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TRIANGULATION OF PLANAR DOMAINS DEFINED BY SYSTEM OF INEQUALITIES

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Abstract. In this work we present a method for constructing a triangulation of a planar, generally multiply connected, domain defined by a system of inequalities. The main idea of the approach is as follows. At the first step, a standard triangulation over a rectangular grid of a certain polygon lying in the given domain is constructed. At the second step, the boundary nodes of the obtained triangulation are shifted to the boundary of the domain by the gradient descent method. We prove that the sines of the angles of the triangles in such a triangulation do not tend to zero (the triangles do not degenerate) as their diameters tend to zero, provided that the given domain has a smooth boundary.

Keywords: triangulation, gradient descent method, quality of triangulation.

Mathematics Subject Classification: 51N20, 51M15

1. Introduction

Let $\{P_i\}_{i=1}^m$ be a finite set of points in the plane $\mathbb{R}^2$. A triangulation of this set of points is a collection of nondegenerate triangles $\mathcal{T} = \{T_j\}_{j=1}^N$ such that

1) the vertices of the triangles belong to the set of points $\{P_i\}_{i=1}^m$ and each point P_i is a vertex of at least one triangle;
2) the triangles either have a common vertex, or a common edge, or have no common points.

By $\alpha(\mathcal{T})$ we denote the minimal sine of the angles of all triangles in the triangulation $\mathcal{T}$.

We observe that each point P_i is a vertex of some collection of triangles from the triangulation $\mathcal{T}$. If the sum of the angles at the vertex P_i over all such triangles is equal to 2π, then the node P_i is called an interior node of the triangulation. Otherwise, the node is called a boundary node.

Let Ω be a bounded domain in $\mathbb{R}^2$. A triangulation of the domain Ω is a triangulation of an arbitrary finite set of points lying in the closure of the domain Ω. There are several approaches to constructing a computational triangular mesh (triangulation). For example, one can take finitely many points in $\overline{\Omega}$ and form a triangulation based on them by using one of appropriate algorithms (see, for example, [17], [18], [11]). There are also other methods that employ either variational principles, or partition the domain (or its boundary) into simple parts, or solving various boundary value problems, etc. (see, for example, [1], [15], [6], [2], [8], [16], [7], [12]).

We propose a somewhat different approach, which is as follows. We assume that the domain Ω is given by a system of inequalities. As an initial triangulation of the domain, we take a triangulation constructed on the basis of a rectangular grid of the plane. This triangulation includes triangles whose vertices fall into the domain. After that, all boundary nodes of the initial triangulation are shifted to the boundary of the domain by the gradient descent method. Note that the gradients here are computed for the functions defining the boundary of the

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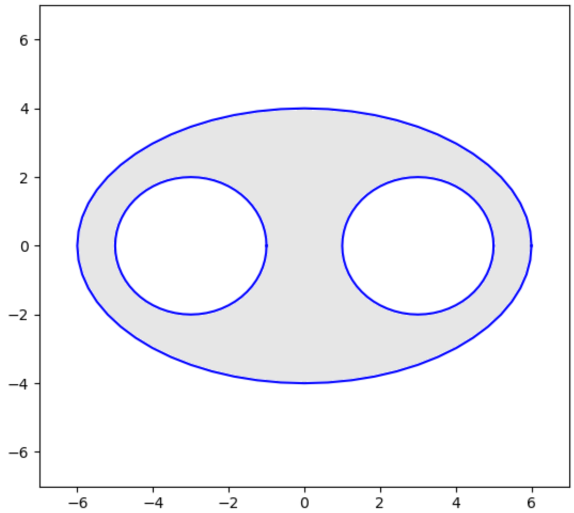

FIGURE 1. Domain defined by system of inequalities (2.1).

domain. The approach involving the shift of boundary nodes to the boundary of the domain is not new. For example, it was applied to construction of a spatial grid in [22]. The difference of our algorithm is that in the cited work the domain was given by only one inequality and the shift of the boundary nodes to the boundary of the domain is implemented somewhat differently.

We mention that as a numerical characteristic responsible for the quality of the triangulation, we consider the minimal sine of the angles of the triangles of the triangulation. It should be noted that the sines of the angles of a triangle significantly affect the error in the computation of a function and its derivatives when they are approximated by piecewise polynomial functions on this triangle (see [21], [19], [20], [13], [14], [3], [4], [10], [9]).

2. Description of algorithm

Consider the partition of the (x, y)-plane by the lines $x = ih$, $y = jh$, where $i, j = 0, \pm 1, \pm 2, \dots$ and h is an arbitrary positive number. Suppose that continuous functions $F_1(x, y), \dots, F_p(x, y)$ are defined for all x, y, which define the domain

$$\Omega = \{(x, y) : F_i(x, y) < 0 \quad \forall\, i = 1, \dots, p\}.$$

An example of such a domain is shown in Figure 1. In this example, the domain is defined by the inequalities

$$\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} - 1 < 0, \\ 4 - (x-3)^2 - y^2 < 0, \\ 4 - (x+3)^2 - y^2 < 0. \end{cases} \tag{2.1}$$

We construct the initial triangulation as follows. We consider a square with vertices (ih, jh), $((i + 1)h, jh)$, $(ih, (j + 1)h)$, $((i + 1)h, (j + 1)h)$. If all vertices of such a square belong to the domain Ω, then we include in the triangulation the two triangles obtained by dividing the square along one of its diagonals. The choice of the diagonal will be described later. If only three vertices of this square are in the domain Ω, then we include the triangle formed by these vertices into the triangulation. In the remaining cases, such a square is not used for constructing the triangulation. Let $\{P_i = (x_i, y_i)\}_{i=1}^m$ be the set of nodes of the constructed triangulation, and let $\mathcal{T}_h = \{T_k\}_{k=1}^N$ be all its triangles.

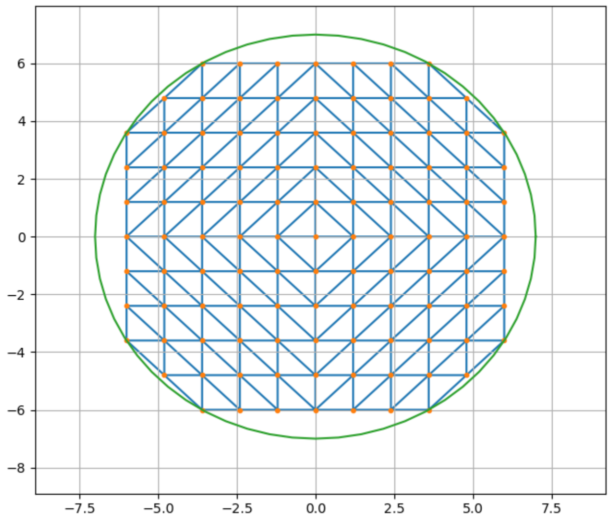

FIGURE 2. Initial triangulation of the circle $x^2 + y^2 - 49 < 0$.

The constructed initial triangulation (see Figure 2) does not, generally speaking, approximate the boundary $\partial\Omega$ of the domain, since the sides of the triangles of the triangulation $\mathcal{T}_h$ are either parallel to the coordinate axes or form an angle of opening $\dfrac{\pi}{4}$ with them. In order for the triangulation to approximate well the boundary $\partial\Omega$, we perform the following transformations. In all triangles T_k that have vertices P_i being boundary nodes of the triangulation $\mathcal{T}_h$, we replace these vertices by the nearest point $P_i' \in \partial\Omega$. Thus, the point $P_i' \in \partial\Omega$ is determined by the identity

$$|P_i - P_i'| = \mathrm{dist}(P_i, \partial\Omega).$$

We obtain a new triangulation $\mathcal{T}^*{}_h$, which, in addition to interior points of the domain Ω, contain nodes lying on $\partial\Omega$. Note that the points P_i' can be approximately computed by the gradient descent method. If the point P_i' lies on the part of the boundary $\partial\Omega$ described by the equation $F_l(x,y) = 0$, $l = 1, \ldots, m$, then we let $P_i^{(0)} = P_i$ and

$$P_i^{(n+1)} = P_i^{(n)} - \delta \cdot F_l(P_i^{(n)}) \nabla F_l(P_i^{(n)}), \qquad n = 0, 1, 2, 3, \ldots. \tag{2.2}$$

where $\delta > 0$ is the gradient descent step. As P_i' we take some approximation computed by formula (2.2).

Let us now comment on the method for choosing the diagonal that divides the squares into two triangles. When constructing the initial triangulation, there can arise a situation, in which all vertices of some triangle are boundary nodes of the triangulation. In this case, when these nodes are shifted to the boundary of the domain, the triangle will be almost degenerate for small values of h. To avoid this, we draw the diagonal in the square as follows. Let Q be the center of the square, and let Q' be the point of the boundary $\partial\Omega$ closest to it. Suppose that the point Q' belongs to the part of the boundary defined by the equation $F_l(x,y) = 0$. Then we take the diagonal of the square that forms the smallest angle with the gradient vector $\nabla F_l(Q)$.

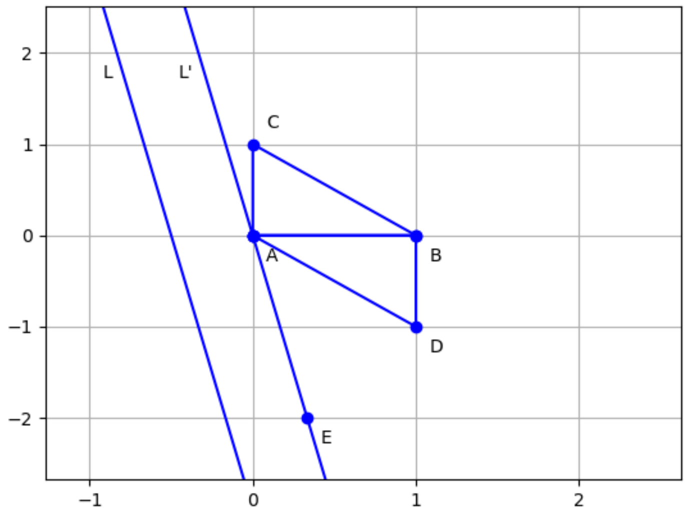

FIGURE 3. The case, when AB is a leg of triangle T_k

3. JUSTIFICATION OF ALGORITHM

Let us show that with the chosen triangulation method, in the limit as $h \to 0$, degenerate triangles do not arise. In order to do this, we provide the proof of the fact that as $h \to 0$, the quantity $\alpha(\mathcal{T}^*{}_h)$ remains separated from zero by some positive number.

We first consider the particular case where the domain Ω is a half–plane, the boundary of which is a straight line L. Note that a node $P_i = (x_i, y_i)$ of the initial triangulation $\mathcal{T}_h$ is interior if and only if the points $(x_i - h, y_i)$, $(x_i + h, y_i)$, $(x_i, y_i - h)$, $(x_i, y_i + h)$ lie in the domain Ω.

Suppose that some triangle T_k of the initial triangulation $\mathcal{T}_h$ has two vertices A and B, which are boundary nodes of the triangulation, and the third vertex C is an interior node. Denote by A' and B' the projections of the points A and B onto the line L. We are going to verify the inequality

$$|A' - B'| \geqslant \frac{1}{\sqrt{2}}|A - B|. \tag{3.1}$$

We first suppose that AB is a leg of the triangle T_k. Then the angle between the segment AB and the line L cannot be greater than $\frac{\pi}{4}$. Indeed, suppose the contrary. Consider the line L' parallel to L and passing through the point A (see Figure 3). If the angle $\angle BAE$, $E \in L'$, is greater than $\frac{\pi}{4}$, then the point D lies in the half–plane Ω. Consequently, the vertex B cannot be a boundary node of the triangulation $\mathcal{T}_h$ and we arrive at a contradiction.

Suppose now that AB is the hypotenuse of the triangle T_k. Then the angle between the segment AB and the line L also cannot be greater than $\frac{\pi}{4}$. Indeed, suppose the contrary. Consider the line L' parallel to L and passing through the point A (see Figure 4). If the angle $\angle BAE$, $E \in L'$, is greater than $\frac{\pi}{4}$, then the point D lies in the half-plane Ω. Consequently, the vertex B cannot be a boundary node of the triangulation $\mathcal{T}_h$ and we again arrive at a contradiction.

Thus, the angle between the segment AB and the line L does not exceed $\frac{\pi}{4}$. This immediately implies inequality (3.1). The subsequent reasoning is the same for both a leg and the hypotenuse. Note that the distance from the third vertex C to the line L satisfies the inequality

$$\operatorname{dist}(C, L) \geqslant \frac{h}{\sqrt{2}}. \tag{3.2}$$

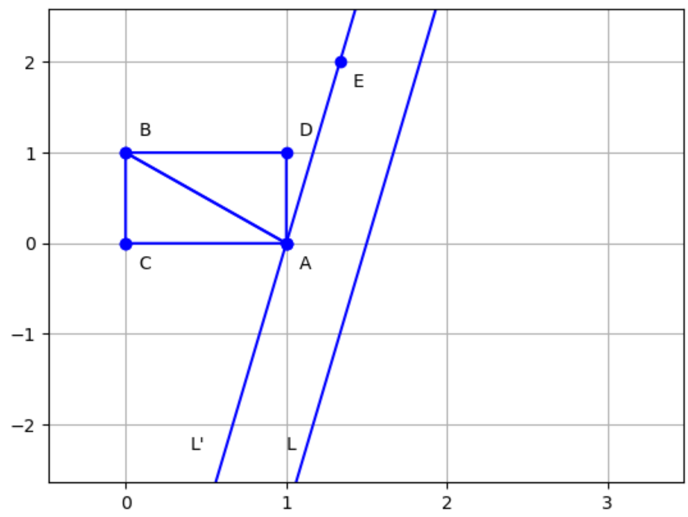

FIGURE 4. The case, when AB is the hypothenuse of triangle T_k.

On the other hand,

$$\mathrm{dist}(C, L) \leqslant |C - A| + |A - A'| \leqslant 2h.$$

Let us estimate from above the distances from C to A' and from C to B'. By the triangle inequality we obtain

$$|C - A'| \leqslant |C - A| + |A - A'| \leqslant h\sqrt{2} + h = (1 + \sqrt{2})h.$$

Then

$$\sin \angle CA'B' = \frac{\mathrm{dist}(C, L)}{|C - A'|} \geqslant \frac{\frac{h}{\sqrt{2}}}{(1 + \sqrt{2})h} = \frac{1}{2 + \sqrt{2}}.$$

Similarly,

$$\sin \angle CB'A' = \frac{\mathrm{dist}(C, L)}{|C - B'|} \geqslant \frac{\frac{h}{\sqrt{2}}}{(1 + \sqrt{2})h} = \frac{1}{2 + \sqrt{2}}.$$

Using the sine theorem, we find the lower bound for the angle $\angle A'CB'$:

$$\sin \angle A'CB' = \frac{|A' - B'|}{|C - A'|} \cdot \sin \angle CB'A' \geqslant \frac{\frac{h}{\sqrt{2}}}{(1 + \sqrt{2})h} \cdot \frac{1}{2 + \sqrt{2}} = \frac{1}{(2 + \sqrt{2})^2}.$$

Consider now the case where the triangle T_k has only one vertex C which is a boundary node of the triangulation $\mathcal{T}_h$, while the vertices A and B are interior. For definiteness, we assume that the vertex C is the right–angle vertex of the triangle T_k. Denote by C' the orthogonal projection of the point C onto the line L. In this case, the triangulation nodes H and D lie in the half–plane Ω (see Figure 5). And since the point C is a boundary node of the triangulation, at least one of the points G or E (or both) are not in Ω. Note that the distance from the point C' to the line AB is at least $\dfrac{h}{\sqrt{2}}$.

Let us estimate the sines of the angles of the triangle $\Delta ABC'$. The altitude of this triangle drawn from the vertex C' is at least $\dfrac{h}{\sqrt{2}}$, and $|C' - A| \leqslant |C' - C| + |C - A| \leqslant 2h$. Consequently, $\sin \angle C'AB \geqslant \frac{1}{2\sqrt{2}}$. Similarly, we obtain $\sin \angle C'BA \geqslant \frac{1}{2\sqrt{2}}$. Applying the sine theorem to the triangle $\Delta ABC'$, we arrive at the inequality

$$\sin \angle AC'B = \frac{|A - B|}{|C' - B|} \cdot \sin \angle C'AB \geqslant \frac{h\sqrt{2}}{2h} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{4} \geqslant \frac{1}{(2 + \sqrt{2})^2}.$$

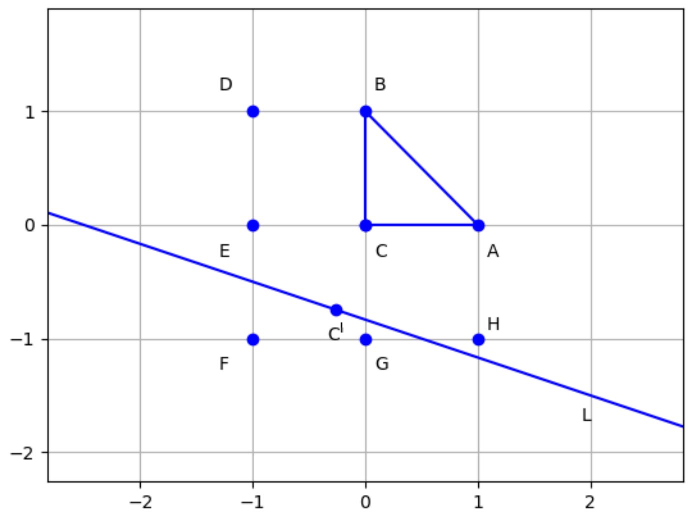

FIGURE 5. The case when C is a boundary node of the initial triangulation.

The case where the vertex C is not at the right angle is treated similarly. The only difference is that $|A - B| = h$ and, therefore,

$$\sin \angle AC'B = \frac{|A - B|}{|C' - B|} \cdot \sin \angle C'AB \geqslant \frac{h}{2h} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{4\sqrt{2}} \geqslant \frac{1}{(2 + \sqrt{2})^2}.$$

Thus, we have proved the following theorem.

Theorem 3.1. *If the domain Ω is a half–plane with the boundary L, then the triangulation $\mathcal{T}^*{}_h$ constructed from the initial triangulation $\mathcal{T}_h$ satisfies the inequality*

$$\alpha(\mathcal{T}^*{}_h) \geqslant \frac{1}{(2 + \sqrt{2})^2}. \tag{3.3}$$

Estimate (3.3) holds for an arbitrary domain with a smooth boundary $\partial\Omega$ only in the limit as $h \to 0$. This follows from the fact that for sufficiently small h, the boundary in the neighborhood of the boundary nodes is, up to second-order smallness, a straight line segment, and in all the above arguments we considered only finitely many cells of the square grid. On this base, we can formulate the following statement.

Theorem 3.2. *If the domain Ω has a smooth boundary $\partial\Omega$, then the triangulation $\mathcal{T}^*{}_h$ constructed from the initial triangulation $\mathcal{T}_h$ satisfies the inequality*

$$\liminf_{h \to 0} \alpha(\mathcal{T}^*{}_h) \geqslant \frac{1}{(2 + \sqrt{2})^2}. \tag{3.4}$$

4. EXAMPLES OF TRIANGULATION CONSTRUCTION

In this section, we present examples of constructing triangulations for various multiply connected domains by using the method described in the preceding sections.

Example 4.1. *Let the domain Ω be defined by the system of inequalities*

$$\begin{cases} x^2 + y^2 - 49 < 0, \\ 1 - |x| - |y| < 0, \\ 1 - (x-3)^2 - (y-3)^2 < 0, \\ 1 - (x+3)^2 - (y-3)^2 < 0, \\ 1 - (x+3)^2 - (y+3)^2 < 0, \\ 1 - (x-3)^2 - (y+3)^2 < 0. \end{cases} \tag{4.1}$$

In Figure 6 we present the triangulation of this domain with the step $h = 0.6$.

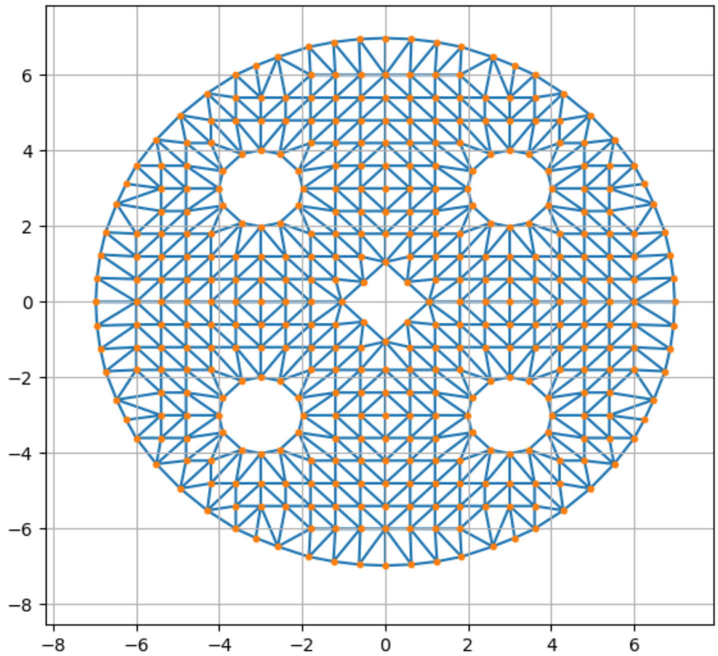

FIGURE 6. Triangulation of a domain defined by inequalities (4.1).

Example 4.2. *We consider the domain defined by the system of inequalities (2.1). To construct a triangulation for this domain, a smaller step $h = 0.24$ was chosen because of the small gap between the outer contour (ellipse) and the inner circles (see Figure 7).*

Each triangulation can be slightly adjusted using the standard method based on making every interior node the centre of the set of vertices of the polygon composed of the triangles surrounding this node. The adjustment itself then looks as follows. We range over all interior nodes P_i and for each of them we determine the vertices $P_{i_1}, \dots, P_{i_s}$ that are adjacent to P_i. After that, the vertex P_i is replaced by

$$P_i' = \frac{1}{s} \sum_{j=1}^{s} P_{i_j}.$$

This procedure (the range over all interior nodes of the triangulation) is repeated several times until the required accuracy is achieved. For example, the triangulation shown in Figure 6, after this transformation, will take the form shown in Figure 8.

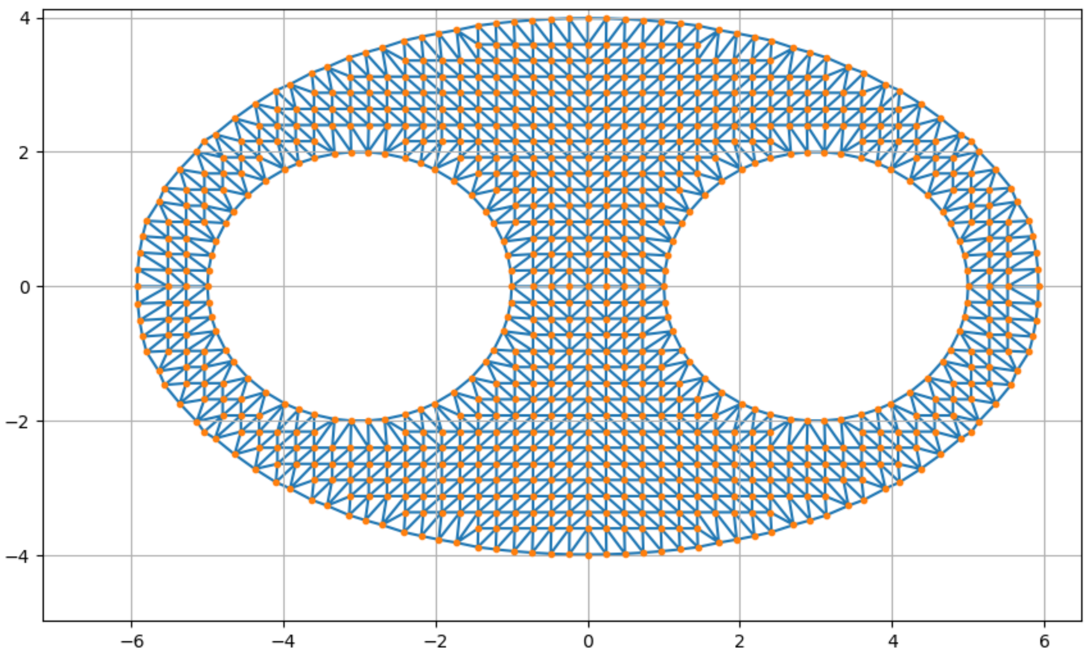

FIGURE 7. Triangulation of a domain defined by inequalities (2.1).

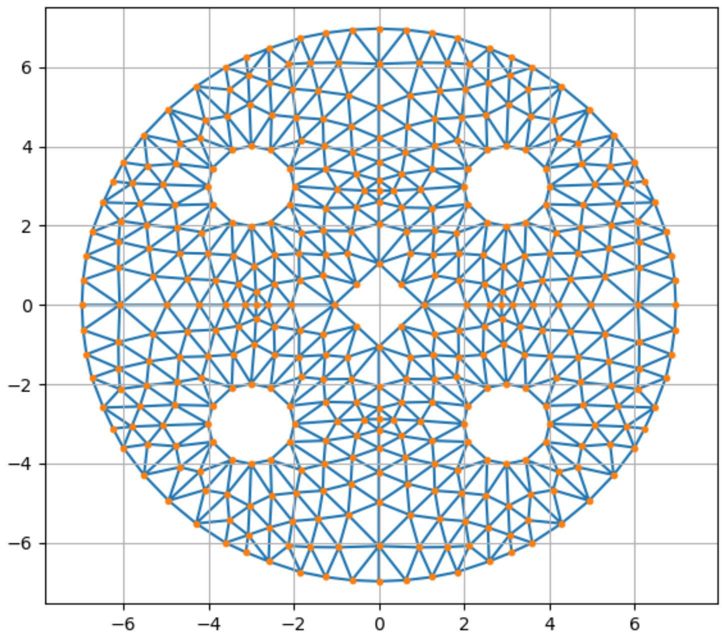

FIGURE 8. Adjusted triangulation of domain (4.1).

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