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ON SOLVABILITY OF PURSUIT AND EVASION PROBLEMS FOR DIFFERENTIAL GAMES WITH HILFER FRACTIONAL DERIVATIVES

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Abstract. In this work, in a Banach space, we sovability the solvability of pursuit and evasion problems for differential games with Hilfer fractional derivatives of order α, $0 < \alpha < 1$, of type μ, $0 \leqslant \mu \leqslant 1$. Using Pontryagin first method and the theorem on strict separation, we prove two theorems on sufficient conditions for the solvability of pursuit and evasion problems on an interval. We solve one game problem describing the relaxation process in the glass formation of supercooled liquids.

Keywords: differential game with Hilfer fractional derivatives, Banach space, pursuit problem, evasion problem on an interval.

Mathematics Subject Classification: 34B45, 81Q15

1. Introduction. Statement of pursuit and evasion problems

In the field of differential game theory, the foundational results were obtained by academicians Pontryagin [15] and Krasovskii [7].

While academician N.N. Krasovskii investigated positional differential games, academician L.S. Pontryagin, considering the differential game separately from the pursuer's point of view and from the evader's point of view, splits the game problem into two different problems: the pursuit problem and the evasion problem. This approach made it possible to apply Pontryagin first method to wider classes of games. For example, the solvability of the pursuit problem using the first method was investigated in the works:

– [4], [16], [1], when geometric and integral constraints are imposed on the players' controls;

– [3], [8], [9], when the differential game is described by an equation of delay or neutral type;

– [13], [11], [12], when the dynamics of the game is described by a differential equation in an infinite–dimensional Banach space;

– [17], [2], [10], [14], when the dynamics of the game is described by a fractional-order differential equation.

In a Banach space X, we consider a differential game described by a fractional order differential equation

$$\begin{cases} D_{0+}^{\alpha,\mu} x(t) = Ax(t) + F(t, u(t), v(t)), \\ I_{0+}^{(1-\alpha)(1-\mu)} x(0) = x_0 \end{cases} \tag{1.1}$$

and a closed terminal set $M \subset X$, where the game ends. In game (1.1), $t \in (0, T]$, $T > 0$, $x(t) \in X$, Y, Z are Banach spaces, $U((0, T], Y)$ is the set of measurable mappings from $(0, T]$

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to Y, $u(\cdot) \in U((0,T],Y)$ is the pursuer's controls, $v(\cdot) \in U((0,T],Z)$ is the evader's controls, $D_{0+}^{\alpha,\mu}$ is the Hilfer fractional derivative of order α, $0 < \alpha < 1$, and type μ, $0 \leqslant \mu \leqslant 1$, A is a linear closed operator with dense domain in X that generates a strongly continuous semigroup $S(t)$, $t \geqslant 0$, and $I_{0+}^{(1-\alpha)(1-\mu)}$ is the fractional integral of order $\rho = (1-\alpha)(1-\mu)$. We assume that for any admissible $u(\cdot)$ and $v(\cdot)$ the mapping $t \to F(t,u(t),v(t))$ is locally integrable and Cauchy problem (1.1) has a solution [19]:

$$x(t) = S_{\alpha,\mu}(t)x_0 + \int_0^t K_\alpha(t-s)F(s,u(s),v(s))\,ds, \qquad t \in (0,T], \tag{1.2}$$

where the integral is understood in the Bochner sense [18],

$$K_\alpha(t) = t^{\alpha-1}P_\alpha(t), \qquad P_\alpha(t) = \int_0^\infty \alpha s M_\alpha(s) S(t^\alpha s)\,ds, \qquad S_{\alpha,\mu}(t) = I_{0+}^{\mu(1-\alpha)}K_\alpha(t),$$

and the Wright function

$$M_\alpha(z) = \sum_{n=1}^\infty \frac{(-z)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \qquad 0<\alpha<1, \qquad z \in \mathbb{C},$$

satisfies the equation

$$\int_0^\infty s^\sigma M_\alpha(s)\,ds = \frac{\Gamma(1+\sigma)}{\Gamma(1+\alpha\sigma)}, \qquad s \geqslant 0,$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

is the Gamma function.

Remark 1.1. *By [19], the fractional integral of order p with lower limit a of a function $f : [a,\infty) \to X$ is defined by the formula*

$$I_{a+}^p f(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{f(s)}{(t-s)^{1-p}}\,ds, \qquad t > a, \qquad p > 0.$$

The Riemann — Liouville derivative of order p of a function $f : [a,\infty) \to X$ is defined by the formula

$$D_{a+}^p f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{p+1-n}}\,ds, \qquad t>a, \qquad n-1<p<n.$$

The Caputo derivative of order p of a function $f : [a,\infty) \to X$ is defined by the formula

$${}^C D_{a+}^p f(t) = D_{a+}^p\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \qquad t>a, \qquad n-1<p<n.$$

If $f \in \mathbb{C}^n([a,\infty),X)$, then

$${}^C D_{a+}^p f(t) = \frac{1}{\Gamma(n-p)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{p+1-n}}\,ds, \qquad t>a, \qquad n-1<p<n.$$

Remark 1.2. *By [19], the Hilfer derivative of order α, $0 < \alpha < 1$, and type μ, $0 \leqslant \mu \leqslant 1$, with lower limit a of a function $f : [a, \infty) \to X$ is defined by the formula*

$$D^{\alpha,\mu}_{a+} f(t) = I^{\mu(1-\alpha)}_{a+} \frac{d}{dt} I^{(1-\alpha)(1-\mu)}_{a+} f(t).$$

Remark 1.3. *For $\mu = 0$, $0 \leqslant \alpha \leqslant 1$ and $a = 0$ we obtain the Riemann – Liouville derivative:*

$$D^{\alpha,0}_{0+} f(t) = \frac{d}{dt} I^{1-\alpha}_{0+} f(t) = D^{\alpha}_{0+} f(t).$$

When $\mu = 1$, $0 \leqslant \alpha \leqslant 1$ and $a = 0$, we obtain the Caputo derivative:

$$D^{\alpha,1}_{0+} f(t) = I^{1-\alpha} \frac{d}{dt} f(t) = {}^{C}D^{\alpha}_{0+} f(t).$$

For differential game (1.1), we give the following definitions and the statement of the pursuit problem in the Pontryagin sense.

Definition 1.1. *In game (1.1), from the initial point $x_0 \in X \backslash M$, pursuit completion is possible if there exists a number $T = T(x_0) > 0$ such that for any evasion control $v(\cdot) \in U([0, \infty), Z)$, by knowing equation (1.1) and the quantities x_0 and $v(s)$, $0 \leqslant s \leqslant t$, one can choose a pursuit control $u(\cdot) \in U([0, \infty), Y)$ such that the solution $x(\cdot)$ of problem (1.1) corresponding to the controls $u(\cdot)$ and $v(\cdot)$ satisfies the inclusion $x(T_1) \in M$ for some $T_1 \in (0, T]$. In this case, the number T is called the guaranteed pursuit time. Here, the choice of $v(t)$ is made on the basis of information about x_0 and $v(t)$.*

Pursuit problem. Find the set of initial points from which in game (1.1) pursuit completion is possible in the sense of Definition 1.1.

For differential game (1.1), we give the following definitions and the statement of the evasion problem on an interval in the Pontryagin sense.

Definition 1.2. *In game (1.1), from the initial point $x_0 \in X \setminus M$, evasion on the interval $[0, T_0)$ is possible if for all $t \in [0, T_0)$ and for any pursuit control $u(\cdot) \in U([0, T_0), Y)$, there exists an evasion control $v(\cdot) \in U([0, T_0), Z)$ such that the solution $x(\cdot)$ of problem (1.1) satisfies $x(t) \notin M$. Here, the choice of $v(t)$ is made according to the formula $v(t) = w(u(t - \varepsilon))$, where $\varepsilon \in (0, T_0)$. The mapping $w : Y \to Z$ is such that for every measurable mapping $u(\cdot)$ the superposition $w(u(\cdot))$ is measurable.*

Evasion problem. Find the set of initial points from which in game (1.1) evasion on the interval $[0, T_0)$ is possible in the sense of Definition 1.2.

2. Main results

The theorem gives sufficient conditions for the solvability of pursuit problem.

Theorem 2.1. *Let the following conditions be satisfied:*

1) *X, Y are separable Banach spaces;*
2) *the initial point $x_0 \in X \setminus M$ is such that for some $T > 0$ the inclusion holds*

$$S_{\alpha,\mu}(T)x_0 \in M - \int_0^T \bigcap_{v \in Z} K_\alpha(T - s) F(s, Y, v)\, ds; \tag{2.1}$$

3) *the mapping $F : [0, \infty) \times Y \times Z \to X$ satisfies the Carathéodory conditions, i.e., it is measurable in $t \in [0, \infty)$ and continuous in $(u, v) \in Y \times Z$, and there exists a locally*

summable function $\eta : [0,\infty) \to [0,\infty)$ such that for all $t \in [0,\infty), u \in Y, v \in Z$ the inequality $\|F(t,u,v)\| \leqslant \eta(t)$ holds.

Then from the initial point x_0 in game (1.1) pursuit completion is possible in time T.

Proof. By (2.1), there exist a point $m \in M$ and an integrable selector $w(\cdot)$ of the mapping

$$s \to \bigcap_{v\in Z} K_\alpha(T-s)F(s,Y,v) \tag{2.2}$$

such that the identity holds

$$S_{\alpha,\mu}(T)x_0 = m - \int_0^T w(s)\,ds. \tag{2.3}$$

Since X and Y are separable spaces, mappings (2.2) and $s \to w(s)$ are measurable, and the mapping $(s,u,v) \to F(s,u,v)$ is continuous in (u,v) and measurable in s. Then by Lemma 1 in [11] on a measurable selector, for any measurable evasion control $v(\cdot) \in U([0,T],Z)$ there exists a measurable pursuit control $u(\cdot) \in U([0,T],Y)$ such that the identity

$$w(s) = K_\alpha(T-s)F(s,u(s),v(s)) \tag{2.4}$$

holds for almost each $s \in [0,T]$.

Therefore, by (2.3), (2.4), the solution $x(\cdot)$ of problem (1.1) corresponding to the controls $u(\cdot)$, $v(\cdot)$ satisfies

$$\begin{aligned} x(T) &= S_{\alpha,\mu}(T)x_0 + \int_0^T K_\alpha(T-s)F(s,u(s),v(s))\,ds \\ &= m - \int_0^T w(s)\,ds + \int_0^T K_\alpha(T-s)F(s,u(s),v(s))\,ds \\ &= m - \int_0^T K_\alpha(T-s)F(s,u(s),v(s))\,ds + \int_0^T K_\alpha(T-s)F(s,u(s),v(s))\,ds = m \in M, \end{aligned}$$

that is, $x(T) \in M$. This is why, from the initial point x_0 in game (1.1), pursuit completion is possible in time T. The proof is complete. □

The following theorem gives sufficient conditions for the solvability of the evasion problem on an interval.

Theorem 2.2. *Let Assumptions 1) – 3) of Theorem 2.1 and the following conditions hold*

4) *the terminal set M is convex and compact in the weak topology $\sigma(X,X^*)$ [5];*

5) *the multivalued mapping*

$$t \to \int_0^t \bigcap_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds, \qquad t \in [0,T]$$

is convex–valued and closed in the weak topology $\sigma(X,X^*)$;

6) *there exists a mapping $w : Y \to Z$ such that for all $u \in Y$, $t \in [0,T]$, $0 \leqslant s \leqslant t$ the inclusion holds*

$$K_\alpha(t-s)F(s,u,w(u)) \in \bigcap_{v\in Z} K_\alpha(t-s)F(s,Y,v)$$

and for every measurable $u(\cdot)$ the superposition $w(u(\cdot))$ is measurable.

Then

1) *from the initial point x_0 in game (1.1), pursuit completion is possible in time*

$$T_0 = \min\{T > 0 : \textit{for which inclusion}\,(2.1)\ \textit{holds}\};$$

2) *from the initial point x_0 in the game (2.1) evasion on the interval $[0, T_0)$ is possible.*

Proof. 1) By Condition 4), the terminal set M is compact, and by Condition 5), the mapping $t \to \int\limits_0^t \bigcap\limits_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds$ is closed in weak topology $\sigma(X, X^*)$. Then the mapping $t \to M - \int\limits_0^t \bigcap\limits_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds$ is sequentially closed in the weak topology $\sigma(X, X^*)$ [13].

Then, by the continuity of the mapping $t \to S_{\alpha,\mu}(t)x_0$, by (2.1) and Lemma 2 in [13], the set $\{T > 0 : \text{for which inclusion}\,(2.1)\ \text{holds}\}$ is closed. And this means that the inclusion holds

$$S_{\alpha,\mu}(T_0)x_0 \in M - \int\limits_0^{T_0} \bigcap_{v\in Z} K_\alpha(T_0-s)F(s,Y,v)\,ds. \tag{2.5}$$

Consequently, by Theorem 2.1, from the initial point x_0 in game (1.1), pursuit completion is possible in time T_0.

2) By the definition of the number T_0, we conclude that for all $t \in [0, T_0)$ the relation

$$\left(S_{\alpha,\mu}(t)x_0 + \int\limits_0^t \bigcap_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds\right) \bigcap M = \varnothing \tag{2.6}$$

holds. If $\varepsilon_t \in (0, T_0)$, then the values $v(t)$ of the evasion control $v(\cdot)$ on $(0, \varepsilon_t)$ are chosen arbitrarily, and in a special way for $t \in (\varepsilon_t, T_0)$: $v(t) = w(u(t-\varepsilon_t))$. Then for the solution $x(t)$ of problem (1.1) corresponding to the chosen controls $u(\cdot)$, $v(\cdot)$ on $(0, T_0)$, we have

$$\begin{aligned} x(t) =& S_{\alpha,\mu}(t)x_0 + \int\limits_0^t K_\alpha(t-s)F(s,u(s),v(s))\,ds \\ =& S_{\alpha,\mu}(t)x_0 + \int\limits_0^{\varepsilon_t} K_\alpha(t-s)F(s,u(s),v(s))\,ds \\ &+ \int\limits_{\varepsilon_t}^t K_\alpha(t-s)F(s,u(s),w(u(s-\varepsilon_t)))\,ds. \end{aligned} \tag{2.7}$$

By (2.6), conditions 4), 5), and according to the theorem on strict separation [5] in the locally convex topological space X with the weak topology $\sigma(X, X^*)$, the compact set M and the closed set $S_{\alpha,\mu}(t)x_0 + \int\limits_0^t \bigcap\limits_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds$ are strictly separated for all $t \in [0, T_0)$, i.e., for all $t \in [0, T_0)$ one can find a continuous linear functional $\varphi_t \in X^*$ and numbers c_t and $\delta_t > 0$ such that

$$\varphi_t(x) \leqslant c_t - \delta_t < c_t \leqslant \varphi(m) \tag{2.8}$$

for all $x \in S_{\alpha,\mu}(t)x_0 + \int\limits_0^t \bigcap\limits_{v\in Z} K_\alpha(t-s)F(s,Y,v)\,ds$ and $m \in M$.

If $\overline{w}(\cdot)$ is an arbitrary integrable selector of the mapping

$$s \to \bigcap_{v \in Z} K_\alpha(t-s)F(s,Y,v), \quad s \in (0,\varepsilon_t],$$

by Condition 6), (2.7) and (2.8) we find

$$\begin{aligned}
\varphi_t(x(t)) =& \varphi_t\left(S_{\alpha,\mu}(t)x_0 + \int_0^{\varepsilon_t} K_\alpha(t-s)F(s,u(s),v(s))\,ds\right. \\
&\left. + \int_{\varepsilon_t}^{t} K_\alpha(t-s)F(s,u(s),w(u(s-\varepsilon_t)))\,ds\right) \\
=& \varphi_t\left(S_{\alpha,\mu}(t)x_0 + \int_0^{\varepsilon_t} \overline{w}(s)\,ds - \int_0^{\varepsilon_t} \overline{w}(s)\,ds + \int_0^{\varepsilon_t} K_\alpha(t-s)F(s,u(s),v(s))\,ds\right. \\
&\left. + \int_{\varepsilon_t}^{t} K_\alpha(t-s)F(s,u(s),w(u(s-\varepsilon_t)))\,ds\right) \\
=& \varphi_t\left(S_{\alpha,\mu}(t)x_0 + \int_0^{\varepsilon_t} \overline{w}(s)\,ds + \int_{\varepsilon_t}^{t} K_\alpha(t-s)F(s,u(s),w(u(s-\varepsilon_t)))\,ds\right) \\
&+ \varphi_t\left(\int_0^{\varepsilon_t} K_\alpha(t-s)F(s,u(s),v(s))\,ds - \int_0^{\varepsilon_t} \overline{w}(s)\,ds\right) \\
\leqslant& c_t - \delta_t + \int_0^{\varepsilon_t} \varphi_t\left(K_\alpha(t-s)F(s,u(s),v(s)) - \overline{w}(s)\right)\,ds
\end{aligned}$$

By the absolute continuity of the Lebesgue integral [6], there exists a number $\varepsilon_t = \theta_t > 0$ such that

$$\left|\int_0^{\theta_t} \varphi_t\left(K_\alpha(t-s)F(s,u(s),v(s)) - \overline{w}(s)\right)\,ds\right| < \frac{\delta_t}{2}.$$

Hence,

$$\varphi_t(x(t)) \leqslant c_t - \delta_t + \frac{\delta_t}{2} = c_t - \frac{\delta_t}{2} < c_t \leqslant \varphi_t(m).$$

Therefore, for all $t \in [0,T_0)$, the number $\varepsilon_t = \theta_t > 0$ and the corresponding control $v(\cdot)$ can be chosen in such a way that $x(t) \notin M$, which proves the possibility of evasion on the interval $[0,T_0)$. The proof is complete. □

3. Example

In a separable reflexive Banach space X, consider a differential game with the Caputo derivative

$$y'(t) + bD^{0,5}y(t) = -\overline{u} + \overline{v} \tag{3.1}$$

with initial point $y(0) = y_0 \in X$, $\|y_0\| > 0$ and terminal set $M_1 = 0$. We suppose that $t \geqslant 0$, $b \neq 0$ is a real number, the measurable pursuit controls $\overline{u} = \overline{u}(t)$ and evasion controls $\overline{v} = \overline{v}(t)$

satisfy the relations $\overline{u}(t) \in Y = \{y : \|y\| \leqslant \rho\} = S_\rho$ is the ball of radius ρ centered at 0, $\overline{v}(t) \in Z = \{z : \|z\| \leqslant \sigma\} = S_\sigma$, $\rho > \sigma > 0$.

We let

$$x_1 = y, \qquad x_2 = D^{0,5}x_1, \qquad D^{0,5}x_2 = D^{0,5}D^{0,5}x_1 = x_1' = y'$$

and rewrite game (3.1) as

$$\begin{cases} D^{0,5}x_1 = x_2, \\ D^{0,5}x_2 = -bx_2 - \overline{u}(t) + \overline{v}(t) \end{cases}$$

or

$$D^{0,5}x = Ax - u + v \tag{3.2}$$

with the initial condition

$$x(0) = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

and the terminal set

$$M = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 = 0 \right\},$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & -b \end{pmatrix}, \qquad u = \begin{pmatrix} 0 \\ \overline{u} \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ \overline{v} \end{pmatrix}.$$

In our example, $\mu = 1$, $\alpha = 0.5$, $I_{0+}^{(1-\alpha)(1-\mu)} = I$ is the identity operator, A is a linear bounded operator that generates a strongly continuous group $S(t) = e^{At}$. By [19], [17], we use the Laplace transform to obtain

$$\begin{aligned} S_{0,5;1}(t) =& I_{0+}^{0,5}K_{0,5}(t) = \frac{1}{\Gamma(0,5)} \int\limits_0^t \frac{K_{0,5}(s)}{(t-s)^{0,5}}\, ds = \frac{1}{\sqrt{\pi}} \int\limits_0^t \frac{s^{-0,5}P_{0,5}(s)}{(t-s)^{0,5}}\, ds \\ =& \frac{1}{\sqrt{\pi}} \int\limits_0^t [s(t-s)]^{-0,5} \left(\int\limits_0^\infty 0,5 \cdot \theta \cdot M_{0,5}(\theta) S(s^{0,5}\theta) d\theta \right) ds \\ =& \frac{1}{2\sqrt{\pi}} \int\limits_0^t [s(t-s)]^{-0,5} \left(\int\limits_0^\infty \theta \cdot \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-0,5n)} e^{s^{0,5}\theta A} d\theta \right) ds \\ =& \sum_{n=0}^\infty \frac{(At^{0,5})^n}{\Gamma(0,5n+1)} = E_2(At^{0,5}; 1) \end{aligned} \tag{3.3}$$

is the generalized Mittag–Leffler function;

$$\begin{aligned} K_{0,5}(t-s) =& (t-s)^{-0,5}P_{0,5}(t-s) = (t-s)^{-0,5} \int\limits_0^\infty 0,5 \cdot \theta \cdot M_{0,5}(\theta) S((t-s)^{0,5}\theta) d\theta \\ =& \frac{1}{2}(t-s)^{0,5} \int\limits_0^\infty \theta \cdot \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-0,5n)} e^{(t-s)^{0,5}\theta A} d\theta \\ =& (t-s)^{0,5} \sum_{n=0}^\infty \frac{(A(t-s)^{0,5})^n}{\Gamma(0,5n+0,5)} \\ =& (t-s)^{0,5} \cdot E_2(A(t-s)^{0,5}; 0,5). \end{aligned} \tag{3.4}$$

For our example, by (3.3) and (3.4), inclusion (2.1) becomes

$$E_2(AT^{0,5};1)x_0 \in M + \int_0^T \bigcap_{v\in\tilde{Z}} (T-s)^{0,5} E_2(A(T-s)^{0,5};0,5)(\tilde{Y}-v)\,ds, \tag{3.5}$$

where

$$\tilde{Z} = \begin{pmatrix} 0 \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ S_\sigma \end{pmatrix} = \tilde{S}_\sigma, \qquad \tilde{Y} = \begin{pmatrix} 0 \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ S_\rho \end{pmatrix} = \tilde{S}_\rho.$$

Since

$$E_2(z;1) = e^{z^2}\operatorname{erfc}(-z)$$

and

$$E_2(z;0,5) = zE_2(z;1) + \frac{1}{\Gamma(0,5)} = ze^{z^2}\operatorname{erfc}(-z) + \frac{1}{\sqrt{\pi}},$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$$

is the complementary error function [17], we have

$$\begin{aligned}
E_2(AT^{0,5};1) &= \sum_{n=0}^{\infty} \frac{(AT^{0,5})^n}{\Gamma(n\cdot 0,5+1)} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} I + \ldots + \begin{pmatrix} 0 & \dfrac{T^{0,5n}\cdot b^{n-1}}{\Gamma(n\cdot 0,5+1)} \\ 0 & \dfrac{T^{0,5n}\cdot b^{n}}{\Gamma(n\cdot 0,5+1)} \end{pmatrix} I + \ldots \\
&= \begin{pmatrix} 1 & b^{-1}E_2(bT^{0,5};1) - b^{-1} \\ 0 & E_2(bT^{0,5};1) \end{pmatrix} I \\
&= \begin{pmatrix} 1 & b^{-1}[e^{b^2T}\operatorname{erfc}(-bT^{0,5})] - b^{-1} \\ 0 & e^{b^2T}\operatorname{erfc}(-bT^{0,5}) \end{pmatrix} I,
\end{aligned}$$

$$\begin{aligned}
E_2(A(T-s)^{0,5};0,5) &= \sum_{n=0}^{\infty} \frac{(A(T-s)^{0,5})^n}{\Gamma(n\cdot 0,5+0,5)} \\
&= \frac{1}{\Gamma(0,5)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \ldots + \frac{(T-s)^{0,5n}}{\Gamma(n\cdot 0,5+0,5)} \begin{pmatrix} 0 & b^{n-1} \\ 0 & b^n \end{pmatrix} + \ldots \\
&= \begin{pmatrix} \frac{1}{\sqrt{\pi}} & b^{-1}E_2(b(T-s)^{0,5};0,5) - \frac{1}{b\sqrt{\pi}} \\ 0 & E_2(b(T-s)^{0,5};0,5) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{\pi}} & (T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5}) \\ 0 & b(T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5}) + \frac{1}{\sqrt{\pi}} \end{pmatrix}.
\end{aligned}$$

If

$$\pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1$$

and

$$A \overset{*}{-} B = C = \{c : c + B \subset A\}$$

is the geometric difference, we find

$$\pi E_2(AT^{0,5};1)x_0 = \pi E_2(AT^{0,5};1)\begin{pmatrix} y_0 \\ 0 \end{pmatrix} = \pi \begin{pmatrix} y_0 \\ 0 \end{pmatrix} = y_0, \qquad \pi M = 0,$$

$$
\begin{aligned}
&\bigcap_{v\in\tilde Z}\pi(T-s)^{-0,5}E_2(A(T-s)^{0,5};0,5)(\tilde Y-v)\\
&\qquad=(T-s)^{-0,5}\pi E_2(A(T-s)^{0,5};0,5)\tilde S_\rho \overset{*}{-} (T-s)^{-0,5}\pi E_2(A(T-s)^{0,5};0,5)\tilde S_\sigma\\
&\qquad=(T-s)^{-0,5}\pi\begin{pmatrix}\frac{1}{\sqrt{\pi}} & (T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})\\ 0 & b(T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})+\frac{1}{\sqrt{\pi}}\end{pmatrix}\begin{pmatrix}0\\ S_\rho\end{pmatrix}\\
&\qquad\overset{*}{-}(T-s)^{-0,5}\pi\begin{pmatrix}\frac{1}{\sqrt{\pi}} & (T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})\\ 0 & b(T-s)^{0,5}e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})+\frac{1}{\sqrt{\pi}}\end{pmatrix}\begin{pmatrix}0\\ S_\sigma\end{pmatrix}\\
&\qquad=e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})S_\rho \overset{*}{-} e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})S_\sigma\\
&\qquad=e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})S_{\rho-\sigma}=S_{(\rho-\sigma)e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})}
\end{aligned}
$$

This is why inclusion (3.5) becomes

$$
\begin{aligned}
y_0\in\int_0^T S_{(\rho-\sigma)e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})}\,ds &=S_{(\rho-\sigma)\int_0^T e^{(T-s)b^2}\operatorname{erfc}(-b(T-s)^{0,5})\,ds}\\
&=S_{(\rho-\sigma)\left(\frac{e^{b^2T}}{b^2}\operatorname{erfc}(-b\sqrt{T})-\frac{1}{b^2}-\frac{2\sqrt{T}}{b\sqrt{\pi}}\right)}=S_{r(T)},
\end{aligned}
$$

where

$$
r(t)=(\rho-\sigma)\left(\frac{e^{b^2t}}{b^2}\operatorname{erfc}(-b\sqrt{t})-\frac{1}{b^2}-\frac{2\sqrt{t}}{b\sqrt{\pi}}\right)
$$

is a positive and increasing function, since $r(0)=0$ and $r'(t)>0$ for all $t>0$. On the other hand, it is known that $1\geqslant\operatorname{erfc}x>0$ for $x\geqslant0$, and $2\geqslant\operatorname{erfc}x>1$ when $x<0$. Therefore, for $b>0$ and $\|y_0\|=r_0$, there exists $t=t_1$ such that

$$
r(t_1)=(\rho-\sigma)\left(\frac{e^{b^2t_1}}{b^2}\operatorname{erfc}(-b\sqrt{t_1})-\frac{1}{b^2}-\frac{2\sqrt{t_1}}{b\sqrt{\pi}}\right)>(\rho-\sigma)\left(\frac{e^{b^2t_1}}{b^2}-\frac{1}{b^2}-\frac{2\sqrt{t_1}}{b\sqrt{\pi}}\right)\geqslant r_0,
$$

and for $b<0$ there exists $t=t_2$ such that

$$
r(t_2)>(\rho-\sigma)\left(-\frac{1}{b^2}-\frac{2\sqrt{t_2}}{b\sqrt{\pi}}\right)\geqslant r_0.
$$

Consequently, for any initial point y_0, there exists $T>0$ such that the inclusion

$$
y_0\in S_{r(T)} \tag{3.6}
$$

holds. Since the ball $S_{r(T)}$ is a closed set, inclusion (3.6) also holds for $T=T_0=\min\{T>0:$ inclusion (3.6) holds$\}$.

Thus, we have proved that Condition 2) of Theorem 2.1 is satisfied. Conditions 1) and 3) are obvious. Conditions 4) and 5) hold because in a reflexive space balls are weakly compact. Condition 6) follows from the fact that as the mapping w one can take the mapping given by the formula $w=v(u)=\dfrac{e}{\|e\|}\alpha_0\sigma$ for $u=\dfrac{e}{\|e\|}\alpha_0\rho$, where $0\leqslant\alpha_0\leqslant1$.

Hence,

1) In game (3.1), from any initial point y_0, pursuit completion is possible in time

$$
T_0=\min\{T>0:\|y_0\|=r(T_0)\}.
$$

Here the pursuit control $\overline{u}(\cdot)$ is chosen by the formula

$$\overline{u}(s) = \frac{\sqrt{\pi} e^{-(T_0-s)b^2}}{2T_0 \int\limits_{-b(T_0-s)^{0,5}}^{\infty} e^{-t^2} dt} y_0 + \overline{v}(s), \qquad 0 < s \leqslant T_0.$$

2) in the game (3.1), from any initial point y_0, evasion on the interval $[0, T_0)$ is possible.

4. Conclusion

In the present work, we provide sufficient conditions for the solvability of the pursuit problem in time T_0 and sufficient conditions for the solvability of the evasion problem on a finite interval $(0, T_0)$, when the dynamics of the game is described by a fractional order differential equation with Hilfer derivatives. We solve one game problem, which describes the relaxation process in the glass formation of supercooled liquids. The obtained results generalize the finite–dimensional result in [10] and the infinite–dimensional result in [14].

These results can be used in the mathematical theory of controlled processes occurring under conflict or uncertainty conditions, and in solving game problems whose dynamics are described by differential equations in suitable Banach spaces.

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