e^{\omega t} t^{\alpha-1},$$

$$\|z_f(t)\|_{\mathcal{Z}} \leqslant \frac{C_1 e^{\omega t}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|f(s)\|_{\mathcal{Z}}\,ds \leqslant C_2 e^{\omega t} t^{\alpha-\gamma},$$

$$\|J^{(1-2\beta)(2-\alpha)+1}S(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant K \int_0^t \frac{(t-s)^{(1-2\beta)(2-\alpha)}s^{-(1-\beta)(2-\alpha)}}{\Gamma((1-2\beta)(2-\alpha)+1)} e^{\omega s}\,ds \leqslant C_3 e^{\omega t} t^{1-\beta(2-\alpha)}$$

for some $\omega \geqslant 0$. Since $1-\beta(2-\alpha) > 0$, we have $J^{(1-2\beta)(2-\alpha)+1}S(0) = 0$. Therefore,

$$\|J^{(1-\beta)(2-\alpha)}z_f(t)\|_{\mathcal{Z}} = \left\| \int_0^t J^{(1-2\beta)(2-\alpha)+1}S(t-s)f(s)\,ds \right\|_{\mathcal{Z}} \leqslant C_4 e^{\omega t} t^{2-\beta(2-\alpha)-\gamma}$$

and hence $J^{(1-\beta)(2-\alpha)}z_f(0) = 0$.

We then find

$$D^1 J^{(1-2\beta)(2-\alpha)+1}S(t)z_0 = J^{1-\beta(2-\alpha)}D^1 J^{(1-\beta)(2-\alpha)}S(t)z_0 + \frac{t^{-\beta(2-\alpha)}z_0}{\Gamma(1-\beta(2-\alpha))},$$

and by the identity $D^1 J^{(1-\beta)(2-\alpha)}S(0) = 0$, obtained in the proof of Theorem 2.2 (see [4]), we get

$$\|D^1 J^{(1-2\beta)(2-\alpha)+1}S(t)z_0\|_{\mathcal{Z}} \leqslant C_5 e^{\omega t} t^{-\beta(2-\alpha)}\|z_0\|_{\mathcal{Z}},$$

$$D^{1-(1-\beta)(2-\alpha)}z_f(t) = D^1 \int_0^t J^{(1-2\beta)(2-\alpha)+1}S(t-s)f(s)\,ds = J^{(1-2\beta)(2-\alpha)+1}S(0)f(t)$$
$$+ \int_0^t D^1 J^{(1-2\beta)(2-\alpha)+1}S(t-s)f(s)\,ds = \int_0^t D^1 J^{(1-2\beta)(2-\alpha)+1}S(t-s)f(s)\,ds.$$

This is why

$$\|D^{1-(1-\beta)(2-\alpha)}z_f(t)\|_{\mathcal{Z}} \leqslant C_5 e^{\omega t}\int\limits_0^t (t-s)^{-\beta(2-\alpha)}\|f(s)\|_{\mathcal{Z}}\,ds \leqslant C_6 e^{\omega t} t^{1-\beta(2-\alpha)-\gamma},$$

and hence $D^{1-(1-\beta)(2-\alpha)}z_f(0) = 0$, since $1-\beta(2-\alpha)-\gamma > 0$.

By definition, the Hilfer derivative has the form (2.1), i.e. in this case

$$D^{\alpha,\beta}h(t) = D^{2-\beta(2-\alpha)}(J^{(1-\beta)(2-\alpha)}h(t) - D^{-(1-\beta)(2-\alpha)}h(0) - D^{1-(1-\beta)(2-\alpha)}h(0)t).$$

Using this identity, the identities

$$J^{(1-\beta)(2-\alpha)}S(0) = I, \qquad D^1 J^{(1-\beta)(2-\alpha)}S(0) = 0$$

and the condition $f(s) \in D_A$ for $s > 0$, for $t \in (0,T]$ we obtain

$$\begin{aligned} A\int\limits_0^t J^{1-\beta(2-\alpha)}S(t-s)f(s)\,ds &= \int\limits_0^t J^{1-\beta(2-\alpha)}AS(t-s)f(s)\,ds \\ &= \int\limits_0^t J^{1-\beta(2-\alpha)}D^{2-\beta(2-\alpha)}(J^{(1-\beta)(2-\alpha)}S(t-s)-I)f(s)\,ds \\ &= \int\limits_0^t J^{1-\beta(2-\alpha)}D^1 J^{\beta(2-\alpha)}D^1(J^{(1-\beta)(2-\alpha)}S(t-s)-I)f(s)\,ds \\ &= \int\limits_0^t J^{1-\beta(2-\alpha)}D^1 J^{\beta(2-\alpha)}D^1 J^{(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds \\ &= \int\limits_0^t D^1 J^{1-\beta(2-\alpha)}J^{\beta(2-\alpha)}D^1 J^{(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds \\ &= \int\limits_0^t D^1 J^{(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds. \end{aligned} \tag{3.2}$$

The last integral converges since $\gamma < 1$, and therefore $z_f(t) \in D_A$ for all $t \in (0,T]$.

Finally, since

$$D^{-(1-\beta)(2-\alpha)}z_f(0) = D^{1-(1-\beta)(2-\alpha)}z_f(0) = 0,$$

we have

$$\begin{aligned} D^{\alpha,\beta}z_f(t) =& D^2 J^{2-\alpha}\int\limits_0^t J^{1-\beta(2-\alpha)}S(t-s)f(s)\,ds = D^2\int\limits_0^t J^{1+(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds \\ =& D^1[J^{1+(1-\beta)(2-\alpha)}S(0)f(t)] + D^1\int\limits_0^t J^{(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds \\ =& J^{(1-\beta)(2-\alpha)}S(0)f(t) + \int\limits_0^t D^1 J^{(1-\beta)(2-\alpha)}S(t-s)f(s)\,ds = f(t) + Az_f(t) \end{aligned} \tag{3.3}$$

by (3.2) and the identities

$$J^{(1-\beta)(2-\alpha)}S(0) = I, \qquad J^{1+(1-\beta)(2-\alpha)}S(0) = 0.$$

The proof is complete. □

4. Equation with stationary perturbation

Theorem 4.1. *Let $\alpha \in (1,2)$, $\beta \in [0,1]$, $A \in \mathcal{C}_{\alpha,\beta}(K,\omega)$, $B \in \mathcal{L}(\mathcal{Z})$. Then*

$$A + B \in \mathcal{C}_{\alpha,\beta}(K_2, \omega + K^{\frac{1}{\alpha}}\|B\|^{\frac{1}{\alpha}}_{\mathcal{L}(\mathcal{Z})}\Gamma(1-(1-\beta)(2-\alpha))^{\frac{1}{\alpha}})$$

for some $K_2 > 0$.

Proof. According to Theorem 3.1, the solution of the Cauchy–type problem

$$J^{(1-\beta)(m-\alpha)}z(0) = z_0, \qquad D^{1-(1-\beta)(m-\alpha)}z(0) = z_1$$

for the equation

$$D^{\alpha,\beta}v(t) = (A+B)v(t), \tag{4.1}$$

in the case of existence reads

$$v(t) = S_A(t)z_0 + \int_0^t J^{1-\beta(2-\alpha)}S_A(t-s)Bv(s)\,ds,$$

where $S_A(t)$ are the operators of the resolving family of the unperturbed equation (2.3). Therefore, the operators of the resolving family of equation (4.1) should read

$$S_{A+B}(t) = S_A(t) + \int_0^t J^{1-\beta(2-\alpha)}S_A(t-s)BS_{A+B}(s)\,ds, \tag{4.2}$$

if we take into consideration the boundedness of the operators on both sides of this identity and the fact that D_A is dense in $\mathcal{Z}$. We seek $S_{A+B}(t)$ by the method of successive approximations in the form

$$S_{A+B}(t) = \sum_{n=0}^{\infty} S_n(t), \tag{4.3}$$

where $S_0(t) = S_A(t)$,

$$S_{n+1}(t) = \int_0^t J^{1-\beta(2-\alpha)}S_0(t-s)BS_n(s)\,ds, \qquad n \in \mathbb{N}.$$

At the same time for $t > 0$ we have

$$\begin{aligned}\|J^{1-\beta(2-\alpha)}S_0(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant& K\int_0^t \frac{(t-s)^{-\beta(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))}e^{\omega s}s^{-(1-\beta)(2-\alpha)}\,ds\\ \leqslant& \frac{Ke^{\omega t}t^{\alpha-1}}{\Gamma(1-\beta(2-\alpha))}\mathcal{B}(1-\beta(2-\alpha), 1-(1-\beta)(2-\alpha))\\ =& \frac{K_1 e^{\omega t}t^{\alpha-1}}{\Gamma(\alpha)},\end{aligned} \tag{4.4}$$

where $\mathcal{B}$ is the Euler beta function, $K_1 = K\Gamma(1-(1-\beta)(2-\alpha))$. By induction one can prove the inequalities

$$\|S_n(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{K_1^{n+1}\|B\|^n_{\mathcal{L}(\mathcal{Z})}e^{\omega t}t^{n\alpha-(1-\beta)(2-\alpha)}}{\Gamma(n\alpha-(1-\beta)(2-\alpha)+1)}, \qquad t>0, \qquad n\in\mathbb{N}_0.$$

Indeed, for $n=0$ the inequality holds; further, for $t>0$ and provided that this inequality holds for a given $n\in\mathbb{N}$ we have

$$\begin{aligned}\|S_{n+1}(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant & \int\limits_0^t \frac{K_1e^{\omega(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{K_1^{n+1}\|B\|^{n+1}_{\mathcal{L}(\mathcal{Z})}e^{\omega s}s^{n\alpha-(1-\beta)(2-\alpha)}}{\Gamma(n\alpha-(1-\beta)(2-\alpha)+1)}ds \\ \leqslant & \frac{K_1^{n+2}\|B\|^{n+1}_{\mathcal{L}(\mathcal{Z})}e^{\omega t}t^{(n+1)\alpha-(1-\beta)(2-\alpha)}}{\Gamma(\alpha)\Gamma(n\alpha-(1-\beta)(2-\alpha)+1)}\mathcal{B}(\alpha,n\alpha-(1-\beta)(2-\alpha)+1) \\ = & \frac{K_1^{n+2}\|B\|^{n+1}_{\mathcal{L}(\mathcal{Z})}e^{\omega t}t^{(n+1)\alpha-(1-\beta)(2-\alpha)}}{\Gamma((n+1)\alpha-(1-\beta)(2-\alpha)+1)}.\end{aligned}$$

This is why series (4.3) converges and

$$\begin{aligned}\|S_{A+B}(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant & \sum_{n=0}^{\infty}\frac{K_1^{n+1}\|B\|^n_{\mathcal{L}(\mathcal{Z})}e^{\omega t}t^{n\alpha-(1-\beta)(2-\alpha)}}{\Gamma(n\alpha-(1-\beta)(2-\alpha)+1)} \\ = & K_1t^{-(1-\beta)(2-\alpha)}e^{\omega t}E_{\alpha,1-(1-\beta)(2-\alpha)}(t^\alpha K_1\|B\|_{\mathcal{L}(\mathcal{Z})}) \\ \leqslant & K_2t^{-(1-\beta)(2-\alpha)}e^{t(\omega+K_1^{\frac{1}{\alpha}}\|B\|^{\frac{1}{\alpha}}_{\mathcal{L}(\mathcal{Z})})}, \qquad t>0,\end{aligned}$$

for some $K_2>0$.

For $\operatorname{Re}\lambda > \omega + K_1^{\frac{1}{\alpha}}\|B\|^{\frac{1}{\alpha}}_{\mathcal{L}(\mathcal{Z})}$ by (4.2) we obtain

$$\begin{aligned}\widehat{S}_{A+B}(\lambda) = & \int\limits_0^\infty e^{-\lambda t}S_{A+B}(t)dt = \int\limits_0^\infty e^{-\lambda t}S_A(t)dt + \int\limits_0^\infty e^{-\lambda t}\int\limits_0^t J^{1-\beta(2-\alpha)}S_A(t-s)BS_{A+B}(s)dsdt \\ = & \lambda^{1-\beta(2-\alpha)}(\lambda^\alpha-A)^{-1} + \int\limits_0^\infty\int\limits_s^\infty e^{-\lambda t}J^{1-\beta(2-\alpha)}S_A(t-s)BS_{A+B}(s)dtds \\ = & \lambda^{1-\beta(2-\alpha)}(\lambda^\alpha-A)^{-1} + \int\limits_0^\infty\int\limits_0^\infty e^{-\lambda(\tau+s)}J^{1-\beta(2-\alpha)}S_A(\tau)BS_{A+B}(s)d\tau\, ds \\ = & \lambda^{1-\beta(2-\alpha)}(\lambda^\alpha-A)^{-1} + (\lambda^\alpha-A)^{-1}B\widehat{S}_{A+B}(\lambda),\end{aligned}$$

since

$$\mathfrak{L}[J^{1-\beta(2-\alpha)}S_A](\lambda) = (\lambda^\alpha-A)^{-1}.$$

This yields

$$\lambda^{\beta(2-\alpha)-1}(\lambda^\alpha-A-B)\widehat{S}_{A+B}(\lambda) = I.$$

Since

$$\|B\|_{\mathcal{L}(\mathcal{Z})} < \frac{(\operatorname{Re}\lambda-\omega)^\alpha}{K_1} \leqslant \frac{(\operatorname{Re}\lambda-\omega)^{\alpha-1+\beta(2-\alpha)}|\lambda|^{1-\beta(2-\alpha)}}{K_1} = \|(\lambda^\alpha-A)^{-1}\|^{-1}_{\mathcal{L}(\mathcal{Z})},$$

we have

$$\|(\lambda^\alpha-A)^{-1}B\|_{\mathcal{L}(\mathcal{Z})} < 1,$$

the operator

$$\lambda^\alpha - A - B = (\lambda^\alpha-A)(I-(\lambda^\alpha-A)^{-1}B)$$

is invertible and the identity

$$\widehat{S}_{A+B}(\lambda)=\lambda^{1-\beta(2-\alpha)}(\lambda^{\alpha}-A-B)^{-1}$$

holds. Then for all $\operatorname{Re}\lambda>\omega+K_1^{\frac{1}{\alpha}}\|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}}$, $n\in\mathbb{N}_0$, we have

$$\begin{aligned}\left\|\frac{d^n}{d\lambda^n}[\lambda^{1-\beta(2-\alpha)}(\lambda^{\alpha}-A-B)^{-1}]\right\|_{\mathcal{L}(\mathcal{Z})}&=\left\|\int\limits_0^{\infty}(-t)^n e^{-\lambda t}S_{A+B}(t)dt\right\|_{\mathcal{L}(\mathcal{Z})}\\&\leqslant K_2\int\limits_0^{\infty}t^{n-(1-\beta)(2-\alpha)}e^{t(\omega+K_1^{\frac{1}{\alpha}}\|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}}-\operatorname{Re}\lambda)}dt\\&=\frac{K_2\Gamma(n-(1-\beta)(2-\alpha)+1)}{(\operatorname{Re}\lambda-\omega-K_1^{\frac{1}{\alpha}}\|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}})^{n-(1-\beta)(2-\alpha)+1}}.\end{aligned}$$

Therefore,

$$A+B\in\mathcal{C}_{\alpha,\beta}(K_2,\omega+K_1^{\frac{1}{\alpha}}\|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}}).$$

The proof is complete. □

Remark 4.1. *It is clear that a solution to the problem*

$$J^{(1-\beta)(m-\alpha)}z(0)=z_0,\qquad D^{1-(1-\beta)(m-\alpha)}z(0)=z_1$$

for the equation

$$D^{\alpha,\beta}z(t)=(A+B)z(t)$$

for $\alpha\in(1,2)$, $\beta\in[0,1]$, $A\in\mathcal{C}_{\alpha,\beta}(K,\omega)$, $B\in\mathcal{L}(\mathcal{Z})$, $z_0,z_1\in D_A$ *is the function*

$$z(t)=S_{A+B}(t)z_0+J^1S_{A+B}(t)z_1.$$

Corollary 4.1. *Under the assumptions of Theorem 4.1*

$$\|S_{A+B}(t)-S_A(t)\|_{\mathcal{L}(\mathcal{Z})}$$
$$\leqslant K_1e^{\omega t}t^{-(1-\beta)(2-\alpha)}\left(E_{\alpha,1-(1-\beta)(2-\alpha)}(t^{\alpha}K_1\|B\|_{\mathcal{L}(\mathcal{Z})})-\frac{1}{\Gamma(1-(1-\beta)(2-\alpha))}\right).$$

Proof. It follows from the proof of Theorem 4.1, by construction,

$$S_{A+B}(t)-S_A(t)=\sum_{n=1}^{\infty}S_n(t),$$

this is why

$$\|S_{A+B}(t)-S_A(t)\|_{\mathcal{L}(\mathcal{Z})}\leqslant\sum_{n=1}^{\infty}\frac{K_1^{n+1}\|B\|_{\mathcal{L}(\mathcal{Z})}^n e^{\omega t}t^{n\alpha-(1-\beta)(2-\alpha)}}{\Gamma(n\alpha-(1-\beta)(2-\alpha)+1)},$$

and this completes the proof. □

Since $D^{\alpha,1}$ is the Gerasimov — Caputo derivative operator, and $D^{\alpha,0}$ is the Riemann — Liouville derivative operator, by letting $\beta=1$ and $\beta=0$ in the formulation of Theorem 4.1, we obtain statements for equations with the Gerasimov — Caputo and Riemann — Liouville derivatives, respectively.

Corollary 4.2. *Let* $\alpha\in(1,2)$, $A\in\mathcal{C}_{\alpha,1}(K,\omega)$, $B\in\mathcal{L}(\mathcal{Z})$. *Then for some* $K_2>0$

$$A+B\in\mathcal{C}_{\alpha,1}(K_2,\omega+K^{\frac{1}{\alpha}}\|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}})$$

for some $K_2>0$.

Corollary 4.3. *Let $\alpha \in (1,2)$, $A \in \mathcal{C}_{\alpha,0}(K,\omega)$, $B \in \mathcal{L}(\mathcal{Z})$. Then for some $K_2 > 1$*

$$A + B \in \mathcal{C}_{\alpha,0}(K_2, \omega + K^{\frac{1}{\alpha}} \|B\|_{\mathcal{L}(\mathcal{Z})}^{\frac{1}{\alpha}} \Gamma(\alpha-1)^{\frac{1}{\alpha}})$$

for some $K_2 > 0$.

5. Equation with non–stationary perturbation

For $B \in C_\gamma(\mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$ we denote $b_t := \max\limits_{s\in[0,t]} \|s^\gamma B(s)\|_{\mathcal{L}(\mathcal{Z})}$ for $t > 0$, then for $s \in (0,t]$ we have $\|B(s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant b_t s^{-\gamma}$.

Theorem 5.1. *Let $\alpha \in (1,2)$, $\beta \in [0,1]$, $A \in \mathcal{C}_{\alpha,\beta}(K,\omega)$, $B \in C_\gamma(\mathbb{R}_+; \mathcal{L}(\mathcal{Z}))$, $\operatorname{im} B(t) \subset D_A$ for $t \in \mathbb{R}_+$, $\gamma < \alpha - 1$. Then for each $z_0 \in \mathcal{Z}$ the Cauchy problem*

$$J^{(1-\beta)(2-\alpha)} z(0) = z_0, \qquad D^{1-(1-\beta)(2-\alpha)} z(0) = 0 \tag{5.1}$$

for the equation

$$D^{\alpha,\beta} z(t) = (A + B(t)) z(t), \qquad t \in \mathbb{R}_+, \tag{5.2}$$

has a unique solution, which reads

$$z(t) = Z_{A+B}(t) z_0 := \sum_{n=0}^{\infty} Z_n(t) z_0, \tag{5.3}$$

where $Z_0(t) = S_A(t)$,

$$Z_{n+1}(t) = \int\limits_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds, \qquad t \in \mathbb{R}_+, \qquad n \in \mathbb{N}_0.$$

Moreover, there exists $C > 0$ such that for all $t > 0$ we have

$$\|Z_{A+B}(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant K t^{-(1-\beta)(2-\alpha)} e^{\omega t + C b_t t^{\alpha-\gamma}}, \tag{5.4}$$

$$\|Z_{A+B}(t) - S_A(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant K t^{-(1-\beta)(2-\alpha)} e^{\omega t} (e^{C b_t t^{\alpha-\gamma}} - 1). \tag{5.5}$$

Proof. By the asymptotic representation for the Gamma function

$$\Gamma(u) \sim \frac{\sqrt{2\pi} u^u}{\sqrt{u} e^u}, \qquad |\arg u| < \pi$$

for some $C > 0$ and all $k \in \mathbb{N}$ we have

$$\frac{\Gamma((k-1)(\alpha-\gamma) - (1-\beta)(2-\alpha) - \gamma + 1)}{\Gamma(k(\alpha-\gamma) - (1-\beta)(2-\alpha) + 1)} \leqslant C_1 (k(\alpha-\gamma) + 1 - (1-\beta)(2-\alpha))^{-\alpha} \leqslant C_2 k^{-\alpha}.$$

Here we have taken into consideration that for sufficiently large $k \in \mathbb{N}$ we have

$$\gamma < \frac{1 - (1-\beta)(2-\alpha)}{k} + \left(1 - \frac{1}{k}\right)\alpha.$$

By induction we can prove the inequalities

$$\|Z_n(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{K K_1^n C_2^n b_t^n e^{\omega t} t^{n(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{(n!)^\alpha}, \qquad t > 0, \qquad n \in \mathbb{N}_0, \tag{5.6}$$

where $K_1 := K\Gamma(1-(1-\beta)(2-\alpha))$, as above. Indeed, for $n=0$ this inequality follows from the definition of $S_A(t)$. For $n \in \mathbb{N}$, taking into consideration inequality (4.4) from (5.6) for a given $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\|Z_{n+1}(t)\|_{\mathcal{L}(\mathcal{Z})} &\leqslant \int\limits_0^t \frac{K_1 e^{\omega(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|B(s)\|_{\mathcal{L}(\mathcal{Z})} \frac{KK_1^n C_2^n b_t^n e^{\omega s} s^{n(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{(n!)^\alpha}\, ds \\
&\leqslant \int\limits_0^t \frac{K_1 e^{\omega(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{KK_1^n C_2^n b_t^{n+1} e^{\omega s} s^{n(\alpha-\gamma)-(1-\beta)(2-\alpha)-\gamma}}{(n!)^\alpha}\, ds \\
&\leqslant \frac{KK_1^{n+1} C_2^n b_t^{n+1} e^{\omega t} t^{(n+1)(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{\Gamma(\alpha)(n!)^\alpha} \mathcal{B}(\alpha, n(\alpha-\gamma)-(1-\beta)(2-\alpha)-\gamma+1) \\
&= \frac{KK_1^{n+1} C_2^n b_t^{n+1} e^{\omega t} t^{(n+1)(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{(n!)^\alpha} \frac{\Gamma(n(\alpha-\gamma)-(1-\beta)(2-\alpha)-\gamma+1)}{\Gamma((n+1)(\alpha-\gamma)-(1-\beta)(2-\alpha)+1)} \\
&\leqslant \frac{KK_1^{n+1} C_2^{n+1} b_t^{n+1} e^{\omega t} t^{(n+1)(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{(n!)^\alpha (n+1)^\alpha} \\
&= \frac{KK_1^{n+1} C_2^{n+1} b_t^{n+1} e^{\omega t} t^{(n+1)(\alpha-\gamma)-(1-\beta)(2-\alpha)}}{((n+1)!)^\alpha}.
\end{aligned}$$

It follows from inequalities (5.6) that series (5.3) converges in the space $\mathcal{L}(\mathcal{Z})$ uniformly on compact subsets of the variable t. Therefore, the operator–valued function $Z_{A+B}(t)$ is strongly continuous on $\mathbb{R}_+$. Moreover,

$$\|z(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant Ke^{\omega t} t^{-(1-\beta)(2-\alpha)} \sum_{n=0}^{\infty} \frac{K_1^n C_2^n b_t^n t^{n(\alpha-\gamma)}}{(n!)^\alpha} \|z_0\|_{\mathcal{Z}},$$

and this implies inequalities (5.4) and (5.5).

We have

$$J^{(1-\beta)(2-\alpha)} S_A(0) z_0 = z_0, \qquad D^{1-(1-\beta)(2-\alpha)} S_A(0) z_0 = 0,$$

by (5.6) for $t>0$, $n \in \mathbb{N}$, we also have

$$\|J^{(1-\beta)(2-\alpha)} Z_n(t)\|_{\mathcal{L}(\mathcal{Z})} \leqslant \frac{KK_1^n C_2^n b_t^n e^{\omega t} t^{n(\alpha-\gamma)}}{(n!)^\alpha} \mathcal{B}((1-\beta)(2-\alpha), n(\alpha-\gamma)-(1-\beta)(2-\alpha)+1),$$

and this is why $J^{(1-\beta)(2-\alpha)} Z_n(0) z_0 = 0$. Moreover,

$$\begin{aligned}
J^{(1-\beta)(2-\alpha)} Z_{n+1}(t) &= \int\limits_0^t J^{1-\beta(2-\alpha)} J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds, \\
D^{1-(1-\beta)(2-\alpha)} Z_{n+1}(t) &= J^{1-\beta(2-\alpha)} J^{(1-\beta)(2-\alpha)} S_A(0) B(t) Z_n(t) \\
&\quad + \int\limits_0^t D^1 J^{1-\beta(2-\alpha)} J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds \\
&= J^{1-\beta(2-\alpha)} \int\limits_0^t D^1 J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds \\
&\quad + \int\limits_0^t \frac{(t-s)^{-\beta(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))} B(s) Z_n(s)\, ds,
\end{aligned}$$

$$\left\| \int_0^t \frac{(t-s)^{-\beta(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))} B(s) Z_n(s)\, ds \right\|_{\mathcal{L}(\mathcal{Z})}$$
$$\leqslant \int_0^t \frac{(t-s)^{-\beta(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))} \frac{KK_1^n C_2^n b_t^{n+1} e^{\omega s} s^{n(\alpha-\gamma)-(1-\beta)(2-\alpha)-\gamma}}{(n!)^\alpha}\, ds$$
$$\leqslant \frac{KK_1^n C_2^n b_t^{n+1} e^{\omega t} t^{n(\alpha-\gamma)-1+\alpha-\gamma}}{(n!)^\alpha} \frac{\Gamma(n(\alpha-\gamma)-(1-\beta)(2-\alpha)-\gamma+1)}{\Gamma(n(\alpha-\gamma)+\alpha-\gamma)}.$$

Therefore, taking into account the uniform convergence of series (5.3) and the inequality $\gamma < \alpha - 1$, we have

$$J^{(1-\beta)(2-\alpha)} z(0) = z_0, \qquad D^{1-(1-\beta)(2-\alpha)} z(0) = 0.$$

By (5.3) and the uniform convergence of the series proved above, we find

$$\begin{aligned} z(t) = Z_{A+B}(t) z_0 :=& S_A(t) z_0 + \sum_{n=0}^{\infty} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) Z_n(s) z_0\, ds \\ =& S_A(t) z_0 + \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds. \end{aligned} \tag{5.7}$$

Since

$$J^{(1-\beta)(2-\alpha)} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds|_{t=0}$$
$$= \sum_{n=0}^{\infty} J^{(1-\beta)(2-\alpha)} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds|_{t=0}$$
$$= J^{1-\beta(2-\alpha)} \sum_{n=0}^{\infty} \int_0^t J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds|_{t=0} = 0,$$

$$D^1 J^{(1-\beta)(2-\alpha)} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds|_{t=0}$$
$$= \sum_{n=0}^{\infty} D^1 J^{(1-\beta)(2-\alpha)} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds|_{t=0}$$
$$= \sum_{n=0}^{\infty} \int_0^t D^1 J^{1-\beta(2-\alpha)} J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) Z_n(s)\, ds|_{t=0} = 0,$$
$$= J^{1-\beta(2-\alpha)} \sum_{n=0}^{\infty} \int_0^t \left(D^1 J^{(1-\beta)(2-\alpha)} S_A(t-s) - \frac{(t-s)^{-\beta(2-\alpha)}}{\Gamma(1-\beta(2-\alpha))} \right) B(s) Z_n(s)\, ds|_{t=0} = 0,$$

by (5.7) we obtain the identity

$$D^{\alpha,\beta} z(t) = AS_A(t) z_0 + D^\alpha \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds. \tag{5.8}$$

By the condition $\operatorname{im} B(t) \subset D_A$ for $t > 0$, as in the proof of identities (3.2), (3.3), we obtain

$$\int_0^t J^{1-\beta(2-\alpha)} A S_A(t-s) B(s) z(s)\, ds = \int_0^t D^1 J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) z(s)\, ds,$$

at the same time,

$$\|D^1 J^{(1-\beta)(2-\alpha)} S_A(t-s) B(s) z(s)\|_{\mathcal{L}(\mathcal{Z})} \leqslant C_3 s^{-(1-\beta)(2-\alpha)-\gamma}, \qquad -(1-\beta)(2-\alpha)-\gamma > -1,$$

since $\gamma < \alpha - 1$, and the latter integral hence converges

$$\int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds \in D_A, \qquad t > 0,$$

and by analogy with the proof of identity (3.3) we prove the identity

$$D^{\alpha,\beta} \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds = A \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s) B(s) z(s)\, ds + B(t) z(t).$$

Let us establish the uniqueness of the solution. We observe that

$$\begin{aligned} &J^{\beta(2-\alpha)} (J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - z_0)|_{t=0} = 0, \\ &D^1 J^{\beta(2-\alpha)} (J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - z_0)|_{t=0} = J^{\beta(2-\alpha)} D^1 (J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - z_0)|_{t=0} \\ &\qquad = J^{\beta(2-\alpha)} D^1 J^{(1-\beta)(2-\alpha)} S_A(0) z_0 = 0, \end{aligned}$$

this is why

$$\begin{aligned} J^\alpha D^{\alpha,\beta} S_A(t) z_0 =& J^\alpha D^2 J^{\beta(2-\alpha)} (J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - z_0) \\ =& D^2 J^\alpha J^{\beta(2-\alpha)} (J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - z_0) \\ =& S_A(t) z_0 - \frac{t^{-(1-\beta)(2-\alpha)} z_0}{\Gamma(1-(1-\beta)(2-\alpha))} = J^\alpha A S_A(t) z_0. \end{aligned}$$

Thus, for each $z_0 \in D_A$,

$$\begin{aligned} z_0 =& J^{(1-\beta)(2-\alpha)} S_A(t) z_0 - J^{(1-\beta)(2-\alpha)} J^\alpha A S_A(t) z_0 \\ =& (J^{(1-\beta)(2-\alpha)} S_A(t) - g_\alpha * A J^{(1-\beta)(2-\alpha)} S_A(t)) z_0, \end{aligned}$$

where $g_\beta(t) := t^{\beta-1}/\Gamma(\beta)$ and the Laplace convolution $*$ is defined by the identity

$$(f * g)(t) = \int_0^t f(t-s) g(s)\, ds.$$

Suppose some function x satisfies equation (5.2) and conditions (5.1) with $z_0 = 0$. Then

$$\begin{aligned} &J^\alpha D^{\alpha,\beta} x(t) = J^\alpha D^2 J^{\beta(2-\alpha)} J^{(1-\beta)(2-\alpha)} x(t) = x(t) = J^\alpha (A+B) x(t), \\ &1 * x = (J^{(1-\beta)(2-\alpha)} S_A - g_\alpha * A J^{(1-\beta)(2-\alpha)} S_A) * x \\ &\quad = J^{(1-\beta)(2-\alpha)} S_A * (x - g_\alpha * Ax) = J^{(1-\beta)(2-\alpha)} S_A * J^\alpha B x \\ &\quad = J^{(1-\beta)(2-\alpha)+\alpha} S_A * Bx. \end{aligned}$$

Thus,

$$x = D^1 J^{(1-\beta)(2-\alpha)+\alpha} S_A * Bx = J^{(1-\beta)(2-\alpha)+\alpha-1} S_A * Bx = J^{1-\beta(2-\alpha)} S_A * Bx$$

or

$$x(t) = \int_0^t J^{1-\beta(2-\alpha)} S_A(t-s)B(s)x(s)\,ds := (Fx)(t). \tag{5.9}$$

For the solution of equation (5.9), by inequality (4.4) for $t > 0$ we have

$$\|(Fz)(t)\|_{\mathcal{Z}} \leqslant \int_0^t \frac{K\Gamma(1-(1-\beta)(2-\alpha))e^{\omega(t-s)}(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|B(s)\|_{\mathcal{L}(\mathcal{Z})}\|z(s)\|_{\mathcal{Z}}\,ds$$
$$\leqslant \frac{K\Gamma(1-(1-\beta)(2-\alpha))\Gamma(1-(1-\beta)(2-\alpha)-\gamma)}{\Gamma(\alpha-(1-\beta)(2-\alpha)-\gamma+1)} e^{\omega t} t^{\alpha-(1-\beta)(2-\alpha)-\gamma} b_t \|z\|_{C_{(1-\beta)(2-\alpha)}((0,t];\mathcal{Z})}.$$

Therefore,

$$\|Fz\|_{C_{(1-\beta)(2-\alpha)}((0,T_1];\mathcal{Z})} \leqslant \frac{K\Gamma(1-(1-\beta)(2-\alpha))\Gamma(1-(1-\beta)(2-\alpha)-\gamma)e^{\omega T_1}}{\Gamma(\alpha-(1-\beta)(2-\alpha)-\gamma+1)}$$
$$\cdot T_1^{\alpha-\gamma} b_T \|z\|_{C_{(1-\beta)(2-\alpha)}((0,T_1];\mathcal{Z})}.$$

For sufficiently small $T_1 \leqslant T$, the operator F is contracting in the Banach space $C_{(1-\beta)(2-\alpha)}((0,T_1];\mathcal{Z})$, and by the contraction mapping theorem we get $z \equiv 0$ on $(0,T_1]$. If $T_1 < T$, we take the space

$$C^{T_1}_{(1-\beta)(2-\alpha)}((0,T_2];\mathcal{Z}) := \{z \in C_{(1-\beta)(2-\alpha)}((0,T_2];\mathcal{Z}) : z(t) = 0, \quad t \in (0,T_1]\}$$

for $T_2 = 2^{(\alpha-\gamma)^{-1}}T_1$, then

$$\|Fz\|_{C^{T_1}_{(1-\beta)(2-\alpha)}((0,T_2];\mathcal{Z})} \leqslant \frac{K\Gamma(1-(1-\beta)(2-\alpha))\Gamma(1-(1-\beta)(2-\alpha)-\gamma)}{\Gamma(\alpha-(1-\beta)(2-\alpha)-\gamma+1)}$$
$$\cdot e^{\omega T_1}(2T_1^{\alpha-\gamma} - T_1^{\alpha-\gamma}) b_T \|z\|_{C^{T_1}_{(1-\beta)(2-\alpha)}((0,T_2];\mathcal{Z})}.$$

Therefore the operator F is contracting in the Banach space $C^{T_1}_{(1-\beta)(2-\alpha)}((0,T_2];\mathcal{Z})$ and $z \equiv 0$ on $(0,T_2]$. After a finite number n of steps we obtain $2^{(\alpha-\gamma)^{-1}n}T_1 \geqslant T$. The proof is complete. $\square$

Analogously to Theorem 3.1, with the operators $S_A(t)$ replaced by the operators $Z_{A+B}(t)$, the following statement can be proved.

Theorem 5.2. *Let $\alpha \in (1,2]$, $\beta \in [0,1]$, $A \in \mathcal{C}_{\alpha,\beta}$, $B \in C_\gamma(\mathbb{R}_+;\mathcal{L}(\mathcal{Z}))$, $f \in C_\gamma((0,T];\mathcal{Z})$, $\operatorname{im} B(t) \subset D_A$, $f(t) \in D_A$ for $t \in \mathbb{R}_+$, $\gamma < \alpha - 1$. Then the Cauchy type problem*

$$D^{-(1-\beta)(2-\alpha)}z(0) = z_0, \qquad D^{1-(1-\beta)(2-\alpha)}z(0) = z_1$$

for the equation

$$D^{\alpha,\beta}z(t) = (A + B(t))z(t) + f(t)$$

has a unique solution for all $z_0, z_1 \in D_A$. This solution reads

$$z(t) = Z_{A+B}(t)z_0 + J^1 Z_{A+B}(t)z_1 + \int_0^t J^{1-\beta(2-\alpha)} Z_{A+B}(t-s)f(s)\,ds.$$

6. Example

For $\alpha \in (1,2)$, $\beta = 0$, consider the Cauchy–type problem

$$D_t^\alpha v(\xi,t) = \frac{\partial^2 v}{\partial \xi^2}(\xi,t) + \int\limits_0^{2\pi} b(\xi,\eta,t)v(\eta,t)d\eta, \qquad (\xi,t) \in (0,2\pi)\times\mathbb{R}_+, \tag{6.1}$$

$$v(0,t) = v(2\pi,t), \qquad \frac{\partial v}{\partial \xi}(0,t) = \frac{\partial v}{\partial \xi}(2\pi,t), \qquad t \in \mathbb{R}_+, \tag{6.2}$$

$$D_t^{\alpha-2+k} v(\xi,0) = v_k(\xi), \qquad \xi \in (0,2\pi), \quad k = 0,1, \tag{6.3}$$

where the subscript t denotes the variable with respect to which the fractional derivative or fractional integral is calculated. We choose the space $\mathcal{Z} = L_2(0,2\pi)$ and the operator $A = \rho D_\xi^2 = \rho\frac{\partial^2}{\partial \xi^2}$ acting in it with the domain

$$D_A = \{w \in H^2(0,2\pi) : w(0) = w(2\pi),\ D_\xi^1 w(0) = D_\xi^1 w(2\pi)\},$$

where $H^2(0,2\pi) := W_2^2(0,2\pi)$ is the Sobolev space. By the embedding theorems for the functions from this space and their first derivatives, the value of the trace at a point is well–defined.

It is easy to find the spectrum $\sigma(A) = \{-k^2 : k \in \mathbb{Z}\}$ and the corresponding orthonormal system of eigenfunctions $\{(2\pi)^{-\frac{1}{2}}e^{ik\xi} : k \in \mathbb{Z}\}$, which forms a basis in $L_2(0,2\pi)$. Therefore, for $\operatorname{Re}\gamma > 0$,

$$(\lambda^\gamma - A)w(\xi) = \sum_{k\in\mathbb{Z}}(\lambda^\gamma + k^2)\left\langle w, \frac{e^{ik\xi}}{\sqrt{2\pi}}\right\rangle \frac{e^{ik\xi}}{\sqrt{2\pi}}, \qquad (\lambda^\gamma - A)^{-1}w(\xi) = \sum_{k\in\mathbb{Z}} \frac{\left\langle w, \frac{e^{ik\xi}}{\sqrt{2\pi}}\right\rangle \frac{e^{ik\xi}}{\sqrt{2\pi}}}{\lambda^\gamma + k^2}.$$

For the operator cosine function we have

$$\mathfrak{L}\left[\sum_{k\in\mathbb{Z}}\cos(kt)\left\langle \cdot, \frac{e^{ik\xi}}{\sqrt{2\pi}}\right\rangle \frac{e^{ik\xi}}{\sqrt{2\pi}}\right] = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\lambda + ik} + \frac{1}{\lambda - ik}\right)\left\langle \cdot, \frac{e^{ik\xi}}{\sqrt{2\pi}}\right\rangle \frac{e^{ik\xi}}{\sqrt{2\pi}}$$
$$= \sum_{k\in\mathbb{Z}}\frac{\lambda}{\lambda^2 + k^2}\left\langle \cdot, \frac{e^{ik\xi}}{\sqrt{2\pi}}\right\rangle \frac{e^{ik\xi}}{\sqrt{2\pi}} = \lambda(\lambda^2 - A)^{-1}.$$

Therefore, by [3, Lm. 5.8], the operator A generates an operator cosine function, i.e. $A \in \mathcal{A}_{2,0}$, and hence, by [6, Thm. 5], $A \in \mathcal{A}_{\alpha,0}$. By Theorem 5.1 we immediately obtain the following result.

Theorem 6.1. *Let $\alpha \in (1,2)$, $b(\xi,\eta,t) \in C_\gamma(\mathbb{R}_+; D_A \times L_2(0,2\pi))$, $\gamma < \alpha - 1$. Then for all $v_0, v_1 \in D_A$ problem (6.1)–(6.3) has a unique solution.*

BIBLIOGRAPHY

1. A.R. Volkova, V.E. Fedorov, D.M. Gordievskikh. *On solvability of some classes of equations with Hilfer derivative in Banach spaces* // Chelyabinskiĭ Fiz.-Mat. Zh. **7**:1, 11–19 (2022). https://doi.org/10.47475/2500-0101-2022-17101
2. A.M. Nakhushev. *Fractional calculus and its applications.* Fizmatlit, Moskva (2003). (in Russian).
3. H.O. Fattorini. *Ordinary differential equations in linear topological spaces. I.* // J. Differ. Eq. **5**:1, 72–105 (1969). https://doi.org/10.1016/0022-0396(69)90105-3
4. V.E. Fedorov, W.-Sh. Du, M. Kostić, M.V. Plekhanova, A.S. Skorynin. *Criterion of the existence of a strongly continuous resolving family for a fractional differential equation with the Hilfer derivative* // Fractal and Fractional **9**:2, 81 (2025). https://doi.org/10.3390/fractalfract9020081
5. V.E. Fedorov, A.S. Skorynin. *Analytic resolving families of operators for linear equations with Hilfer derivative* // J. Math. Sci. **277**:3, 385–402 (2023).

6. V.E. Fedorov, M.V. Plekhanova, D.A. Vershinina. *Subordination principle for equations with Riemann — Liouville derivative* // Lobachevskii J. Math. **46**:11, 5578–5588 (2025). https://doi.org/10.1134/S1995080225611543
7. R. Hilfer. *Applications of Fractional Calculus in Physics.* World Scientific, Singapore (2000).
8. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam (2006).
9. I. Podlubny. *Fractional Differential Equations.* Academic Press San Diego, CA (1999).
10. V.E. Tarasov. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media.* Springer, Berlin; Higher Education Press, Beijing (2011).
11. V.V. Uchaykin. *Fractional Derivatives for Physicists and Engineers. Vol. I. Background and Theory.* Springer, Heidelberg; Higher Education Press, Beijing (2013).
12. V.V. Uchaykin. *Fractional Derivatives for Physicists and Engineers. Vol. I. Applications.* Springer, Singapore; Higher Education Press, Beijing (2025).

Vladimir Evgenievich Fedorov,
Chelyabinsk State University,
Bratiev Kashirinych str. 129,
454001, Chelyabinsk, Russia
E-mail: kar@csu.ru

Anton Alexandrovich Bykov,
Yugra State University,
Chekhova str. 16,
Khanty–Mansi Autonomous Okrug — Yugra
628011, Khanty–Mansiysk, Russia
E-mail: bykov_antony@mail.ru