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OPERATOR ESTIMATES FOR ELLIPTIC PROBLEMS IN DOMAINS WITH FAST LOCALLY PERIODICALLY OSCILLATING BOUNDARY AND ROBIN BOUNDARY CONDITION: HOMOGENIZED DIRICHLET CONDITION

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Abstract. We consider a general elliptic operator in a multidimensional domain with a fast locally periodically oscillating boundary. On this oscillating boundary we impose the Robin condition with a strictly positive coefficient. It is supposed that the amplitude of the boundary oscillations is much larger than the linear size of the periodicity cell. Under such assumptions, we show that the homogenization leads to the Dirichlet condition on the limiting boundary. Our main result are W_2^1– and L_2– operator estimates.

Keywords: elliptic problem, fast oscillating boundary, operator estimate, Robin condition.

Mathematics Subject Classification: 35B27, 35B40

1. Introduction

Problems in domains with rapidly oscillating boundaries are one of the main objects of study in the homogenization theory. Such problems were studied in many works, see, for instance, [1], [11], [13]–[20], [32]–[35]. The classical results provide a classification of possible homogenized problems and establish the convergence of solutions of perturbed problems to ones of homogenized problems for given right hand sides in equations and boundary conditions. There are also many estimates for the convergence rate once the right hand sides are fixed.

The mentioned classical results indicate the presence of weak or strong resolvent convergence. There are also stronger results on the presence of uniform operator convergence, that is, convergence of resolvents in the operator norm. In terms of the original problems, this means convergence of solutions of perturbed problems to the homogenized solutions uniformly in the right hand sides. The corresponding estimates for the convergence rate are called operator estimates. For problems with a rapidly oscillating boundary, such results were obtained in several papers. In [16, Ch. III, Sect. 4] the operator estimate was established for a scalar operator in a two-dimensional domain with the Robin condition on a periodically oscillating boundary, and the amplitude and period of the oscillations were proportional to the same small parameter. The differences of resolvents was estimated as an operator from L_2 into W_2^1. In [36], [37] similar estimates were obtained for the linear system of Stokes equations and the Poisson equation in a multidimensional domain with a Dirichlet boundary condition on a compact locally periodically oscillating boundary. In [21] operator estimates were derived for a self–adjoint second order elliptic operator in a planar strip with one of the classical boundary conditions on a rapidly and periodically oscillating boundary, the period and amplitude of the oscillations were independent small parameters. Possible homogenized problems were described and W_2^1–operator estimates were established. In [6], similar results were obtained for general

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planar domains with an oscillating boundary, which was a graph of an arbitrary bounded function with small values and oscillating in an arbitrary, not necessarily periodic way. At this boundary, the Dirichlet or Neumann condition was imposed; it was also preserved under the homogenization. Operator W_2^1– and L_2–operator estimates were proved. The cases of alternating Dirichlet and Neumann boundary conditions on an oscillating boundary were considered in [2], [5], the alternation could be at one point [2] or a frequent one [5]. The homogenized problem either kept the alternation of the boundary conditions [2], or it was replaced by the Dirichlet condition [5]. The corresponding W_2^1– and L_2–operator estimates were proved, and it turned out that the presence of the alternation changes significantly the convergence rate both in comparison with problems with an oscillating boundary without the alternation, and in comparison with problems with frequent alternations on a smooth boundary. In [3], [4], a general multidimensional problem with an oscillating boundary of an arbitrary structure was considered. The Dirichlet or Neumann condition was set at such a boundary. General, very weak conditions for the geometry of the boundary were obtained, under which averaging was carried out and W_2^1– and L_2–operator estimates were obtained. These conditions were fulfilled for a wide class of essentially non–periodically oscillating boundaries, as well as for many other geometric boundary perturbations. We note that operator estimates are also known for problems with frequently alternating boundary conditions and with perforation both over the entire domain and along a given manifold, see [7]–[10], [22]–[31].

In this paper, we consider an elliptic operator in a multidimensional domain with a fast locally periodically oscillating boundary. On this boundary we impose the Robin condition with a strictly positive coefficient. It is assumed that the amplitude of the oscillations is significantly larger than the linear size of the corresponding periodicity cell. Under these conditions, it is shown that the homogenization leads to the Dirichlet condition at the limiting boundary. The main obtained result is W_2^1– and L_2–operator estimates for the considered problem.

2. Problem and main results

Let $x = (x_1, \ldots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$, $n \geqslant 2$, and Ω be some bounded or unbounded domain in $\mathbb{R}^n$, the boundary of which has the smoothness C^1. By Γ we denote one of connected components of the boundary $\partial\Omega$, and we suppose that it has the smoothness C^3, has no edge, and the domain Ω on one side of the surface Γ.

We suppose that on the surface Γ we can select global variables $s = (s_1, \ldots, s_{n-1})$, $s \in S \subseteq \mathbb{R}^{n-1}$, and all eigenvalues of the Gram matrix associated with these variables are uniformly separated from zero. The set S is supposed to be closed.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward normal to the boundary of Γ, and τ be the distance from the point to the boundary measured in the direction of the normal ν. We suppose that the variables (τ, s) are well–defined in the domain $\Pi_{2\tau_0}$, where $\tau_0 > 0$ is some fixed number, and we denote

$$\Pi_r := \{x \in \Omega :\, \operatorname{dist}(x, \Gamma) < r\}.$$

In addition we suppose that the derivatives of variables x with respect to (τ, s) and variables (τ, s) with respect to x up to the second order are uniformly bounded in the domain Π_r.

We choose an arbitrary periodic lattice in the space $\mathbb{R}^{n-1}$ with a periodicity cell $\square$ and consider the real function $\gamma = \gamma(s, \xi)$ defined on the Cartesian product of $S \times \mathbb{R}^{n-1}$ and $\square$–periodic by ξ for each value of $s \in S$. We assume that for each s the function $\gamma(\xi, s)$ is an element of the Hölder space $C^{1+\alpha}(\mathbb{R}^{n-1})$, and the mapping $s \mapsto \gamma(s, \xi)$, acting from S to $C^{1+\alpha}(\mathbb{R}^{n-1})$, is an element of the space $W_\infty^1(S)$, namely,

$$\operatorname*{ess\,sup}_{s\in S} \|\gamma(s,\,\cdot\,)\|_{C^{1+\alpha}(\overline{\square})} + \operatorname*{ess\,sup}_{s\in S} \|\nabla_s\gamma(s,\,\cdot\,)\|_{C^{1+\alpha}(\overline{\square})} < +\infty. \tag{2.1}$$

We also suppose that

$$0 \leqslant \gamma(s,\xi) \leqslant 1, \quad s \in S, \quad \xi \in \mathbb{R}^{n-1}, \tag{2.2}$$

$$\langle|\nabla_\xi\gamma(\,\cdot\,,s)|^2\rangle \geqslant c_1 > 0, \quad s\in S, \tag{2.3}$$

where c_1 is a fixed constant, and for an arbitrary $\square$–periodic function $h=h(\xi)$ we denote

$$\langle h\rangle := \frac{1}{\operatorname{mes}\square}\int\limits_{\square} h(\xi)\,d\xi.$$

By ε and $\eta(\varepsilon)$ we denote a small positive parameter and a small positive function of this parameter and assume that convergence holds

$$\frac{\varepsilon}{\eta(\varepsilon)}\to 0, \quad \varepsilon\to +0. \tag{2.4}$$

Along the surface Γ we define the fast oscillating surface

$$\Gamma_\varepsilon := \Big\{x\in\mathbb{R}^n:\, \tau=-\eta(\varepsilon)\gamma\big(s,s\varepsilon^{-1}\big)\Big\}.$$

Condition (2.4) means that the amplitude of the oscillations of the surface Γ_ε is significantly larger than the linear size of the periodicity cell. By Ω_ε, we denote the domain with the boundaries of $\partial\Omega\setminus\Gamma$ and Γ_ε. In view of condition (2.2) and the smallness of the function $\eta(\varepsilon)$, it is clear that the domain Ω_ε differs a little from Ω in the sense of the embedding

$$\Omega_\varepsilon\subseteq\Omega, \qquad \Omega\setminus\Omega_\varepsilon\subset\overline{\Pi_\eta},$$

where we have denoted

$$\Pi_r := \big\{x\in\mathbb{R}:\, \operatorname{dist}(x,\Gamma)<r,\ x\in\Omega\big\}, \quad r>0.$$

On the domain Ω we define complex–valued functions $A_{ij}=A_{ij}(x)$, $A_j=A_j(x)$, $A_0=A_0(x)$, $i,j=1,\ldots,n$, which obey the conditions

$$\operatorname{Re}\sum_{i,j=1}^{n} A_{ij}z_i\overline{z}_j \geqslant c_2|z|^2, \qquad A_{ij}\in W^1_\infty(\Omega), \qquad A_j,\, A_0\in L_\infty(\Omega), \tag{2.5}$$

where $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$, and c_2 is some positive constant independent of x and z. Let $a=a(x)$ be some real function defined on Π_{τ_0} and obeying the conditions

$$a\in L_\infty(\Pi_{r_0}), \qquad a(x)\geqslant a_0>0 \quad \text{for almost each} \quad x\in\Pi_{\tau_0}, \tag{2.6}$$

where a_0 is some constant.

Using $\mathring{W}_2^1(\Omega,\Upsilon)$, we denote the Sobolev space of functions having a zero trace on the surface of $\Upsilon\subset\Omega$. In the space of $L_2(\Omega_\varepsilon)$ define the sesquilinear form

$$\begin{aligned}\mathfrak{h}_\varepsilon(u,v) = &\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)} + \frac{1}{2}\sum_{j=1}^{n}\left(A_j\frac{\partial u}{\partial x_j},v\right)_{L_2(\Omega_\varepsilon)}\\ &-\frac{1}{2}\sum_{j=1}^{n}\left(u,\frac{\partial}{\partial x_j}\overline{A_j}v\right)_{L_2(\Omega_\varepsilon)} + (A_0u,v)_{L_2(\Omega_\varepsilon)} + (au,v)_{L_2(\Gamma_\varepsilon)}\end{aligned}$$

on the domain $\mathfrak{D}(\mathfrak{h}_\varepsilon):=\mathring{W}_2^1(\Omega_\varepsilon,\partial\Omega\setminus\Gamma)$. Using the above conditions on the functions A_{ij}, A_j, A_0 and a, it is easy to verify that the introduced form is sectorial and closed. This is why by the first representation theorem [12, Chapter VI, §2.1], there exists a unique m-sectorial operator associated with the form $\mathfrak{h}_\varepsilon$. We denote this operator by $\mathcal{H}_\varepsilon$.

Similarly, in the space $L_2(\Omega)$ we define the sesquilinear form

$$\begin{aligned}\mathfrak{h}_0(u,v) = &\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_i}\right)_{L_2(\Omega)} + \frac{1}{2}\sum_{j=1}^{n}\left(A_j\frac{\partial u}{\partial x_i},v\right)_{L_2(\Omega)}\\ &-\frac{1}{2}\sum_{j=1}^{n}\left(u,\frac{\partial}{\partial x_j}\overline{A_j}v\right)_{L_2(\Omega)} + (A_0u,v)_{L_2(\Omega)}\end{aligned}$$

on the domain $\mathfrak{D}(\mathfrak{h}_0) := \mathring{W}_2^1(\Omega, \partial\Omega)$. This form is also sectorial and closed and by the first representation theorem [12, Ch. VI, Sect. 2.1] there exists a unique m–sectorial operator associated with the form $\mathfrak{h}_0$; we denote this operator by $\mathcal{H}_0$.

We note that both operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ have the same differential expression

$$\hat{\mathcal{H}} := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x_j} + A_0(x).$$

The boundary conditions for the operator $\mathcal{H}_\varepsilon$ read

$$u = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma, \qquad \frac{\partial u}{\partial \boldsymbol{\nu}} + au = 0 \quad \text{on} \quad \Gamma_\varepsilon, \tag{2.7}$$

$$\frac{\partial}{\partial \boldsymbol{\nu}} := -\sum_{i,j=1}^{n} \nu_i A_{ij} \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{n} \nu_j A_j,$$

and the operator $\mathcal{H}_0$ is subject to the Dirichlet condition on $\partial\Omega$. By the standard smoothness improving theorems we easily establish that the domain of operator $\mathcal{H}_\varepsilon$ consists of functions $u \in \mathring{W}_2^2(\Omega_\varepsilon, \partial\Omega \setminus \Gamma)$, which obey boundary conditions (2.7), and the domain of operator $\mathcal{H}_0$ coincides with the space $\mathring{W}_2^2(\Omega, \partial\Omega)$.

By $\|\cdot\|_{X\to Y}$ we denote the norm of a bounded operator acting from a Banach space X into a Banach space Y.

Our main result reads as follows.

Theorem 2.1. *There exists λ_0 independent of ε such that the half–plane $\operatorname{Re}\lambda \leqslant \lambda_0$ is in the resolvent sets of operators $\mathcal{H}_0$ and $\mathcal{H}_\varepsilon$ for all sufficiently small ε. Under condition (2.4), the operator estimate holds*

$$\|(\mathcal{H}_\varepsilon - \lambda)^{-1} - (\mathcal{H}_0 - \lambda)^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega_\varepsilon)} \leqslant C\left(\eta^{\frac{1}{2}}(\varepsilon) + \frac{\varepsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}(\varepsilon)}\right), \tag{2.8}$$

where C is a constant independent of ε and η. If in addition

$$A_j \in W_\infty^1(\Omega), \tag{2.9}$$

then the operator estimate holds

$$\|(\mathcal{H}_\varepsilon - \lambda)^{-1} - (\mathcal{H}_0 - \lambda)^{-1}\|_{L_2(\Omega)\to L_2(\Omega_\varepsilon)} \leqslant C\left(\varepsilon^{\frac{1}{2}} + \eta(\varepsilon) + \frac{\varepsilon}{\eta(\varepsilon)}\right), \tag{2.10}$$

where C is a constant independent of ε and η.

Let us briefly discuss the obtained results. The main nontrivial effect is that under the homogenization, the Robin boundary condition is replaced by the Dirichlet condition. The assumptions providing such an effect are the strict positivity of the coefficient in the Robin condition, see (2.6), and convergence (2.4). Condition (2.3) is also important, but it rather means that the oscillations are non–degenerate, which excludes the independence of the function γ_ε on the fast variable ξ. The initial formulation of the problem also requires specifying the function γ_ε in terms of single local variables on the entire surface of Γ. It is clear that this imposes significant restrictions on the geometric properties of such a surface. Such a condition is fulfilled, for example, for a surface that is a graph of a given function on some hyperplane, or for surfaces having a cylindrical or toroidal shape. The case of sphere–type surfaces involves partitioning the surface by using a suitable atlas, and on each of charts its own locally periodically oscillating function γ_ε is used, and then these functions are glued into a single one by using a suitable partition of unity. However, it unexpectedly turned out that this formulation significantly complicates the problem and makes it comparable in complexity to the case of an arbitrary non–periodically oscillating function. This requires a completely different approach and technique, and we will consider this situation in a next paper.

The main result of this work, Theorem 2.1, states operator W_2^1– and L_2–estimates. First estimate (2.8) describes the convergence rate of the resolvent of the perturbed operator $\mathcal{H}_\varepsilon$ to the resolvent of the homogenized operator $\mathcal{H}_0$ in the norm of operators acting from $L_2(\Omega)$ in $W_2^1(\Omega_\varepsilon)$. In this case, the order of smallness turns out to be independent of the dimension of the domain n. For operator L_2–estimate (2.10), we have to impose a standard condition (2.9) on a higher smoothness of the coefficients at the first derivatives, which previously occurred earlier, see, for example, [3]–[6]. Since the operator norm is weaker here, the convergence rate turns out to be higher. Namely, the convergence rate from the W_2^1–operator evaluation is doubled and the term $\varepsilon^{\frac{1}{2}}$ is additionally added. This term also appeared in the case of the Neumann condition on the oscillating boundary [4], so its appearance here is expected as well.

3. Operator W_2^1–estimate

In this section, we prove our main result, Theorem 2.1. We begin by proving the existence of the number λ_0. Conditions (2.5), (2.6) imply immediately that the estimates hold

$$
\begin{aligned}
&|\operatorname{Im}\mathfrak{h}_\varepsilon(u,u)| \leqslant c_3\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} + c_4\|u\|^2_{L_2(\Omega_\varepsilon)}, && u\in\mathfrak{D}(\mathfrak{h}_\varepsilon),\\
&\frac{c_2}{2}\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} \leqslant \operatorname{Re}\mathfrak{h}_\varepsilon(u,u) + c_5\|u\|^2_{L_2(\Omega_\varepsilon)}, && u\in\mathfrak{D}(\mathfrak{h}_\varepsilon),\\
&|\operatorname{Im}\mathfrak{h}_0(u,u)| \leqslant c_3\|\nabla u\|^2_{L_2(\Omega)} + c_4\|u\|^2_{L_2(\Omega)}, && u\in\mathfrak{D}(\mathfrak{h}_0),\\
&\frac{c_2}{2}\|\nabla u\|^2_{L_2(\Omega)} \leqslant \operatorname{Re}\mathfrak{h}_0(u,u) + c_5\|u\|^2_{L_2(\Omega)}, && u\in\mathfrak{D}(\mathfrak{h}_0).
\end{aligned}
\tag{3.1}
$$

where c_3, c_4, c_5 are some positive constants independent of ε and u. This implies that the numerical ranges of the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are contained in the sector

$$
\left\{z\in\mathbb{C}:\ |\operatorname{Im} z| \leqslant \frac{2c_3}{c_2}\left(\operatorname{Re} z + c_5 + \frac{c_4c_2}{2c_3}\right)\right\},
$$

and therefore the open half-plane $\operatorname{Re}\lambda < -c_5 - \frac{c_4c_2}{2c_3}$ are in the resolvent sets of both operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$. Therefore, we can select λ_0 as follows

$$
\lambda_0 := -c_5 - \frac{c_4c_2}{2c_3} - 1. \tag{3.2}
$$

Let $f\in L_2(\Omega)$ be an arbitrary function $\operatorname{Re}\lambda\leqslant\lambda_0$, and

$$
u_\varepsilon := (\mathcal{H}_\varepsilon-\lambda)^{-1}f, \qquad u_0 := (\mathcal{H}_\varepsilon-\lambda)^{-1}f.
$$

The functions u_ε and u_0 satisfy standard integral identities

$$
\mathfrak{h}_\varepsilon(u_\varepsilon,v) - \lambda(u_\varepsilon,v)_{L_2(\Omega_\varepsilon)} = (f,v)_{L_2(\Omega_\varepsilon)}, \qquad v\in\mathfrak{D}(\mathfrak{h}_\varepsilon), \tag{3.3}
$$

$$
\mathfrak{h}_0(u_0,v) - \lambda(u_0,v)_{L_2(\Omega)} = (f,v)_{L_2(\Omega)}, \qquad v\in\mathfrak{D}(\mathfrak{h}_0). \tag{3.4}
$$

Due to estimates (3.1) and the above choice of λ_0, these identities with $v=u_\varepsilon$ and $v=u_0$ respectively yield the inequalities

$$
\begin{aligned}
&\|u_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} + \|u_\varepsilon\|_{L_2(\Omega_\varepsilon)} + (au_\varepsilon,u_\varepsilon)^{\frac{1}{2}}_{L_2(\Gamma_\varepsilon)} \leqslant c_6\|f\|_{L_2(\Omega)},\\
&\|u_0\|_{W_2^1(\Omega)} + \|u\|_{L_2(\Omega)} \leqslant c_6\|f\|_{L_2(\Omega)},
\end{aligned}
\tag{3.5}
$$

where c_6 is some positive constant independent of ε, u_0, u_ε and f. Moreover, by (2.6) for $\operatorname{Re}\lambda\leqslant\lambda_0$ the estimate holds

$$
\operatorname{Re}\mathfrak{h}_\varepsilon(u,u) - \operatorname{Re}\lambda\|u\|_{L_2(\Omega_\varepsilon)} \geqslant c_7\|u\|^2_{W_2^1(\Omega_\varepsilon)} + a_0\|u\|^2_{L_2(\Gamma_\varepsilon)}, \qquad u\in\mathfrak{D}(\mathfrak{h}_\varepsilon), \tag{3.6}
$$

where c_7 is some positive constant independent of ε and u. By estimate (3.5) and the standard smoothness improving theorems for elliptic boundary value problems we obtain the inequality

$$
\|u_0\|_{W_2^2(\Pi_{r_0})} \leqslant c_8\|f\|_{L_2(\Omega)}, \tag{3.7}
$$

where c_8 is some positive constant independent of ε, u_0, u_ε and f.

Let $\chi = \chi(t)$ be an infinitely differentiable non–increasing cut–off function, which equal to one for $t < -3$ and vanishes for at $t > -2$. We denote

$$\chi_\varepsilon(x) := \begin{cases} \chi\left(\dfrac{\tau}{\eta(\varepsilon)}\right) & \text{in} \quad \overline{\Pi_{3\eta(\varepsilon)}}, \\ \quad 1 & \text{in} \quad \overline{\Omega} \setminus \Pi_{3\eta(\varepsilon)}. \end{cases}$$

It is clear that the function χ_ε is infinitely differentiable in $\Omega \cup \Omega_\varepsilon$, vanishes identically on $\Pi_{2\eta(\varepsilon)}$ and satisfies the estimate and identity

$$|\nabla \chi_\varepsilon(x)| \leqslant C\eta^{-1}(\varepsilon), \qquad \nabla_{s_k}\chi_\varepsilon(x) = 0, \qquad x \in \Pi_{3\eta(\varepsilon)} \setminus \Pi_{2\eta(\varepsilon)}, \tag{3.8}$$

where C is some constant independent of ε, η, x, s. This is why the functions

$$v_\varepsilon := u_\varepsilon - \chi_\varepsilon u_0 \tag{3.9}$$

and $\chi_\varepsilon v_\varepsilon$ belongs to the domains of both forms $\mathfrak{h}_\varepsilon$ and $\mathfrak{h}_0$ and can serve as the test functions in (3.3), (3.4).

We write integral identity (3.3) with $v = v_\varepsilon$ and integral identity (3.4) with $v = v_\varepsilon\chi_\varepsilon$ and take into consideration that by the definition of function χ_ε the identity holds

$$\begin{aligned} &\mathfrak{h}_0(u_0, v_\varepsilon\chi_\varepsilon) = \mathfrak{h}_0(u_0\chi_\varepsilon, v_\varepsilon) + F_\varepsilon, \\ &F_\varepsilon := \sum_{i,j=1}^{n} \left(A_{ij}\frac{\partial u_0}{\partial x_j}, v_\varepsilon \frac{\partial \chi_\varepsilon}{\partial x_i} \right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})} - \sum_{i,j=1}^{n} \left(A_{ij}\frac{\partial \chi_\varepsilon}{\partial x_j} u_0, \frac{\partial v_\varepsilon}{\partial x_i} \right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})} \\ &\qquad - \sum_{j=1}^{n} \left(A_j u_0, v_\varepsilon \frac{\partial \chi_\varepsilon}{\partial x_j} \right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}. \end{aligned} \tag{3.10}$$

We deduct the latter obtained identity from the former one

$$\mathfrak{h}_\varepsilon(v_\varepsilon, v_\varepsilon) - \lambda\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = \big(f, (1-\chi_\varepsilon), v_\varepsilon\big)_{L_2(\Omega_\varepsilon)} + F_\varepsilon. \tag{3.11}$$

The left hand side of this identity satisfies estimate (3.6) with $u = v_\varepsilon$.

Our further aim is to estimate the right hand side of identity (3.11). To do this, we shall need an auxiliary statement.

Lemma 3.1. *The estimate hold*

$$\|u\|_{L_2(\Pi_{3\eta(\varepsilon)}\cap\Omega_\varepsilon)} \leqslant C\eta^{\frac{1}{2}}(\varepsilon)\|u\|_{W_2^1(\Omega_\varepsilon)}, \qquad u \in W_2^1(\Omega_\varepsilon), \tag{3.12}$$

$$\|u\|_{L_2(\Pi_{3\eta(\varepsilon)})} \leqslant C\eta^{\frac{3}{2}}(\varepsilon)\|u\|_{W_2^2(\Omega)}, \qquad u \in \mathring{W}_2^2(\Omega,\Gamma), \tag{3.13}$$

$$\int\limits_\Gamma +|u|^2\big|_{\tau=\eta(\varepsilon)\gamma_\varepsilon(s)}\,ds \leqslant C\|u\|_{W_2^1(\Omega_\varepsilon)}, \qquad u \in W_2^1(\Omega_\varepsilon), \tag{3.14}$$

where C are some constants independent of ε and u.

Proof. The proof of the first two estimates was given in [4, Lm. 2.2]. The third estimate can be easily derived from the obvious identity

$$|u|^2\big|_{\tau=\eta(\varepsilon)\gamma_\varepsilon(s)} = \int\limits_{\tau_0}^{\eta(\varepsilon)\gamma_\varepsilon(s)} \frac{\partial}{\partial\tau}|u|^2\left(1-\chi\left(\frac{3\tau}{\tau_0}\right)\right)d\tau.$$

□

This lemma, relations (3.8), inequality (3.7) and estimates (3.13) with $u=u_0$, (3.12) with $u=v_\varepsilon$ allow us to estimate most of the terms on the right hand side in (3.11):

$$
\begin{aligned}
&\left|\big(f,(1-\chi_\varepsilon),v_\varepsilon\big)_{L_2(\Omega_\varepsilon)}\right| \leqslant C\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Pi_{3\eta(\varepsilon)})} \leqslant C\eta^{\frac{1}{2}}(\varepsilon)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)},\\
&\left|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial\chi_\varepsilon}{\partial x_i}u_0,\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\right| \leqslant C\eta^{-1}(\varepsilon)\|u_0\|_{L_2(\Pi_{3\eta(\varepsilon)})}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\\
&\hphantom{\left|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial\chi_\varepsilon}{\partial x_i}u_0,\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\right|} \leqslant C\eta^{\frac{1}{2}}(\varepsilon)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)},\\
&\left|\sum_{j=1}^{n}\left(A_j u_0, v_\varepsilon\frac{\partial\chi_\varepsilon}{\partial x_j}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\right| \leqslant C\eta^{-1}(\varepsilon)\|u_0\|_{L_2(\Pi_{3\eta(\varepsilon)})}\|v_\varepsilon\|_{L_2(\Pi_{3\eta(\varepsilon)})}\\
&\hphantom{\left|\sum_{j=1}^{n}\left(A_j u_0, v_\varepsilon\frac{\partial\chi_\varepsilon}{\partial x_j}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\right|} \leqslant C\eta(\varepsilon)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.
\end{aligned}
\tag{3.15}
$$

Hereinafter by C we denote various inessential constants independent of ε,f,u_ε,u_0,v_ε and the spatial variables.

We rewrite the remaining term in the right hand side of (3.11) as

$$
\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_0}{\partial x_j}, v_\varepsilon\frac{\partial\chi_\varepsilon}{\partial x_i}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})} = \sum_{i,j=1}^{n}\int\limits_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})} A_{ij}\frac{\partial u_0}{\partial x_j}\overline{v_\varepsilon}\frac{\partial\tau}{\partial x_i}\frac{\partial\chi_\varepsilon}{\partial\tau}\,dx.
$$

By the monotonicity of the function χ the derivative $\frac{\partial\chi_\varepsilon}{\partial\tau}$ is non–positive. In view of this fact and the above identity we obtain the estimate

$$
\left|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_0}{\partial x_j}, v_\varepsilon\frac{\partial\chi_\varepsilon}{\partial x_i}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\right| \leqslant C\int\limits_{\Gamma} ds \int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)} |\nabla u_0||v_\varepsilon|\frac{\partial(1-\chi_\varepsilon)}{\partial\tau}\,d\tau, \tag{3.16}
$$

and the integrand in the integral in the right hand side is non–negative. To continue the estimation, we need to make auxiliary calculations and to prove one lemma.

Let $x=\rho(s)$ be the parametric equation of surface S, then the equation for the oscillating boundary Γ_ε reads

$$
x=\rho_\varepsilon(s):=\rho(s)+\eta(\varepsilon)\gamma\left(s,\frac{s}{\varepsilon}\right)\nu(s),
$$

where $\nu=\nu(s)$ is the introduced above unit normal to Γ. The tangential vectors to the surface Γ_ε read

$$
\frac{\partial\rho_\varepsilon}{\partial s_k}=\frac{\partial\rho}{\partial s_k}-\eta\frac{\partial\gamma_\varepsilon}{\partial s_k}\nu-\eta\gamma\frac{\partial\nu}{\partial s_k},\qquad \frac{\partial\gamma_\varepsilon}{\partial s_k}:=\left(\frac{\partial\gamma}{\partial s_k}+\varepsilon^{-1}\frac{\partial\gamma}{\partial\xi_k}\right).
$$

It is clear that the vectors $\frac{\partial\rho}{\partial s_k}$ and ν, as well as ν and $\frac{\partial\nu}{\partial s_k}$ are orthogonal. In view of this fact and the obvious identity

$$
\frac{\partial\rho}{\partial s_k}\cdot\frac{\partial\nu}{\partial s_j}=-\frac{\partial^2\rho}{\partial^2 s_k}\cdot\nu,
$$

where $\cdot$ is the scalar product in $\mathbb{R}^n$, it is easy to verify that

$$
\frac{\partial\rho_\varepsilon}{\partial s_k}\cdot\frac{\partial\rho_\varepsilon}{\partial s_j}=\frac{\partial\rho}{\partial s_k}\cdot\frac{\partial\rho}{\partial s_j}+2\eta\gamma_\varepsilon\frac{\partial^2\rho}{\partial s_i\partial s_j}\cdot\nu-\eta^2\gamma_\varepsilon\frac{\partial\nu}{\partial s_k}\cdot\frac{\partial\nu}{\partial s_j}+\eta^2\frac{\partial\gamma_\varepsilon}{\partial s_k\partial s_j}\frac{\partial\gamma_\varepsilon}{\partial s_j}.
$$

This and the made assumptions on the surface G imply that the Gram matrix of the surface Γ_ε associated with the variables s and denote by $\Gamma_\varepsilon=\Gamma_\varepsilon(s)$ is represented as

$$
G_\varepsilon(s)=M_0\left(s,\frac{s}{\varepsilon},\varepsilon\right)+\frac{\eta^2}{\varepsilon^2}M_1\left(s,\frac{s}{\varepsilon}\right).
$$

Here

$$
M_0(s,\xi,\varepsilon)=G_0(s)+\eta(\varepsilon)\gamma(s,\xi)M_2(s)+\eta^2\gamma^2(s,\xi)M_3(s),
$$

$$M_1(s,\xi,\varepsilon):=\big(\nabla_\xi\gamma(s,\xi)+\varepsilon\nabla_s\gamma(s,\xi)\big)\big(\nabla_\xi\gamma(s,\xi)+\varepsilon\nabla_s\gamma(s,\xi)\big)^t,$$

where $G_0(s)$ is the Gram matrix of surface Γ, the matrices M_2, M_3 are uniformly bounded and Hermitian, the matrix M_1 is non–negative. These identities and conditions on the Gram matrix G_0 imposed in the formulation of problem, guarantee the estimate

$$c_9+\frac{\eta^2(\varepsilon)}{\varepsilon^2}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)\leqslant G_\varepsilon(s)\leqslant c_9^{-1}+\frac{\eta^2(\varepsilon)}{\varepsilon^2}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)$$

where c_9 is some fixed positive constant independent of s and ε. Therefore, the eigenvalues of the matrix G_ε are estimated from below by the sums of c_9 and the eigenvalues of the non-negative matrix M_1. It immediately implies that

$$\begin{aligned}\big(\det G_\varepsilon(s)\big)^{\frac14}\geqslant&\left(c_9^{n-1}+c_9\frac{\eta^2(\varepsilon)}{\varepsilon^2}\operatorname{Tr}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)\right)^{\frac14}\\ \geqslant& c_{10}\left(1+\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\left(\operatorname{Tr}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)\right)^{\frac14}\right),\end{aligned}\tag{3.17}$$

where c_{10} is a fixed positive constant independent of s and ε, as well as

$$\big(\det G_\varepsilon(s)\big)^{\frac14}\leqslant c_{11}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}},\tag{3.18}$$

where c_{11} is a fixed positive constant independent of s and ε. We note that by condition (2.3) the estimate holds

$$c_{12}\langle|\nabla_\xi\gamma(s,\,\cdot\,)|^2\rangle\leqslant\langle\operatorname{Tr}M_1(s,\,\cdot\,,\varepsilon)\rangle\leqslant c_{13}\langle|\nabla_\xi\gamma(s,\,\cdot\,)|^2\rangle,\tag{3.19}$$

where c_{12}, c_{13} are some fixed positive constants independent of s and ε.

We introduce an auxiliary function

$$\theta(s,\xi,\varepsilon):=\operatorname{Tr}M_1(s,\xi,\varepsilon)-\langle\operatorname{Tr}M_1(s,\,\cdot\,,\varepsilon)\rangle,\qquad\langle\theta(s,\,\cdot\,,\varepsilon)\rangle=0,\tag{3.20}$$

and consider an auxiliary equation

$$\Delta_\xi\Theta(s,\xi,\varepsilon)=\theta(s,\xi,\varepsilon),\qquad\xi\in\mathbb{R}^{n-1}.\tag{3.21}$$

By the second identity in (3.20) this equation is solvable in the class of $\square$–periodic functions with respect to ξ and has a unique solution satisfying the condition $\langle\Theta(s,\,\cdot\,,\varepsilon)\rangle=0$. For this solution, due to the smoothness of the function γ and the standard Schauder estimates the following estimate holds

$$\|\Theta(s,\,\cdot\,,\varepsilon)\|_{C^{2+\alpha}(\overline{\square})}\leqslant c_{12}\|\gamma_\perp(s,\xi)\|_{C^\alpha(\overline{\square})},\tag{3.22}$$

where c_{12} is a constant independent of ε, $s\in S$ and θ. The problem

$$\Delta_\xi\Theta_i(s,\xi,\varepsilon)=\frac{\partial\theta}{\partial s_i}(s,\xi),\qquad\xi\in\mathbb{R}^{n-1},$$

also have a unique $\square$–periodic solution satisfying the condition $\langle\Theta_i(s,\,\cdot\,)\rangle=0$. This solution is an element of the space $C^{2+\alpha}(\square)$ and the estimate holds

$$\|\Theta_i(s,\,\cdot\,,\varepsilon)\|_{C^{2+\alpha}(\overline{\square})}\leqslant c_{12}\left\|\frac{\partial\theta}{\partial s_i}(s,\xi)\right\|_{C^\alpha(\overline{\square})}.\tag{3.23}$$

For small δ we consider the function

$$\tilde\Theta_i(s,\xi,\varepsilon):=\frac{\Theta(s+\delta e_i,\xi,\varepsilon)-\Theta(s+\delta e_i,\xi,\varepsilon)}{\delta}-\Theta_i(s,\xi,\varepsilon),\qquad e_i:=(0,\dots,0,1,0,\dots,0),$$

where the vector e_i has 1 on ith position and it is supposed that $s+\delta e_i$, $s\in S$. Such a function obviously solves the equation

$$\Delta_\xi\tilde{\Theta}_i(s,\xi,\varepsilon)=\frac{\theta(s+\delta e_i,\xi,\varepsilon)-\theta(s+\delta e_i,\xi,\varepsilon)}{\delta}-\frac{\partial\theta}{\partial s_i}(s,\xi,\varepsilon),\qquad \xi\in\mathbb{R}^{n-1},$$

obeys the condition $\langle\tilde{\Theta}_i(s,\,\cdot\,)\rangle=0$ and the estimate

$$\|\tilde{\Theta}_i\|_{C^{2+\alpha}(\overline{\square})}\leqslant c_{12}\left\|\frac{\theta(s+\delta e_i,\,\cdot\,,\varepsilon)-\theta(s+\delta e_i,\,\cdot\,,\varepsilon)}{\delta}-\frac{\partial\theta}{\partial s_i}(s,\,\cdot\,,\varepsilon)\right\|_{C^\alpha(\overline{\square})}.$$

Passing to the limits as $\delta\to 0$ in this estimate, by condition (2.1) we conclude that $\frac{\partial\Theta}{\partial s_i}=\Theta_i$ and in view of (3.22), (3.23)

$$\operatorname*{ess\,sup}_{s\in S}\|\Theta(s,\,\cdot\,,\varepsilon)\|_{C^{2+\alpha}(\overline{\square})}+\operatorname*{ess\,sup}_{s\in S}\|\nabla_s\Theta(s,\,\cdot\,,\varepsilon)\|_{C^{2+\alpha}(\overline{\square})}<C, \tag{3.24}$$

where the constant C is independent of ε.

Problem (3.21) and estimates (3.19) imply the chain of relations

$$\begin{aligned}1=&\frac{1+\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\langle\operatorname{Tr}M_1(s,\,\cdot\,,\varepsilon)\rangle}{1+\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\langle\operatorname{Tr}M_1(s,\,\cdot\,,\varepsilon)\rangle}\leqslant\frac{1+c_{13}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\left(\operatorname{Tr}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)-\Delta_\xi\Theta\left(s,\frac{s}{\varepsilon},\varepsilon\right)\right)}{1+c_{12}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\langle|\nabla_\xi\gamma(s,\,\cdot\,)|^2\rangle}\\
=&\frac{1+c_{13}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\operatorname{Tr}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)-c_{13}\eta^{\frac12}(\varepsilon)\varepsilon^{\frac12}\sum\limits_{j=1}^{n-1}\frac{\partial}{\partial s_i}\frac{\partial\Theta}{\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)+c_{13}\eta^{\frac12}(\varepsilon)\varepsilon^{\frac12}\sum\limits_{j=1}^{n-1}\frac{\partial^2\Theta}{\partial s_i\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)}{1+c_{12}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\langle|\nabla_\xi\gamma(s,\,\cdot\,)|^2\rangle}.\end{aligned}$$

Here $\frac{\partial\Theta^2}{\partial s_i\partial\xi_i}$ stands for the partial derivative in s_i and ξ_i of the function $\Theta(s,\xi)$, at which $\xi=\frac{s}{\varepsilon}$ is substituted, while $\frac{\partial}{\partial s_i}\frac{\partial\Theta}{\partial\xi_i}$ denotes the partial derivative $\frac{\partial\Theta}{\partial\xi_i}(s,\xi)$, at which we substitute $\xi=\frac{s}{\varepsilon}$ and then make the total differentiation in s_i. In view of conditions (2.1), (2.3) and estimate (3.17), we hence obtain

$$\begin{aligned}1\leqslant&\frac{1+c_{14}c_{13}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\left(\operatorname{Tr}M_1\left(s,\frac{s}{\varepsilon},\varepsilon\right)\right)^{\frac14}-c_{13}\eta^{\frac12}(\varepsilon)\varepsilon^{\frac12}\sum\limits_{j=1}^{n-1}\frac{\partial}{\partial s_i}\frac{\partial\Theta}{\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)+c_{13}\eta^{\frac12}(\varepsilon)\varepsilon^{\frac12}\sum\limits_{j=1}^{n-1}\frac{\partial^2\Theta}{\partial s_i\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)}{1+c_{12}\frac{\eta^{\frac12}(\varepsilon)}{\varepsilon^{\frac12}}\langle|\nabla_\xi\gamma(s,\,\cdot\,)|^2\rangle}\\
\leqslant& c_{15}\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\big(\det G_\varepsilon(s)\big)^{\frac14}-c_{15}\varepsilon\sum_{j=1}^{n-1}\frac{\partial}{\partial s_i}\frac{\partial\Theta}{\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)+c_{15}\varepsilon\sum_{j=1}^{n-1}\frac{\partial^2\Theta}{\partial s_i\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right),\end{aligned}$$

where c_{14}, c_{15} are some constants independent of ε, η, s.

We return back to estimate (3.16). In view of the above relations and estimate (3.24) we have

$$\begin{aligned}\int\limits_\Gamma ds\int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)}|\nabla u_0||v_\varepsilon|\frac{\partial(1-\chi_\varepsilon)}{\partial\tau}\,d\tau\leqslant& c_{15}\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\int\limits_\Gamma ds\int\limits_{-3\eta(\varepsilon)}^{-\eta(\varepsilon)\gamma_\varepsilon(s)}|\nabla u_0||v_\varepsilon|\frac{\partial(1-\chi_\varepsilon)}{\partial\tau}\big(\det G_\varepsilon(s)\big)^{\frac14}\,d\tau\\
&+c_{15}\varepsilon\sum_{j=1}^{n-1}\int\limits_\Gamma ds\int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)}\frac{\partial\chi_\varepsilon}{\partial\tau}|\nabla u_0||v_\varepsilon|\frac{\partial}{\partial s_i}\frac{\partial\Theta}{\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)\,d\tau\\
&-c_{15}\varepsilon\sum_{j=1}^{n-1}\int\limits_\Gamma ds\int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)}\frac{\partial\chi_\varepsilon}{\partial\tau}|\nabla u_0||v_\varepsilon|\frac{\partial^2\Theta}{\partial s_i\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)\,d\tau.\end{aligned}$$

In the first term in the right hand side we integrate by parts moving the derivative in τ, while in the second term we move the derivatives in s_i. We then obtain

$$
\begin{aligned}
\int\limits_{\Gamma} ds \int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)} |\nabla u_0||v_\varepsilon| \frac{\partial(1-\chi_\varepsilon)}{\partial\tau}\,d\tau \leqslant & c_{15}\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\int\limits_{\Gamma} |\nabla u_0||v_\varepsilon|\Big|_{\tau=-\eta(\varepsilon)\gamma_\varepsilon(s)} \big(\det G_\varepsilon(s)\big)^{\frac14}\,ds \\
&+ c_{15}\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\int\limits_{\Gamma} ds \int\limits_{-3\eta(\varepsilon)}^{\eta(\varepsilon)\gamma_\varepsilon(s)} (1-\chi_\varepsilon)\big(\det G_\varepsilon(s)\big)^{\frac14}\frac{\partial}{\partial\tau}|\nabla u_0||v_\varepsilon|\,d\tau \\
&- c_{15}\varepsilon\sum_{j=1}^{n-1}\int\limits_{\Gamma} ds \int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)} \frac{\partial\chi_\varepsilon}{\partial\tau}\frac{\partial\Theta}{\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)\frac{\partial}{\partial s_i}|\nabla u_0||v_\varepsilon|\,d\tau \\
&- c_{15}\varepsilon\sum_{j=1}^{n-1}\int\limits_{\Gamma} ds \int\limits_{-3\eta(\varepsilon)}^{-2\eta(\varepsilon)} \frac{\partial\chi_\varepsilon}{\partial\tau}|\nabla u_0||v_\varepsilon|\frac{\partial^2\Theta}{\partial s_i\partial\xi_i}\left(s,\frac{s}{\varepsilon}\right)\,d\tau.
\end{aligned}
$$

By estimates (3.7), (3.8), (3.16), (3.12), (3.14), (3.18) this yields

$$
\begin{aligned}
&\bigg|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial u_0}{\partial x_i}, v_\varepsilon\frac{\partial\chi_\varepsilon}{\partial x_j}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\bigg| \\
&\qquad\leqslant C\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\|v_\varepsilon\|_{L_2(\Gamma_\varepsilon)}\bigg(\int\limits_{\Gamma}\Big(|\nabla u_0|\Big|_{\tau=\eta(\varepsilon)\gamma_\varepsilon(s)}\Big)^2 ds\bigg)^{\frac12} \\
&\qquad\quad + C\frac{\varepsilon}{\eta^{\frac12}(\varepsilon)}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \\
&\qquad\leqslant C\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \\
&\qquad\quad + C\frac{\varepsilon}{\eta^{\frac12}(\varepsilon)}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.
\end{aligned}
\tag{3.25}
$$

It follows from this inequality and (3.15) that the right hand side of identity (3.11) admits the estimate

$$
\begin{aligned}
\big|\big(f,(1-\chi_\varepsilon),v_\varepsilon\big)_{L_2(\Omega_\varepsilon)}+F_\varepsilon\big| \leqslant & C\left(\eta^{\frac12}(\varepsilon)+\frac{\varepsilon}{\eta^{\frac12}(\varepsilon)}\right)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \\
&+ C\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Gamma_\varepsilon)}.
\end{aligned}
$$

Then by the inequality (3.6) with $u=v_\varepsilon$ for the left hand side of identity (3.11) we obtain

$$
\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}+\|v_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \leqslant C\left(\eta^{\frac12}(\varepsilon)+\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\right)\|f\|_{L_2(\Omega)}.
\tag{3.26}
$$

We also note that by (3.12), (3.13), (3.7), (3.8) the estimates hold

$$
\|(1-\chi_\varepsilon)u_0\|_{L_2(\Omega_\varepsilon)} \leqslant C\eta^{\frac32}(\varepsilon)\|f\|_{L_2(\Omega)}, \qquad \|\nabla(1-\chi_\varepsilon)u_0\|_{L_2(\Omega_\varepsilon)} \leqslant C\eta^{\frac12}(\varepsilon)\|f\|_{L_2(\Omega)}.
$$

Together with (3.26) and definition (3.9) of the function v_ε this imply

$$
\|u_\varepsilon-u_0\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\left(\eta^{\frac12}(\varepsilon)+\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\right),\|f\|_{L_2(\Omega)},
$$

and this proves estimate (2.8).

4. Operator L_2–estimate

We shall prove operator estimate in the space L_2 by the duality arguments [38], [39] by means of the techniques from works [4], [5], [9].

By straightforward calculations we verify that the adjoint operator $\mathcal{H}_0^*$ has the differential expression

$$\hat{\mathcal{H}}^* := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \overline{A_{ij}(x)} \frac{\partial}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \overline{A_j(x)} + \overline{A_0(x)}$$

and the Dirichlet condition on $\partial\Omega$. The structure of such operator is the same as of the operator $\mathcal{H}_0$, and this is why it possesses the same properties. Namely, it is m–sectoral, its domain is $\mathring{W}^2(\Omega, \partial\Omega)$, and this operator is associated with the sesquilinear form $\overline{\mathfrak{h}_0(v,u)}$. The half–plane $\operatorname{Re}\lambda \leqslant \lambda_0$ is in the resolvent set of the operator $\mathcal{H}_0^*$ with λ_0 from (3.2).

We continue the function v_ε by zero in $\Omega\backslash\Omega_\varepsilon$ and keep the same notation for the continuation. We denote $w := (\mathcal{H}_0^* - \lambda)^{-1} v_\varepsilon$. The function w satisfies estimate (3.7), which in this case reads

$$\|w\|_{W_2^2(\Omega)} \leqslant c_8 \|v_\varepsilon\|_{L_2(\Omega)} = c_8 \|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}. \tag{4.1}$$

The integral identity for the function w with the test function $\chi_\varepsilon v_\varepsilon$ reads

$$\begin{aligned}\sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial \chi_\varepsilon v_\varepsilon}{\partial x_j}, \frac{\partial w}{\partial x_i} \right)_{L_2(\Omega)} - \frac{1}{2} \sum_{j=1}^{n} \left(\chi_\varepsilon v_\varepsilon, \frac{\partial \overline{A_j} w}{\partial x_j} \right)_{L_2(\Omega)} + \frac{1}{2} \sum_{j=1}^{n} \left(A_j \frac{\partial \chi_\varepsilon v_\varepsilon}{\partial x_j}, w \right)_{L_2(\Omega)} \\ + (A_0 v_\varepsilon, \chi_\varepsilon w)_{L_2(\Omega)} - \lambda (v_\varepsilon, \chi_\varepsilon w)_{L_2(\Omega)} = (\chi_\varepsilon v_\varepsilon, v_\varepsilon)_{L_2(\Omega)}.\end{aligned}$$

In view of the obvious identity $(\chi_\varepsilon v_\varepsilon, aw)_{L_2(\Gamma_\varepsilon)} = 0$ and the definition of the cut–off function χ_ε, by direct calculations we easily rewrite the above integral identity to

$$\mathfrak{h}_\varepsilon(v_\varepsilon, \chi_\varepsilon w) - \lambda(v_\varepsilon, \chi_\varepsilon w)_{L_2(\Omega)} + G_\varepsilon = (\chi_\varepsilon v_\varepsilon, v_\varepsilon)_{L_2(\Omega)}, \tag{4.2}$$

where the function G_ε is given by the formula

$$\begin{aligned}G_\varepsilon := &\sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial \chi_\varepsilon}{\partial x_j} v_\varepsilon, \frac{\partial w}{\partial x_i} \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})} - \sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial v_\varepsilon}{\partial x_j}, \frac{\partial \chi_\varepsilon}{\partial x_i} w \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})} \\ &+ \sum_{j=1}^{n} \left(A_j \frac{\partial \chi_\varepsilon}{\partial x_j} v_\varepsilon, w \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})}.\end{aligned}$$

Since the function $\chi_\varepsilon w$ vanishes identically on $\Pi_{2\eta(\varepsilon)} \supset \Omega \setminus \Omega_\varepsilon$, we have the obvious identities

$$\begin{aligned}\mathfrak{h}_\varepsilon(v_\varepsilon, \chi_\varepsilon w) - \lambda(v_\varepsilon, \chi_\varepsilon w)_{L_2(\Omega)} =& \mathfrak{h}_\varepsilon(u_\varepsilon, \chi_\varepsilon w) - \lambda(u_0, \chi_\varepsilon w)_{L_2(\Omega)} \\ &- \big(\mathfrak{h}_0(\chi_\varepsilon u_0, \chi_\varepsilon w) - \lambda(\chi_\varepsilon u_0, \chi_\varepsilon w)_{L_2(\Omega)} \big),\end{aligned}$$

and similar to the derivation of integral identity (3.10) we obtain

$$\mathfrak{h}_\varepsilon(v_\varepsilon, \chi_\varepsilon w) - \lambda(v_\varepsilon, \chi_\varepsilon w)_{L_2(\Omega)} = ((1-\chi_\varepsilon) f, \chi_\varepsilon w)_{L_2(\Omega)} - \tilde{F}_\varepsilon,$$

where we have denoted

$$\begin{aligned}\tilde{F}_\varepsilon := &\sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial u_0}{\partial x_j}, \chi_\varepsilon w_\varepsilon \frac{\partial \chi_\varepsilon}{\partial x_i} \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})} - \sum_{i,j=1}^{n} \left(A_{ij} \frac{\partial \chi_\varepsilon}{\partial x_j} u_0, \frac{\partial \chi_\varepsilon w}{\partial x_i} \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})} \\ &- \sum_{j=1}^{n} \left(A_j u_0, \chi_\varepsilon w \frac{\partial \chi_\varepsilon}{\partial x_j} \right)_{L_2(\Pi_{3\eta(\varepsilon)} \backslash \Pi_{2\eta(\varepsilon)})}.\end{aligned}$$

By (4.2) this implies the formula

$$(\chi_\varepsilon v_\varepsilon, v_\varepsilon)_{L_2(\Omega)} = G_\varepsilon - \tilde{F}_\varepsilon + ((1-\chi_\varepsilon) f, \chi_\varepsilon w)_{L_2(\Omega)}. \tag{4.3}$$

Let us estimate the right hand side of this formula.

Similar to the derivation of estimates (3.15), on the base of Lemma 3.1 and estimates (3.7), (4.1), (3.8), (3.26) we immediately obtain

$$
\begin{aligned}
&\Bigg|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial v_\varepsilon}{\partial x_j},\frac{\partial \chi_\varepsilon}{\partial x_i}w\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\Bigg| + \Bigg|\sum_{j=1}^{n}\left(A_j\frac{\partial \chi_\varepsilon}{\partial x_j}v_\varepsilon, w\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\Bigg| + |\tilde F_\varepsilon|\\
&+ |((1-\chi_\varepsilon)f,\chi_\varepsilon w)_{L_2(\Omega)}| \leqslant C\eta^{\frac12}(\varepsilon)\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\|w\|_{W_2^2(\Omega)} + C\eta(\varepsilon)\|u_0\|_{W_2^2(\Omega_\varepsilon)}\|w\|_{W_2^2(\Omega)}\\
&\qquad + C\eta^{\frac32}(\varepsilon)\|f\|_{L_2(\Omega)}\|w\|_{W_2^2(\Omega)}\\
&\qquad \leqslant C(\eta(\varepsilon)+\varepsilon^{\frac12})\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)},
\end{aligned}
\tag{4.4}
$$

where the constant C is independent of ε, f, v_ε, u_0, w. The remaining term in the function G_ε can be estimated similar to (3.25) by using (3.26), (4.1)

$$
\begin{aligned}
\Bigg|\sum_{i,j=1}^{n}\left(A_{ij}\frac{\partial \chi_\varepsilon}{\partial x_j}v_\varepsilon,\frac{\partial w}{\partial x_i}\right)_{L_2(\Pi_{3\eta(\varepsilon)}\setminus\Pi_{2\eta(\varepsilon)})}\Bigg| \leqslant & C\frac{\varepsilon^{\frac12}}{\eta^{\frac12}(\varepsilon)}\|w\|_{W_2^2(\Omega)}\|v_\varepsilon\|_{L_2(\Gamma_\varepsilon)}\\
&+ C\frac{\varepsilon}{\eta^{\frac12}(\varepsilon)}\|w\|_{W_2^2(\Omega)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\\
\leqslant & C\left(\frac{\varepsilon}{\eta(\varepsilon)}+\varepsilon^{\frac12}\right)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}.
\end{aligned}
$$

By these relations and (4.4), (4.3) we obtain

$$
\left|(\chi_\varepsilon v_\varepsilon, v_\varepsilon)_{L_2(\Omega)}\right| \leqslant C\left(\frac{\varepsilon}{\eta(\varepsilon)}+\varepsilon^{\frac12}\right)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}. \tag{4.5}
$$

We also note that by (3.12), (3.26) the estimate holds

$$
\left|((1-\chi_\varepsilon)v_\varepsilon, v_\varepsilon)_{L_2(\Omega)}\right| \leqslant C\eta^{\frac12}(\varepsilon)\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\|v_\varepsilon\|_{L_2(\Omega)} \leqslant C(\eta(\varepsilon)+\varepsilon^{\frac12})\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Omega)}.
$$

In view (4.5) we now finally obtain

$$
\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2 \leqslant C\left(\varepsilon^{\frac12}+\eta(\varepsilon)+\frac{\varepsilon}{\eta(\varepsilon)}\right)\|f\|_{L_2(\Omega)}\|v_\varepsilon\|_{L_2(\Omega)},
$$

and this proves estimate (2.10).

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