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ON DAMPING CONTROL SYSTEM ON STAR GRAPH WITH GLOBAL TIME–PROPORTIONAL DELAY

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Abstract. We consider the problem of damping a control system with aftereffect described by first–order functional-differential equations on a temporal star graph. The delay in the system is proportional to time and propagates through the interior vertex. A variational problem of minimizing an energy functional is studied taking into consideration the probabilities of scenarios corresponding to different edges. It is established that the optimal trajectory satisfies Kirchhoff type conditions at the interior vertex. The equivalence of the variational problem to a certain boundary value problem for second order functional–differential equations on the graph is proved, and the unique solvability of both problems is established.

Keywords: pantograph equation, functional–differential equation, quantum graph, global delay, temporal graph, variational problem, optimal control.

Mathematics Subject Classification: 93C23, 49J55, 34K35, 34K10

1. Introduction

Krasovskii [6] posed and investigated the problem of damping a controlled system with aftereffect. A system with constant delay described by an equation of retarded type was considered. The more complicated case when the equation is of neutral type, i.e., contains delay also in the higher order terms, was considered by Skubachevskii [9], [21] and later by other authors (see also [1] and the references therein). Recently, Buterin extended this problem to the so–called temporal graphs [12], [3], [4], [5].

Differential operators on graphs, often called quantum graphs, have been extensively studied since the previous century in connection with modelling various processes in complex systems represented as spatial networks [18], [19], [17], [7], [10]. Such models typically involve Kirchhoff type conditions at the interior vertices. In [12], [3], [4], [5] it was shown that optimal trajectories on temporal graphs satisfy analogous conditions.

In [11], a definition of operators on graphs with global delay was proposed. The latter means that the delay propagates through the interior vertices of the graph. In other words, the solution of the equation on an incoming edge serves as the initial function for the equations on the outgoing edges. Global delay became an alternative to the locally nonlocal case considered in [22], where the equation on each edge has its own delay parameter and can be solved independently from the equations on the other edges.

The use of the concept of global delay made it possible to extend the aforementioned problem of damping a controlled system with aftereffect to graphs in [12], [3]. This, in turn, led to the concept of a temporal graph, whose edges, unlike those of a spatial network, are identified

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with time intervals, and each interior vertex is understood as a branching point of the process, giving several different scenarios for its further evolution. Moreover, in [12], [4], a stochastic interpretation of the control system on a temporal tree was proposed. Namely, a system on a tree appears, for example, by replacing the coefficients in the equation on an interval with discrete–time stochastic processes. In [5], a globally nonlocal integro–differential control system on a star graph was investigated.

In the present work, the problem of damping a control system described by the so–called pantograph equation, which has a number of important applications [2], [13], [14], [15], [16], [20], [8], is extended to a temporal star graph. In this case, the delay is not constant but is a time–proportional contraction.

On an interval, this case was considered by Rossovskii in [8] for the equation of neutral type

$$y'(t) + ay'\left(q^{-1}t\right) + by(t) + cy\left(q^{-1}t\right) = u(t), \qquad t > 0, \tag{1.1}$$

where $a, b, c \in \mathbb{R}$ and $q > 1$, and $u(t)$ is a real–valued control function. The state of the system at the initial time is given by the condition

$$y(0) = y_0 \in \mathbb{R}. \tag{1.2}$$

The control problem is formulated as follows. It is required to find $u(t) \in L_2(0,T)$ that brings system (1.1), (1.2) to the equilibrium $y(t) = 0$ for $t \geqslant T$ for some $T > 0$.

In order to do this, it is sufficient to find $u(t)$ that brings the system to the state

$$y(t) = 0, \quad q^{-1}T \leqslant t \leqslant T, \tag{1.3}$$

and then to switch off the control by setting $u(t) \equiv 0$ for $t > T$. Among all possible controls, one seeks a control with minimal energy

$$\int\limits_0^T u^2(t)\,dt.$$

As a result, we obtain a problem of minimizing a quadratic functional

$$\mathcal{J}(y) = \int\limits_0^T \left(y'(t) + ay'\left(q^{-1}t\right) + by(t) + cy\left(q^{-1}t\right)\right)^2 dt \longrightarrow \min \tag{1.4}$$

over the set of functions $y(t) \in W_2^1[0,T]$ satisfying boundary conditions (1.2), (1.3).

The solution of problem (1.2)–(1.4) was obtained in [8]. It is reduced to solving an equivalent boundary value problem for a second order functional–differential equation. In particular, it was established that if the function $y(t) \in W_2^1[0,T]$ satisfies (1.2), (1.3) and minimizes the functional (1.4), then for all $v \in W_2^1[0,T]$ satisfying (1.2) with $y_0 = 0$ and (1.3), the integral identity holds

$$\begin{aligned}
&\int\limits_0^{q^{-1}T} \left(\left(1 + a^2 q\right) y'(t) + ay'\left(q^{-1}t\right) + aqy'(qt)\right) v'(t)dt \\
&\qquad + \int\limits_0^{q^{-1}T} \left(\left(ab - cq^{-1}\right) y'\left(q^{-1}t\right) + \left(cq - abq^2\right) y'(qt)\right. \\
&\qquad \left. + \left(b^2 + c^2 q\right) y(t) + bcy\left(q^{-1}t\right) + bcqy(qt)\right) v(t)dt = 0.
\end{aligned}$$

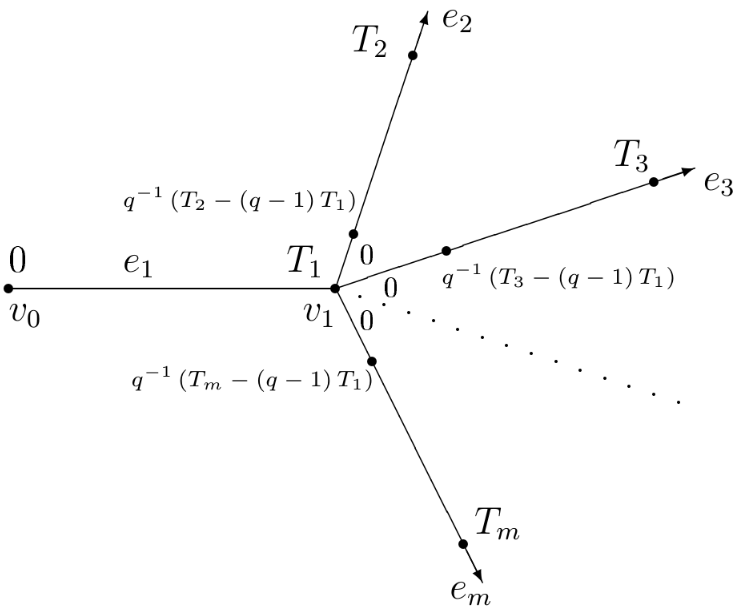

FIGURE 1. Star graph Γ_m.

The latter means that $y(t) \in W_2^1[0,T]$ defines a generalized solution of the boundary value problem

$$\begin{aligned} &-\left(\left(1+a^2q\right)y'(t)+ay'\left(q^{-1}t\right)+aqy'(qt)\right)'+\left(ab-cq^{-1}\right)y'\left(q^{-1}t\right) \\ &+\left(cq-abq^2\right)y'(qt)+\left(b^2+c^2q\right)y(t)+bcy\left(q^{-1}t\right)+bcqy(qt)=0, \qquad 0<t<q^{-1}T, \end{aligned} \tag{1.5}$$

under conditions (1.2) and (1.3). The converse is also true: if $y(t) \in W_2^1[0,T]$ is a generalized solution of problem (1.2), (1.3), (1.5), then y provides a minimum to functional (1.4).

The following theorem establishes the existence and uniqueness of a generalized solution of boundary value problem (1.2), (1.3), (1.5), as well as the unique solvability of variational problem (1.2)–(1.4).

Theorem 1.1 ([8]). *Let $|a| \neq q^{-1/2}$. Then problem (1.2), (1.3), (1.5) has a unique generalized solution $y \in W_2^1[0,T]$.*

In the next section, we give a formulation of the variational problem on a star graph. In particular, this requires an understanding of how global contraction should look on the graph. For simplicity, we restrict ourselves to the case $a = 0$, i.e., the equation of retarded type. In the third section, we establish the equivalence of the variational problem to a certain boundary value problem for second order functional–differential equations on the graph. In the last section, we prove the unique solvability of both problems.

2. Formulation of the variational problem on a star-shaped graph

We consider the graph Γ_m shown in Figure 1. As usual, by a function y on the graph Γ_m we mean a tuple $y = [y_1, \ldots, y_m]$, in which the component y_j is defined on the edge e_j, i.e., $y_j = y_j(t)$, $t \in [0, T_j]$.

Suppose that to the time $t = T_1 > 0$, associated with the only interior vertex v_1 of the graph Γ_m, our control system with time–proportional delay on Γ_m is described by the equation

$$\ell_1 y(t) := y_1'(t) + b_1 y_1(t) + c_1 y_1\left(q^{-1}t\right) = u_1(t), \qquad 0 < t < T_1, \tag{2.1}$$

on the edge e_1 of the graph Γ_m, with the initial condition

$$y_1(0) = y_0. \tag{2.2}$$

At $t = T_1$, i.e., at the vertex v_1, the system branches into $m-1$ independent parallel processes described by the equations

$$\ell_j y(t) := y_j'(t) + b_j y_j(t) + c_j y_j\left(q^{-1}(t - (q-1)T_1)\right) = u_j(t), \qquad t > 0, \qquad j = \overline{2,m}, \tag{2.3}$$

but having a common history determined by equation (2.1) with initial condition (2.2) and the conditions

$$y_j(t) = y_1(t + T_1), \qquad \left(q^{-1} - 1\right)T_1 < t < 0, \qquad j = \overline{2,m}, \tag{2.4}$$

as well as continuity conditions at the vertex v_1, which in this case fits (2.4) as $t \to -0$:

$$y_j(0) = y_1(T_1), \qquad j = \overline{2,m}. \tag{2.5}$$

As in the previous section, we assume that $q > 1$, $y_0 \in \mathbb{R}$ and all $b_j, c_j \in \mathbb{R}$.

In (2.3), the j-th equation is given on the edge e_j of the graph Γ_m, which is, generally speaking, an infinite ray emanating from the vertex v_1. Conditions (2.4) mean that the delay propagates through the vertex v_1.

Cauchy problem (2.1)–(2.5) has a unique solution $y_j(t) \in W_2^1[0, T_j]$, $j = \overline{1,m}$, for each fixed $T_j > 0$, $j = \overline{2,m}$, provided that $u_j(t) \in L_2(0, T_j)$ for $j = \overline{1,m}$. On the edge e_1, it reduces to a Volterra equation of the second kind with a piecewise constant kernel and a free term from $W_2^1[0, T_1]$:

$$y_1(t) = y_0 + \int_0^t u_1(s)\,ds - b_1 \int_0^t y_1(s)\,ds - qc_1 \int_0^{q^{-1}t} y_1(s)\,ds = f(t) + \int_0^t K(t,s) y_1(s)\,ds,$$

where

$$f(t) = y_0 + \int_0^t u_1(s)\,ds, \qquad K(t,s) = \begin{cases} -(b_1 + qc_1), & s \leqslant q^{-1}t, \\ -b_1, & s > q^{-1}t. \end{cases}$$

Then the obtained function $y_1(t)$ provides common initial function (2.4) and common initial condition (2.5) for all equations in (2.3), which can be solved similarly.

Example 2.1. *Let $m = 2$, $b := b_1 = b_2$, $c := c_1 = c_2$, and*

$$y(t) := \begin{cases} y_1(t), & 0 \leqslant t \leqslant T_1, \\ y_2(t - T_1), & t > T_1, \end{cases} \qquad u(t) := \begin{cases} u_1(t), & 0 < t < T_1, \\ u_2(t - T_1), & t > T_1. \end{cases}$$

Then Cauchy problem (2.1)–(2.5) becomes (1.1), (1.2) with $a = 0$.

Assume for definiteness that $T_j > (q-1)T_1$ for all $j = \overline{2,m}$. To damp system (2.1)–(2.5) simultaneously for all scenarios, it is necessary to bring it to the state

$$y_j(t) = 0, \qquad q^{-1}(T_j - (q-1)T_1) \leqslant t \leqslant T_j, \qquad j = \overline{2,m}, \tag{2.6}$$

by choosing suitable controls $u_j(t)$, $j = \overline{1,m}$. Then, letting $u_j(t) \equiv 0$ for $t > T_j$, $j = \overline{2,m}$, we get $y_j(t) = 0$ for the same t and j. In other words, the system will be damped on each outgoing edge. Since such $u_j(t)$ are not unique, we shall seek them by minimizing the efforts $\|u_j\|^2_{L_2(0,T_j)}$. Moreover, similarly to what was done in [12] for the case of constant delay, we can regulate the degree of participation of each $\|u_j\|^2_{L_2(0,T_j)}$ in the corresponding energy functional by choosing a certain positive weight α_j.

Thus, we arrive at the variational problem

$$\mathcal{J}(y) = \sum_{j=1}^{m} \alpha_j \int_0^{T_j} (\ell_j y(t))^2 \, dt \to \min \tag{2.7}$$

under conditions (2.2), (2.4)–(2.6), where $\alpha_j > 0$, $j = \overline{1,m}$, are fixed.

To choose the weights α_j, $j = \overline{1,m}$, one can apply a probabilistic approach in accordance with the interpretation of the control system on a temporal graph proposed in [12]. Namely, to do this, it is necessary to set $\alpha_1 = 1$, and take as α_j, $j = \overline{2,m}$, the probabilities of the scenarios given by the corresponding equations in (2.3), i.e., $\alpha_2 + \ldots + \alpha_m = 1$. The latter identity also ensures the fitting with the interval case if the equations in (2.3) are independent of j, i.e., are artificial copies of a unique possible scenario (see Example 2 in [12]).

Note that conditions (2.4) impose no restrictions on the function $y = [y_1, \ldots, y_m]$. Therefore, we adopt that taking $\mathcal{J}(y)$, as well as $\ell_j y$ for $j = \overline{2,m}$, for any function y on Γ_m, means the application of conditions (2.4). For the sake of brevity we also introduce the notation $\ell y := [\ell_1 y, \ldots, \ell_m y]$.

3. Reduction to boundary value problem

We consider the real Hilbert space $W_2^k(\Gamma_m) = \bigoplus_{j=1}^m W_2^k[0, T_j]$ with the scalar product

$$(y, z)_{W_2^k(\Gamma_m)} = \sum_{j=1}^{m} (y_j, z_j)_{W_2^k[0,T_j]},$$

where $y = [y_1, \ldots, y_m]$, $z = [z_1, \ldots, z_m]$,

$$(f, g)_{W_2^k[a,b]} = \sum_{\nu=0}^{k} \left(f^{(\nu)}, g^{(\nu)}\right)_{L_2(a,b)}$$

is the inner product in $W_2^k[a, b]$, and $(\cdot, \cdot)_{L_2(a,b)}$ is the scalar product in $L_2(a, b)$.

Let $\mathcal{W}$ be the closed subspace of $W_2^1(\Gamma_m)$ consisting of all tuples $[y_1, \ldots, y_m]$ satisfying conditions (2.5), (2.6) and $y_1(0) = 0$.

We also introduce the space $W_2^k(\widetilde{\Gamma}_m) = W_2^k[0, T_1] \oplus \bigoplus_{j=2}^{m} W_2^k[0, q^{-1}(T_j - (q-1)T_1)]$.

Lemma 3.1. *If $y \in W_2^1(\Gamma_m)$ solves variational problem (2.2), (2.4)–(2.7), then*

$$B(y, w) := \sum_{j=1}^{m} \alpha_j \int_0^{T_j} \ell_j y(t)\, \ell_j w(t) \, dt = 0 \qquad w \in \mathcal{W}. \tag{3.1}$$

And vice versa, if for some $y \in W_2^1(\Gamma_m)$ conditions (2.2), (2.5), (2.6) and (3.1) hold, then y is solves problem (2.2), (2.4)–(2.7).

Proof. Let $y \in W_2^1(\Gamma_m)$ be a solution of problem (2.2), (2.4)–(2.7). Then for an arbitrary fixed function $w \in \mathcal{W}$ the sum $y + sw$ belongs to $W_2^1(\Gamma_m)$ for any $s \in \mathbb{R}$ and satisfies conditions (2.2), (2.5), (2.6). It is easy to see that

$$\mathcal{J}(y + sw) = \mathcal{J}(y) + 2sB(y, w) + s^2 \mathcal{J}(w).$$

Since $\mathcal{J}(y + sw) \geqslant \mathcal{J}(y)$ for all $s \in \mathbb{R}$, identity (3.1) holds.

Conversely, let $y \in W_2^1(\Gamma_m)$ satisfy conditions (2.2), (2.5) and (2.6). Then it follows from (3.1) that

$$\mathcal{J}(y+w) = \mathcal{J}(y) + 2B(y,w) + \mathcal{J}(w) \geqslant \mathcal{J}(y) \quad \forall w \in \mathcal{W}.$$

Thus, y provides a minimum to functional (2.7) under conditions (2.2), (2.5) and (2.6). The proof is complete. $\square$

We transform (3.1) by making a change of variables in the terms containing $w_j(q^{-1}(t-(q-1)T_1))$ and $w_1(q^{-1}t)$. As a result, the expression for $B(y,w)$ becomes

$$B(y,w) = \sum_{j=1}^{m} \alpha_j \int_0^{T_j} \ell_j y(t)\, w_j'(t)\, dt + \sum_{j=1}^{m} \alpha_j b_j \int_0^{T_j} \ell_j y(t)\, w_j(t)\, dt$$
$$+ q\left(\alpha_1 c_1 \int_0^{q^{-1}T_1} \ell_1 y(qt)\, w_1(t)\, dt + \sum_{j=2}^{m} \alpha_j c_j \int_{(q^{-1}-1)T_1}^{q^{-1}(T_j-(q-1)T_1)} \ell_j y(qt+(q-1)T_1)\, w_j(t)\, dt\right).$$

Applying (2.4) to $w = [w_1, \ldots, w_m] \in \mathcal{W}$, we get the representation

$$\int_{(q^{-1}-1)T_1}^{q^{-1}(T_j-(q-1)T_1)} \ell_j y(qt+(q-1)T_1)\, w_j(t)\, dt = \int_0^{q^{-1}(T_j-(q-1)T_1)} \ell_j y(qt+(q-1)T_1)\, w_j(t)\, dt$$
$$+ \int_{q^{-1}T_1}^{T_1} \ell_j y(qt-T_1)\, w_1(t)\, dt, \qquad j = \overline{2,m}.$$

Then we rewrite (3.1) in the equivalent form

$$B(y,w) = \sum_{j=1}^{m} \left(\alpha_j \int_0^{T_j} \ell_j y(t)\, w_j'(t)\, dt + \int_0^{T_j} \left(\alpha_j b_j \ell_j y(t) + \widetilde{\ell}_j y(t)\right) w_j(t)\, dt \right) = 0 \qquad \forall w \in \mathcal{W}, \tag{3.2}$$

where

$$\tilde{\ell}_1 y(t) = \begin{cases} q\alpha_1 c_1 \ell_1 y(qt), & 0 < t < q^{-1}T_1, \\ q\sum_{k=2}^{m} \alpha_k c_k \ell_k y(qt-T_1), & q^{-1}T_1 < t < T_1, \end{cases}$$
$$\tilde{\ell}_j y(t) = \begin{cases} q\alpha_j c_j \ell_j y(qt+(q-1)T_1), & 0 < t < q^{-1}(T_j-(q-1)T_1), \\ 0, & q^{-1}(T_j-(q-1)T_1) < t < T_j, \end{cases} \qquad j = \overline{2,m}. \tag{3.3}$$

We denote by $\mathcal{B}$ the boundary value problem for second order functional–differential equations

$$\mathcal{L}_j y(t) := -\alpha_j (\ell_j y)'(t) + \alpha_j b_j \ell_j y(t) + \widetilde{\ell}_j y(t) = 0, \qquad 0 < t < l_j, \qquad j = \overline{1,m}, \tag{3.4}$$

under conditions (2.2), (2.4)–(2.6) and the Kirchhoff type condition

$$\alpha_1 y_1'(T_1) - \sum_{j=2}^{m} \alpha_j y_j'(0) + \beta y_1(T_1) + \gamma y_1\left(q^{-1}T_1\right) = 0, \tag{3.5}$$

where $\widetilde{\ell}_j y\,(t)$ were defined in (3.3), and

$$l_1 := T_1, \qquad l_j := q^{-1}\left(T_j - (q-1)\,T_1\right), \qquad j = \overline{2, m}, \tag{3.6}$$

$$\beta := \alpha_1 b_1 - \sum_{j=2}^{m} \alpha_j b_j, \qquad \gamma := \alpha_1 c_1 - \sum_{j=2}^{m} \alpha_j c_j. \tag{3.7}$$

The following statement holds.

Lemma 3.2. *If $y \in W_2^1(\Gamma_m)$ satisfies conditions (2.2), (2.5), (2.6) and (3.1), then $y \in W_2^2(\widetilde{\Gamma}_m)$ and is a solution of the boundary value problem $\mathcal{B}$. And vice versa, each solution of problem $\mathcal{B}$ satisfies condition (3.1).*

Proof. Let $y \in W_2^1(\Gamma_m)$ satisfy conditions (2.2), (2.5), (2.6) and (3.1). Taking into consideration the fact that (3.1) is equivalent to (3.2) and applying Lemma 2 from [12] to (3.2) together with (3.6), we obtain that $\ell_j y\,(t) \in W_2^1\,[0, l_j]$ and

$$\alpha_1 \ell_1 y\,(T_1) = \sum_{j=2}^{m} \alpha_j \ell_j y\,(0)\,. \tag{3.8}$$

Using (2.1), (2.3) and (3.6), we then obtain $y_j'\,(t) \in W_2^1\,[0, l_j]$ for $j = \overline{1, m}$. Therefore, $y \in W_2^2(\widetilde{\Gamma}_m)$.

In view of (2.1) and (2.3), we rewrite (3.8) in the equivalent form

$$\alpha_1\left(y_1'\,(T_1) + b_1 y_1\,(T_1) + c_1 y_1\left(q^{-1}T_1\right)\right) = \sum_{j=2}^{m} \alpha_j\left(y_j'\,(0) + b_j y_j\,(0) + c_j y_j\left(\left(q^{-1}-1\right)T_1\right)\right), \tag{3.9}$$

where, according to (2.4), the limits

$$y_j\left(\left(q^{-1}-1\right)T_1\right) := \lim_{t \to (q^{-1}-1)T_1 + 0} y_j\,(t) = \lim_{t \to (q^{-1}-1)T_1 + 0} y_1\,(t + T_1) = y_1(q^{-1}T_1),$$

are obviously well–defined. Thus, relation (3.9) together with (2.4), (2.5) yields (3.5) with (3.7). Finally, integrating by parts in (3.2) and using (2.5), (2.6), we obtain

$$B\,(y, w) = w_1\,(T_1)\left(\alpha_1 \ell_1 y\,(T_1) - \sum_{j=2}^{m} \alpha_j \ell_j y\,(0)\right) + \sum_{j=1}^{m} \int_0^{l_j} \mathcal{L}_j y\,(t)\, w_j\,(t)\; dt = 0. \tag{3.10}$$

Due (3.8) and the arbitrary choice of w_j, by (3.10) we obtain (3.4).

Conversely, let y be a solution of problem $\mathcal{B}$. Since (3.5) is equivalent to (3.8), the second identity in (3.10) holds. Integrating by parts in (3.10) we arrive at (3.2). Since (3.2) is equivalent to (3.1), then (3.1) holds. The proof is complete. □

Combining Lemmas 3.1 and 3.2, we obtain the following result.

Theorem 3.1. *A function $y \in W_2^1(\Gamma_m)$ is a solution of variational problem (2.2), (2.4)–(2.7) if and only if $y \in W_2^2(\widetilde{\Gamma}_m)$ and is a solution of the boundary value problem $\mathcal{B}$.*

4. Unique solvability

We are going to establish the unique solvability of the boundary value problem $\mathcal{B}$, and hence, by Theorem 3.1, of variational problem (2.2), (2.4)–(2.7).

First we prove two auxiliary statements.

Lemma 4.1. *There exists C_1 such that*

$$\|\ell w\|^2_{L_2(\Gamma_m)} \leqslant C_1 \|w\|^2_{W_2^1(\widetilde{\Gamma}_m)}, \qquad w \in \mathcal{W}. \tag{4.1}$$

Proof. Using (2.1), (2.3) and the inequality

$$(a_1 + \cdots + a_n)^2 \leqslant n\left(a_1^2 + \cdots + a_n^2\right), \qquad a_1, \ldots, a_n \in \mathbb{R}, \tag{4.2}$$

for $n = 3$ we obtain

$$\begin{aligned} \|\ell w\|^2_{L_2(\Gamma_m)} \leqslant & 3 \sum_{j=1}^{m} \int_0^{T_j} \left(w_j'(t)\right)^2 dt + 3 \sum_{j=1}^{m} b_j^2 \int_0^{T_j} w_j^2(t)\, dt \\ & + 3\left(c_1^2 \int_0^{T_j} w_1^2\left(q^{-1}t\right) dt + \sum_{j=2}^{m} c_j^2 \int_0^{T_j} w_j^2\left(q^{-1}\left(t - (q-1)T_1\right)\right) dt\right). \end{aligned} \tag{4.3}$$

Applying (2.4) to w, we find

$$\begin{aligned} & \int_0^{T_j} w_1^2\left(q^{-1}t\right) dt = q\|w_1\|^2_{L_2(0, q^{-1}T_1)}, \\ & \int_0^{T_j} w_j^2\left(q^{-1}\left(t - (q-1)T_1\right)\right) dt = q\|w_1\|^2_{L_2(q^{-1}T_1, T_1)} \\ & \qquad\qquad + q\|w_j\|^2_{L_2(0, q^{-1}(T_j - (q-1)T_1))}, \quad j = \overline{2, m}. \end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) and applying (2.6), we arrive at (4.1), where the constant C_1 is independent of w. The proof is complete. □

Lemma 4.2. *There exists $C_2 > 0$ such that*

$$\mathcal{J}(w) \geqslant C_2 \|w\|^2_{W_2^1(\widetilde{\Gamma}_m)}, \qquad w \in \mathcal{W}.$$

Proof. Assume the contrary. Suppose that there exists $w_{(n)} \in \mathcal{W}$, $n \in \mathbb{N}$, such that

$$\mathcal{J}\left(w_{(n)}\right) \leqslant \frac{1}{n} \|w_{(n)}\|^2_{W_2^1(\widetilde{\Gamma}_m)}.$$

Without loss of generality we suppose that

$$\|w_{(n)}\|_{W_2^1(\widetilde{\Gamma}_m)} = 1. \tag{4.5}$$

Then

$$\mathcal{J}\left(w_{(n)}\right) \leqslant \frac{1}{n}, \qquad n \in \mathbb{N}. \tag{4.6}$$

Using (2.1), (2.3) and inequality (4.2) for $n = 3$, we obtain

$$\begin{aligned} & \left(w_1'(t)\right)^2 \leqslant 3\left(\left(\ell_1 w(t)\right)^2 + b_1^2 w_1^2(t) + c_1^2 w_1^2\left(q^{-1}t\right)\right), \\ & \left(w_j'(t)\right)^2 \leqslant 3\left(\left(\ell_j w(t)\right)^2 + b_j^2 w_j^2(t) + c_j^2 w_j^2\left(q^{-1}\left(t - (q-1)T_1\right)\right)\right), \qquad j = \overline{2, m}. \end{aligned} \tag{4.7}$$

Integrating (4.7) from 0 to T_j and multiplying by α_j, and then summing over $j = \overline{1,m}$, we arrive at the estimate

$$\begin{aligned}\frac{1}{3}\sum_{j=1}^{m}\alpha_j\int_0^{T_j}\left(w_j'(t)\right)^2 dt \leqslant \sum_{j=1}^{m}\alpha_j\int_0^{T_j}\left(\ell_j w(t)\right)^2 dt \\ +\sum_{j=1}^{m}\alpha_j b_j^2\int_0^{T_j} w_j^2(t)\,dt + \alpha_1 c_1^2\int_0^{T_1} w_1^2\left(q^{-1}t\right) dt \\ +\sum_{j=2}^{m}\alpha_j c_j^2\int_0^{T_j} w_j^2\left(q^{-1}\left(t-(q-1)T_1\right)\right) dt.\end{aligned}\tag{4.8}$$

According to [12, Lm. 5] we have

$$\|w'\|^2_{L_2(\Gamma_m)} \geqslant C_3\|w\|^2_{W_2^1(\widetilde{\Gamma}_m)}, \qquad w \in \mathcal{W}, \tag{4.9}$$

where $C_3 > 0$, $w' = [w_1', \dots, w_m']$. Substituting (4.4) into (4.8) and using estimate (4.9), we obtain

$$\frac{\alpha C_3}{3}\|w\|^2_{W_2^1(\widetilde{\Gamma}_m)} \leqslant \mathcal{J}(w) + K\|w\|^2_{L_2(\Gamma_m)}, \qquad \alpha := \min_{j=\overline{1,m}}\alpha_j > 0. \tag{4.10}$$

Since $W_2^1(\Gamma_m)$ is compactly embedded in $L_2(\Gamma_m)$, by (4.5) there exists a subsequence $\{w_{(n_k)}\}_{k\in\mathbb{N}}$ that is fundamental in $L_2(\Gamma_m)$. Then by inequality (4.10) we obtain

$$\frac{\alpha C_3}{3}\|w_{(n_k)} - w_{(n_l)}\|^2_{W_2^1(\widetilde{\Gamma}_m)} \leqslant \mathcal{J}\left(w_{(n_k)} - w_{(n_l)}\right) + K\|w_{(n_k)} - w_{(n_l)}\|^2_{L_2(\Gamma_m)}.$$

Moreover, using (4.2) for $n = 2$ and (4.6), we have

$$\mathcal{J}\left(w_{(n_k)} - w_{(n_l)}\right) \leqslant \frac{2}{n_k} + \frac{2}{n_l}.$$

Thus, the sequence $\{w_{(n_k)}\}_{k\in\mathbb{N}}$ is fundamental in $W_2^1(\widetilde{\Gamma}_m)$ and converges to some function $w_{(0)} \in \mathcal{W}$.

By Lemma 4.1, the convergence of $w_{(n_k)}$ to $w_{(0)}$ in $W_2^1(\widetilde{\Gamma}_m)$ implies the convergence of $\ell w_{(n_k)}$ to $\ell w_{(0)}$ in $L_2(\Gamma_m)$. Consequently, in view of (4.6), we obtain

$$\|\ell w_{(0)}\|^2_{L_2(\Gamma_m)} = \lim_{k\to\infty}\|\ell w_{(n_k)}\|^2_{L_2(\Gamma_m)} \leqslant \frac{1}{\alpha}\lim_{k\to\infty}\mathcal{J}\left(w_{(n_k)}\right) = 0,$$

that is, $\ell w_{(0)} = 0$. Thus, $w_{(0)}$ is a solution of the Cauchy problem (2.1)–(2.5) with $y_0 = 0$ and $u_j(t) \equiv 0$, $j = \overline{1,m}$. By the uniqueness of the solution of the Cauchy problem, we have $w_{(0)} \equiv 0$, which contradicts (4.5). The proof is complete. □

We proceed to the main result of this section.

Theorem 4.1. *The boundary value problem $\mathcal{B}$ has a unique solution $y \in W_2^1(\Gamma_m) \cap W_2^2(\widetilde{\Gamma}_m)$. Moreover, there exists C such that*

$$\|y\|_{W_2^1(\Gamma_m)} \leqslant C|y_0|. \tag{4.11}$$

Proof. We introduce a function $\Phi = [\Phi_1, \dots, \Phi_m] \in W_2^1(\Gamma_m)$ such that

$$\Phi_1(t) = \begin{cases} y_0\left(1 - \frac{qt}{T_1}\right), & 0 \leqslant t < q^{-1}T_1, \\ 0, & q^{-1}T_1 \leqslant t \leqslant T_1, \end{cases} \qquad \Phi_j(t) \equiv 0, \qquad j = 2, \dots, m.$$

By Lemma 3.2, for a function $y \in W_2^1(\Gamma_m)$ satisfying conditions (2.2), (2.5), (2.6) to be a solution of the boundary value problem $\mathcal{B}$, it is necessary and sufficient that condition (3.1) hold.

In other words, y is a solution of the boundary value problem $\mathcal{B}$ if and only if $x := y - \Phi \in \mathcal{W}$ (which is equivalent to conditions (2.2), (2.5), (2.6)) and

$$B(\Phi, w) + B(x, w) = 0, \qquad w \in \mathcal{W}, \tag{4.12}$$

which, in turn, is equivalent to condition (3.1).

We are going to show that there exists a unique function x such that (4.12) holds. Since $B(w, w) = \mathcal{J}(w)$, by Lemmas 4.1 and 4.2 the bilinear form $B(\cdot,\cdot)$ defines on $\mathcal{W}$ an equivalent inner product. Moreover, using (2.3), (2.4), (3.1) and the Cauchy — Bunyakovsky inequality, we obtain the estimate

$$|B(\Phi, w)| = \alpha_1 \left| \int_0^{T_1} \ell_1\Phi(t)\, \ell_1 w(t)\, dt \right| \leqslant M|y_0| \|w\|_{\mathcal{W}}, \tag{4.13}$$

where $\|w\|_{\mathcal{W}} = \sqrt{(w,w)_{\mathcal{W}}}$. Thus, by the Riesz representation theorem for linear continuous functionals in a Hilbert space, there exists a unique function $x \in \mathcal{W}$ such that (4.12) holds. Hence, the boundary value problem $\mathcal{B}$ has a unique solution $y = \Phi + x$.

Finally, it follows from (4.12) and (4.13) that

$$\|x\|_{\mathcal{W}} \leqslant M|y_0|. \tag{4.14}$$

Then, using (4.14) and the expression

$$\|\Phi\|^2_{W_2^1(\Gamma_m)} = \|\Phi_1\|^2_{W_2^1[0,q^{-1}T_1]} = \frac{T_1^2 + 3q^2}{3qT_1} y_0^2,$$

we obtain estimate (4.11). The proof is complete. □

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