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PROBLEM ON EXISTENCE OF LIMIT FOR SPECIAL PRODUCT OF SINES

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Abstract. We consider the problem on the asymptotic behavior of "long" products of sines, the arguments of which are generated by a given infinitely large sequence. We briefly discuss possible branches of the general problem. The main attention is focused on the case, when the considered sequence is a certain subsequence of the sequence of factorials of natural numbers. The scheme of proof of main results is based on the number–theoretic properties of sequences composed of factorials multiplied (or divided) by the Euler number. We employ little–known integral representations for the integer and fractional parts of elements of such sequences. We outline the ideas for developing the proposed approach, and promising aspects are indicated. The research is initiated by a recently arisen question in the theory of boundary value problems on the exact computation of the spectral radius for a parametric family of functional operators.

Keywords: product of sines, factorial, Euler's number, integral representation, spectral radius.

Mathematics Subject Classification: 40A05

1. Introduction

In [8] the following question was posed. Let p be a fixed real number, $p > 1$. What is the limit of the sequence $(b_n(p))^{\frac{1}{n}}$, where

$$b_n(p) \equiv \sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin p^k x| \tag{1.1}$$

as $n \to \infty$? The complete question is not known. According to [10] (see also [6]), for almost all (but not all) values of p, this limit is equal to 1. Short note [1] and regular work [2] were devoted to a rigorous justification of the existence of a limit for each p and an elementary proof of the relation

$$\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} \left|\sin 2^{\nu k} x\right| \right)^{\frac{1}{n}} = \cos \frac{\pi}{2\,(2^\nu + 1)}, \qquad \nu \in \mathbb{N}.$$

The study was continued in [3], where the behavior of sequence (1.1) was studied for irrational numbers p of a special kind with the Pisot property. In [1], [2], [3] the history of the issue was presented in detail accompanied by the list of bibliographic sources.

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A more general problem naturally arises. Let $\mathcal{P} = (p_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers tending to $+\infty$. We denote

$$a_n^{(\mathcal{P})} \equiv \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin p_k x|\right)^{\frac{1}{n}}, \qquad n \in \mathbb{N}. \tag{1.2}$$

It is obvious that

$$0 \leqslant \varliminf_{n\to\infty} a_n^{(\mathcal{P})} \leqslant \varlimsup_{n\to\infty} a_n^{(\mathcal{P})} \leqslant 1.$$

We need to find out whether the limit

$$\lim_{n\to\infty} a_n^{(\mathcal{P})} \tag{1.3}$$

exists and to calculate it in the case of existence.

Apparently, a similar problem, in which an infinitely large sequence is replaced by an infinitesimal one, is also of some interest.

On the contrary, the question becomes meaningless if the limit of the convergent sequence of positive integers $\mathcal{P}$ is nonzero. Indeed, let $p_k \to p_0 > 0$ for $k \to \infty$. Assuming for simplicity that $p_k/(2p_0) \notin \mathbb{N}$ for all numbers k, we get a sequence of positive numbers

$$\left|\sin \frac{\pi p_k}{2p_0}\right| \to 1, \qquad k \to \infty.$$

But then, as is well known, the sequence of geometric means satisfies

$$\left(\prod_{k=1}^{n} \left|\sin \frac{\pi p_k}{2p_0}\right|\right)^{\frac{1}{n}} \to 1, \qquad n \to \infty,$$

which guarantees that quantity (1.3) is well–defined and is equal to one.

Let us return back to the main direction and discuss some possibilities inherent in construction (1.2) with an infinitely large sequence $\mathcal{P}$.

In our opinion, an interesting line related to the asymptotic behavior of quantity (1.2) arises when, as a natural choice, one takes an arbitrary arithmetic progression as the generating sequence $\mathcal{P}$. Even in the "simplest" case, such as $p_k = k$, where $k \in \mathbb{N}$, the problem is not entirely trivial[1] and is partly connected with the question of the asymptotics of the global minimum of the Dirichlet kernel (see [4]).

The specific feature of the geometric progression $\mathcal{P} = \left(p^k\right)_{k\in\mathbb{N}}$ in the context of the problem posed above is that for each value $p > 1$ sequence (1.1) is *submultiplicative*, i.e.

$$b_{m+n}(p) \leqslant b_m(p) b_n(p), \qquad m, n \in \mathbb{N},$$

and, by the Fekete lemma, corresponding sequence (1.2) always has a limit (see [1] for details). It is appropriate here to mention an observation that escaped our attention in [3, Thm. 1]. The point is that $(b_n(p))^{\frac{1}{n}} \leqslant 1$ for all $n \in \mathbb{N}$, and the submultiplicativity of sequence (1.1) implies the identity

$$\lim_{n\to\infty} (b_n(p))^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} (b_n(p))^{\frac{1}{n}}. \tag{1.4}$$

This leads us to the following conclusion: if the parameter $p > 1$ is chosen such that the limit in (1.4) turns out to be equal to unity, then $b_n(p) = 1$ for each index $n \in \mathbb{N}$. This will be the case, for example, whenever p is a transcendental number or p is a Pisot number from the class of quadratic irrationalities introduced in [3].

[1] As it turned out during the preparation of this paper for publication, this problem was solved in [9].

In the general case the situation becomes more complicated because the sequence

$$b_n^{(\mathcal{P})} \equiv \left(a_n^{(\mathcal{P})}\right)^n = \sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin p_k x|\,, \qquad n \in \mathbb{N},$$

may lose the property of submultiplicativity. For example, if $\mathcal{P} = (k!)_{k\in\mathbb{N}}$, then

$$b_2^{(\mathcal{P})} \equiv \sup_{x\in\mathbb{R}} |\sin x \sin 2x| = \frac{4\sqrt{3}}{9} = 0.7698\ldots,$$

$$b_4^{(\mathcal{P})} \equiv \sup_{x\in\mathbb{R}} |\sin x \sin 2x \sin 6x \sin 24x| = 0.6906\ldots,$$

and hence,

$$b_4^{(\mathcal{P})} = 0.6906\ldots > 0.5925\ldots = \left(b_2^{(\mathcal{P})}\right)^2.$$

The interest in the last example motivates the main content of the paper, which constitutes a tentative step in passing in construction (1.2) from a geometric progression to sequences of superexponential growth.

The following exact result holds.

Theorem 1.1. *In each of the cases*

$$\mathcal{P} = ((2k)!)_{k\in\mathbb{N}}\,, \qquad \mathcal{P} = ((4k-3)!)_{k\in\mathbb{N}}\,, \qquad \mathcal{P} = ((4k-1)!)_{k\in\mathbb{N}}$$

limit (1.3) is well–defined and is equal to 1.

The second result is of a preliminary nature.

Theorem 1.2. *When the generating sequence is chosen as $\mathcal{P} = (k!)_{k\in\mathbb{N}}$, the lower limit as $n \to \infty$ of quantity* (1.2) *is no less than* $\dfrac{\sqrt{3}}{2}$.

The question on existence of the limit

$$\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin k!\,x| \right)^{\frac{1}{n}}$$

remains open for the time being, although we have no doubt that the limit in question not only exists but is also equal to unity. However, to justify such a claim, it will be necessary to develop further the method proposed in the present note (see the discussion in the final Section 3).

The formulated results will be proved in the next section.

2. Proof of Theorems 1.1 and 1.2

2.1. Proof of Theorem 1.2. We begin the proof of Theorem 1.2 by verifying the validity of the relation

$$\varliminf_{n\to\infty} \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin k!\,x| \right)^{\frac{1}{n}} \geqslant \frac{\sqrt{3}}{2}. \tag{2.1}$$

Let $u \in \mathbb{R}$. We list the notation, which will be used:

- $\lfloor u \rfloor$ is the floor (integer part) of number u,
- $\lceil u \rceil$ is the ceil of number u,
- $\{u\}$ is the fractional part of u,
- $\lfloor u \rceil$ is the integer nearest to u,
- $\|u\|$ is the distance from u to $\lfloor u \rceil$.

We rewrite (2.1) in an equivalent form

$$\varliminf_{n\to\infty}\left(\sup_{t\in\mathbb{R}}\prod_{k=1}^{n}|\sin k!et|\right)^{\frac{1}{n}} \geqslant \frac{\sqrt{3}}{2}, \tag{2.2}$$

and to verify (2.2) we make use the following statement.

Lemma 2.1. *For each $k\in\mathbb{N}$, in the representation*

$$k!e = \lfloor k!e\rfloor + \{k!e\}$$

the terms satisfy

$$\lfloor k!e\rfloor = 3q_k+\delta_k, \qquad q_k\in\mathbb{N}_0\equiv\mathbb{N}\cup\{0\}, \qquad \delta_k\in\{1,2\}, \tag{2.3}$$

$$\{k!e\}\to 0, \qquad k\to\infty. \tag{2.4}$$

Proof. In [7], there were provided the formulas

$$I_k^{(1)} \equiv \int_0^1 s^k e^{1-s}\,ds = \{k!e\}, \tag{2.5}$$

$$I_k^{(2)} \equiv \int_1^{+\infty} s^k e^{1-s}\,ds = \lfloor k!e\rfloor, \tag{2.6}$$

valid for all $k\in\mathbb{N}$. Hence, the representation

$$k!e = e\int_0^{+\infty} s^k e^{-s}\,ds = I_k^{(1)}+I_k^{(2)}$$

is a decomposition of the number $k!e$ into fractional and integer parts. Identity (2.5) not only implies immediately (2.4), but also shows that the sequence of fractional parts $\{k!e\}$ is completely monotone (in particular, decreasing and convex). Furthermore, integrals (2.6) obey a recurrence rule

$$I_{k+1}^{(2)} = (k+1)\,I_k^{(2)}+1, \qquad k\in\mathbb{N}, \qquad I_1^{(2)}=2, \tag{2.7}$$

which can be verified by integrating by parts. Thanks to (2.7), it is easy to write out the first few terms of the sequence (2.6). The beginning is as follows

$$2,\quad 5,\quad 16,\quad 65,\quad 326,\quad 1957,\quad 13700,\quad 109601,\quad 986410,\quad \ldots \tag{2.8}$$

with the periodic sequence

$$2,\quad 2,\quad 1,\quad 2,\quad 2,\quad 1,\quad 2,\quad 2,\quad 1,\quad \ldots$$

of the remainders of $I_k^{(2)}$ modulo 3. Formula (2.3) is a crude recording of the latter property. A rigorous justification is carried out by using (2.7). For instance, the fact that $I_{3m}^{(2)}$ for each $m\in\mathbb{N}$ gives remainder 1 modulo 3 follows from the formula

$$I_{3m}^{(2)} = 3m\,I_{3m-1}^{(2)}+1, \qquad m\in\mathbb{N}.$$

The proof is complete. □

We return to the proof of Theorem 1.2 and verify estimate (2.2). It is obvious that

$$\left(\sup_{t\in\mathbb{R}}\prod_{k=1}^{n}|\sin k!et|\right)^{\frac{1}{n}} \geqslant \left(\prod_{k=1}^{n}\left|\sin k!e\frac{\pi}{3}\right|\right)^{\frac{1}{n}}, \qquad n\in\mathbb{N}. \tag{2.9}$$

By Lemma 2.1 the sequence of positive numbers $\alpha_k \equiv \left|\sin k!e\frac{\pi}{3}\right|$ tends to $\frac{\sqrt{3}}{2}$ as $k \to \infty$ since

$$k!e\,\frac{\pi}{3} = (3q_k + \delta_k)\,\frac{\pi}{3} + \{k!e\}\,\frac{\pi}{3}, \qquad q_k \in \mathbb{N}_0,$$

for all $k \in \mathbb{N}$ that implies

$$\alpha_k = \sin\left(\frac{\pi\delta_k}{3} + \varepsilon_k\right) \to \frac{\sqrt{3}}{2}, \qquad k \to \infty.$$

We have used the fact that $\varepsilon_k \equiv \{k!e\}\,\frac{\pi}{3} \to 0$ as $k \to \infty$, as well as that $\delta_k = 1$ or 2 (see (2.3), (2.4)). Then

$$\left(\prod_{k=1}^{n} \alpha_k\right)^{\frac{1}{n}} \to \frac{\sqrt{3}}{2}$$

as $n \to \infty$. Passing to the lower limit in (2.9), we obtain (2.2). The proof of Theorem 1.2 is complete.

2.2. Proof of Theorem 1.1. Before proving Theorem 1.1, we establish two lemmas.

Lemma 2.2. *For each $k \in \mathbb{N}$, the integer part $\lfloor (2k)!e \rfloor$ is an odd number, and the integer part $\lfloor (4k-3)!e \rfloor$ gives remainder 2 modulo 4.*

Proof. The claimed properties are clearly seen in the first terms of (2.8). By (2.6), (2.7) we obtain that for each $k \in \mathbb{N}$ the number

$$\lfloor (2k)!e \rfloor = I_{2k}^{(2)} = 2k\, I_{2k-1}^{(2)} + 1$$

is odd, while the number

$$\lfloor (4k-3)!e \rfloor = I_{4k-3}^{(2)} = \begin{cases} 2, & k = 1, \\ (4k-4)\big((4k-3)I_{4k-5}^{(2)} + 1\big) + 2, & k = 2, 3, \ldots, \end{cases}$$

has remainder 2 modulo 4. The proof is complete. □

Lemma 2.3. *For each $k \in \mathbb{N}$ the number $\lfloor (4k-1)!/e \rfloor$ gives remainder 2 modulo 4, and moreover $\{(4k-1)!/e\} \to 0$ as $k \to \infty$.*

Proof. It is easy to verify (see [5]) that

$$I_m^{(3)} \equiv \int_0^1 s^m e^{s-1}\, ds = \left\|\frac{m!}{e}\right\| = (-1)^m \left(\left\lceil \frac{m!}{e} \right\rceil - \frac{m!}{e}\right), \qquad m \in \mathbb{N}. \tag{2.10}$$

In view of the definition (the left hand side of (2.10)) it is clear that $I_m^{(3)} \to 0$ as $m \to \infty$. On the other hand, one can apply the recurrence formula

$$I_{m+1}^{(3)} = 1 - (m+1)I_m^{(3)}, \qquad m \in \mathbb{N}, \qquad I_1^{(3)} = \frac{1}{e}, \tag{2.11}$$

derived by integrating by parts. According to (2.10), for each $k \in \mathbb{N}$ the integral

$$0 < I_{4k-1}^{(3)} \equiv \int_0^1 s^{4k-1} e^{s-1}\, ds$$

is calculated as

$$\frac{(4k-1)!}{e} - \left\lceil \frac{(4k-1)!}{e} \right\rceil = \frac{(4k-1)!}{e} - \left\lfloor \frac{(4k-1)!}{e} \right\rfloor = \left\{ \frac{(4k-1)!}{e} \right\} \to 0, \qquad k \to \infty.$$

Moreover,

$$\left\lfloor \frac{(4k-1)!}{e} \right\rfloor = \begin{cases} 2, & k=1, \\ (4k-2)(16k^2-20k+5) + \frac{(4k-1)!}{(4k-5)!} \left\lfloor \frac{(4k-5)!}{e} \right\rfloor, & k=2,\,3,\,\ldots, \end{cases} \tag{2.12}$$

where the last formula arises once we express $I^{(3)}_{4k-1}$ in terms of $I^{(3)}_{4k-5}$ by using (2.11) and then replace in the resulting relation $I^{(3)}_{4k-1}$ by the difference $(4k-1)!/e - \lfloor (4k-1)!/e \rfloor$, and $I^{(3)}_{4k-5}$ is to be replaced by the difference $(4k-5)!/e - \lfloor (4k-5)!/e \rfloor$. As we seen by (2.12), for each $k \in \mathbb{N}$ the number $\lfloor (4k-1)!/e \rfloor$ is congruent to 2 modulo 4 since

$$(4k-2)(16k^2-20k+5) = 4(k-1)(16k^2-12k+3)+2,$$
$$\frac{(4k-1)!}{(4k-5)!} = 4(k-1)(4k-3)(4k-2)(4k-1).$$

The proof is complete. □

We proceed to the proof of Theorem 1.1.

Let first $\mathcal{P} = ((2k)!)_{k\in\mathbb{N}}$. We rewrite the sequence (1.2) as

$$a_n^{(\mathcal{P})} = \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin (2k)!\,x| \right)^{\frac{1}{n}} = \left(\sup_{t\in\mathbb{R}} \prod_{k=1}^{n} |\sin (2k)!\,e t| \right)^{\frac{1}{n}}, \qquad n \in \mathbb{N}.$$

In this case, for each $n \in \mathbb{N}$ we have

$$a_n^{(\mathcal{P})} \geqslant \left(\prod_{k=1}^{n} \left|\sin (2k)!\,e\,\frac{\pi}{2}\right| \right)^{\frac{1}{n}}.$$

The sequence of positive numbers $\beta_k \equiv \left|\sin (2k)!\,e\frac{\pi}{2}\right|$ tends to 1 as $k\to\infty$ since

$$(2k)!\,e\,\frac{\pi}{2} = \lfloor (2k)!\,e \rfloor \,\frac{\pi}{2} + \{(2k)!\,e\}\,\frac{\pi}{2},$$

where $\lfloor (2k)!\,e \rfloor$ is an odd number by Lemma 2.2, and $\{(2k)!\,e\} \to 0$ as $k\to\infty$ by Lemma 2.1 (see property (2.4) there). But then

$$\left(\prod_{k=1}^{n} \beta_k \right)^{\frac{1}{n}} \to 1$$

as $n\to\infty$. Therefore,

$$1 \geqslant \varlimsup_{n\to\infty} a_n^{(\mathcal{P})} \geqslant \varliminf_{n\to\infty} a_n^{(\mathcal{P})} \geqslant \lim_{n\to\infty} \left(\prod_{k=1}^{n} \beta_k \right)^{\frac{1}{n}} = 1,$$

that is, under the choice $\mathcal{P} = ((2k)!)_{k\in\mathbb{N}}$ limit (1.3) is well–defined and is equal to 1.

Now let $\mathcal{P} = ((4k-3)!)_{k\in\mathbb{N}}$. Then the sequence (1.2) becomes

$$a_n^{(\mathcal{P})} = \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin (4k-3)!\,x| \right)^{\frac{1}{n}} = \left(\sup_{t\in\mathbb{R}} \prod_{k=1}^{n} |\sin (4k-3)!\,e t| \right)^{\frac{1}{n}}, \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ the estimate holds

$$a_n^{(\mathcal{P})} \geqslant \left(\prod_{k=1}^{n} \left|\sin (4k-3)!\,e\,\frac{\pi}{4}\right| \right)^{\frac{1}{n}}.$$

The appeared sequence of positive numbers $\gamma_k = \left|\sin(4k-3)!\,e\,\frac{\pi}{4}\right|$ tends to 1 as $k \to \infty$ since

$$(4k-3)!\,e\,\frac{\pi}{4} = \lfloor (4k-3)!\,e \rfloor\,\frac{\pi}{4} + \{(4k-3)!\,e\}\,\frac{\pi}{4},$$

where each integer part $\lfloor (4k-3)!\,e \rfloor$ gives remainder 2 modulo 4 (by Lemma 2.2), and $\{(4k-3)!\,e\} \to 0$ as $k \to \infty$ by Lemma 2.1 (see (2.4)). Therefore,

$$\left(\prod_{k=1}^{n} \gamma_k\right)^{\frac{1}{n}} \to 1$$

as $n \to \infty$. Hence,

$$1 \geqslant \overline{\lim_{n\to\infty}}\, a_n^{(\mathcal{P})} \geqslant \underline{\lim_{n\to\infty}}\, a_n^{(\mathcal{P})} \geqslant \lim_{n\to\infty} \left(\prod_{k=1}^{n} \gamma_k\right)^{\frac{1}{n}} = 1,$$

that is, under the choice $\mathcal{P} = ((4k-3)!)_{k\in\mathbb{N}}$ limit (1.3) is well–defined and is equal to 1.

Finally, let $\mathcal{P} = ((4k-1)!)_{k\in\mathbb{N}}$ and consider the sequence

$$a_n^{(\mathcal{P})} = \left(\sup_{x\in\mathbb{R}} \prod_{k=1}^{n} |\sin(4k-1)!\,x|\right)^{\frac{1}{n}} = \left(\sup_{t\in\mathbb{R}} \prod_{k=1}^{n} \left|\sin\frac{(4k-1)!}{e}\,t\right|\right)^{\frac{1}{n}}, \qquad n \in \mathbb{N},$$

which obviously satisfies the estimate

$$a_n^{(\mathcal{P})} \geqslant \left(\prod_{k=1}^{n} \left|\sin\frac{(4k-1)!}{e}\,\frac{\pi}{4}\right|\right)^{\frac{1}{n}}, \qquad n \in \mathbb{N}.$$

By Lemma 2.3 in the representation

$$\frac{(4k-1)!}{e}\,\frac{\pi}{4} = \left\lfloor \frac{(4k-1)!}{e} \right\rfloor \frac{\pi}{4} + \left\{ \frac{(4k-1)!}{e} \right\} \frac{\pi}{4}$$

the sine of the first term is equal to ± 1, and the second term is an infinitesimal quantity as $k \to \infty$. Hence,

$$0 < \mu_k \equiv \left|\sin\frac{(4k-1)!}{e}\,\frac{\pi}{4}\right| \to 1, \qquad k \to \infty,$$

and therefore,

$$\left(\prod_{k=1}^{n} \mu_k\right)^{\frac{1}{n}} \to 1, \qquad n \to \infty,$$

and again,

$$1 \geqslant \overline{\lim_{n\to\infty}}\, a_n^{(\mathcal{P})} \geqslant \underline{\lim_{n\to\infty}}\, a_n^{(\mathcal{P})} \geqslant \lim_{n\to\infty} \left(\prod_{k=1}^{n} \mu_k\right)^{\frac{1}{n}} = 1.$$

Thus, the choice $\mathcal{P} = ((4k-1)!)_{k\in\mathbb{N}}$ in (1.2) generates the sequence $a_n^{(\mathcal{P})}$, which converges to 1. The proof is complete.

3. Conclusion

Lemma 2.1 can be proved by using the standard expansion

$$k!e = \sum_{j=0}^{\infty} \frac{k!}{j!} = \sum_{j=0}^{k} \frac{k!}{j!} + \sum_{j=k+1}^{\infty} \frac{k!}{j!}, \qquad k \in \mathbb{N},$$

where the integer part $\lfloor k!e \rfloor = \sum_{j=0}^{k} \frac{k!}{j!}$ has remainder 2 modulo 3, and the fractional part $\{k!e\} = \sum_{j=k+1}^{\infty} \frac{k!}{j!}$ tends to zero as $k \to \infty$. Our approach, based on integral representations (2.5), (2.6), has its advantages and may be useful in other problems. In particular (see the proof of Lemma 2.1), by (2.5) we clearly see additional properties of the infinitesimal sequence of fractional parts $\{k!e\}$. In the same way, integral representation (2.10) works clearly in the proof of Lemma 2.3, although, of course, another scheme is applicable, which involves the alternating expansion

$$\frac{(4k-1)!}{e} = \sum_{j=0}^{\infty} (-1)^j \frac{(4k-1)!}{j!} = \sum_{j=2}^{4k-1} (-1)^j \frac{(4k-1)!}{j!} + \sum_{j=4k}^{\infty} (-1)^j \frac{(4k-1)!}{j!}.$$

Here it is necessary to prove that for each $k \in \mathbb{N}$ the right hand side of the formula

$$\left\lfloor \frac{(4k-1)!}{e} \right\rfloor = \sum_{j=2}^{4k-1} (-1)^j \frac{(4k-1)!}{j!}$$

gives remainder 2 modulo 4, not forgetting to verify the relation

$$\left\{ \frac{(4k-1)!}{e} \right\} = \sum_{j=4k}^{\infty} (-1)^j \frac{(4k-1)!}{j!} \to 0, \qquad k \to \infty.$$

The idea of applying the Euler fundamental constant in the problem of the behavior of a product of sines with arguments generated by the sequence of factorials looks striking, but in such a pure form it is essentially exhausted by the proof of the results presented in the paper. Therefore, it seems useful to us to record precisely this, in itself curious closed fragment of the theory. The proposed method admits a natural development: along with the number e and its classical factorial expansion, sums of the form $\sum_{j=0}^{\infty} \frac{c_j}{j!}$ with specially chosen coefficients c_j deserve attention, as well as sums of more general numerical series of the form $\sum_{j=1}^{\infty} \frac{c_j}{p_j}$ related with the generating sequence $\mathcal{P}$.

The authors plan to prepare a separate paper in which they will implement the considerations outlined here in relation to constructions (1.2) with rapidly growing sequences $\mathcal{P} = (p_k)_{k\in\mathbb{N}}$, including the case $p_k = k!$, $k \in \mathbb{N}$, and other typical cases.

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