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STABILITY AND CONTROLLABILITY RESULTS FOR CAPUTO–TYPE FRACTIONAL DIFFERENTIAL HEMIVARIATIONAL INEQUALITIES WITH THREE–PARAMETER PERTURBATIONS

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Abstract. The aim of this paper is to investigate a perturbed differential system on real Banach spaces that consists of a Caputo–type fractional order evolution equation and a time–dependent constrained hemivariational inequality. We begin by proving the stability result for the mild solution set mapping of the differential system, where the mapping in the evolution equation, as well as both the mappings and the constraint set in the hemivariational inequality are perturbed by different parameters. Additionally, we explore an existence result for an optimal control problem governed by the fractional differential hemivariational inequality with three–parameter perturbations.

Keywords: differential hemivariational inequality, Caputo–type fractional operator, stability, optimal control problem.

Mathematics Subject Classification: 49J40, 47J20, 49J53

1. Introduction

Differential variational inequalities (DVIs) are a class of dynamic systems formed by combining ordinary differential equations with variational inequalities. These systems are crucial in various fields, such as dynamic Nash equilibrium problems, spatial price equilibrium control issues, dynamic oligopolistic network competition, frictional contact problems, and dynamic traffic networks, as highlighted in works [5], [8], [16], [28]. In 2008, Pang and Stewart [25] were the first to introduce and systematically study a class of differential variational inequalities in finite–dimensional Euclidean spaces. Over recent years, significant advancements have been made, leading to sharper results in different types of DVIs, see e.g., [18], [34], [37] and references therein.

Fractional calculus, which naturally generalizes classical integer order calculus, offers an accurate representation of certain physical phenomena in viscoelastic materials, including the fractional Kelvin — Voigt constitutive laws and the fractional Maxwell model [12], [14], [26]. Fractional calculus methods for differential variational inequalities have also been proposed and explored through various fractional differential variational inequalities, see e.g., [13], [22], [34]. In addition, Liu, Zeng and Migórski [19] studied a class of differential hemivariational

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inequalities (DHVIs) that combine a differential equation with a time–dependent hemivariational inequality. The theory of hemivariational inequalities is a generalization of the concept of variational inequalities, which was first introduced by Panagiotopoulos [23], [24]. This theory addresses various mechanical problems involving non–convex and non–smooth energy potentials, relying on the Clarke generalized gradient for locally Lipschitz functions. Recently, a lot of works introduced and investigated various fractional versions of DHVIs in the study of solution properties and numerical analysis to these DHVIs, see e.g., Lu, Liu, Jiang and Luo [20], Zeng, Liu, and Migórski [36], Weng, Chen, Li, and Huang [32], Anh [1], Tam and Wu [29] and Chen, Zhang, Huang, and Xiao [6] and the references therein. In particular, Wu and Chen [33] studied the existence of a novel class of DHVIs with Caputo–type fractional differential operators and fuzzy mappings and provided an illustrative application to a class of frictional quasistatic contact problems for viscoelastic materials.

On the other hand, because the data collected by engineers using different devices often contain errors, achieving perfectly accurate data for practical problems is not feasible. The stability results guarantee that the solution will change continuously when the mapping is slightly perturbed, serving as a useful check to verify that the solution of the original problem is similar to the solution of the perturbed problems. As a result, the stability analysis of solutions for DVIs is always essential. Gwinner [11] studied the stability of the solution set mapping for a specific class of linear differential variational inequalities, focusing on upper convergence when both the constraint set and the associated linear mapping are perturbed. Wang, Li, Li, and Huang [30] investigated the upper semicontinuity and continuity of the solution mapping for DVIs of the mixed type, considering the effects of perturbations in both the mapping and the constraint set. Recently, Guo, Li, Xiao, and Migórski [10] extended these ideas to analyze the stability of solution mappings for a class of DVIs in real Banach spaces, building upon the framework established in [30]. Li and Wang [17] also developed the continuity of the mild solution mapping for DVIs of the set–valued type perturbed by parameters in real Banach spaces. Very recently, Yang, Li, Zhang and Alwadani [35] studied and established the stability results of mild solution mappings for a system composed of a nonlinear evolutionary equation and a set–valued mixed variational inequality in real Banach spaces when both the mapping and constraint set in the inequality are perturbed by various parameters. Besides, Cen, Sousa, and Li [4] also developed the stability result of a class of FDVIs, which is composed of a Caputo–type fractional evolution equation and a time–dependent variational inequality perturbed by two different parameters in the mapping and the constraint set.

Inspired by the previously mentioned studies, in this paper investigates the following perturbed differential system, which is characterized by a Caputo–type fractional differential equation coupled with a time–dependent constrained hemivariational inequality.

Problem 1.1. *Given $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$, find functions $w\colon [0,T] \to W$ and $z\colon [0,T] \to Z$ such that*

$$\begin{cases} {}_0^C\mathbf{D}_\tau^q w(\tau) = Aw(\tau) + H(\tau, w(\tau), z(\tau), \lambda) \ \ a.e.\ \tau \in [0,T], \\ z(\tau) \in \mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta) \ \ a.e.\ \tau \in [0,T], \\ w(0) = w_0, \end{cases}$$

where W, Z are real Banach spaces, Λ, $\mathfrak{G}$ and Ξ are three metric spaces, $q \in (0,1]$, $w(\tau) \in W$, and $z(\tau) \in P(\gamma)$, $P\colon \mathfrak{G} \to 2^Z$, $A : D(A) \subset W \to W$ is an infinitesimal generator of C_0–semigroup $\{\mathcal{T}(\tau)\}_{\tau \geqslant 0}$,

$$H : [0,T] \times W \times Z \times \Lambda \to W, \qquad \mathcal{B} : \Xi \times Z \to Z^*, \qquad \Psi : [0,T] \times W \times Z \to Z^*$$

are given mappings, J° is the generalized directional derivative of a locally Lipschitz functional $J : \Xi \times Z \to \mathbb{R}$; $\mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta)$ stands for the solution sets of the following hemivariational *inequality in Z (for short, $(\mathrm{HVI})_w^{\gamma\zeta}$): given a function $w\colon [0,T] \to W$, find $z(\tau) \in P(\gamma)$ for a.e. $\tau \in [0,T]$ such that*

$$\langle \mathcal{B}(\zeta, z(\tau)) - \Psi(\tau, w(\tau), z(\tau)), v - z(\tau)\rangle_{Z^* \times Z} + J^\circ(\zeta, Nz(\tau); N(v - z(\tau))) \geqslant 0,$$

for all $v \in P(\gamma)$ and a.e. $\tau \in [0,T]$, with $N : Z \to Z$ being a prescribed linear operator.

The perturbed differential system described in Problem 1.1 provides a generalized framework for the corresponding differential systems investigated in [4], [11], [10], [17], [30], [35]. In our formulation, the perturbations affect not only the mapping in the evolution equation but also the mappings and the constraint set in the hemivariational inequality, each governed by different parameters. Furthermore, this system incorporates an evolution equation involving the Caputo–type fractional derivative together with a time–dependent constrained hemivariational inequality in real Banach spaces.

If

$$\begin{aligned} &H(\tau, w, z, \lambda) = H(\tau, w, z), \qquad P(\gamma) = P, \qquad \mathcal{B}(\zeta, z) = \mathcal{B}(z), \\ &J(\zeta, \nu) = J(\nu) \quad \text{for all} \quad (\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi, \quad (w, z) \in W \times Z, \quad \nu \in Z, \quad \tau \in [0,T], \end{aligned}$$

then Problem 1.1 perturbed by different parameters reduces to the following differential system.

Problem 1.2. *Find functions $w\colon [0,T] \to W$ and $z\colon [0,T] \to Z$ such that*

$$\begin{cases} {}_0^C\mathbf{D}_\tau^q w(\tau) = Aw(\tau) + H\left(\tau, w(\tau), z(\tau)\right) \quad a.e. \quad \tau \in [0,T], \\ z(\tau) \in P \quad \textit{for a.e.} \quad \tau \in [0,T] \quad \textit{such that} \\ \langle \mathcal{B}(z(\tau)) - \Psi(\tau, w(\tau), z(\tau)), v - z(\tau)\rangle_{Z^* \times Z} + J^\circ(Nz(\tau); N(v - z(\tau))) \geqslant 0, \\ \textit{for all} \quad v \in P \quad \textit{and a.e.} \quad \tau \in [0,T], \\ w(0) = w_0. \end{cases}$$

Remark 1.1. *(i) It is worth noting that Problem 1.2 can be viewed as a particular case of the fractional differential hemivariational inequality introduced by Wu – Chen [33] without the fuzzy mapping and the delay in the fuzzy differential inclusion.*

(ii) The set–valued solution mapping of Problem 1.1 is given by

$$\Lambda \times \mathfrak{G} \times \Xi \ni (\lambda, \gamma, \zeta) \mapsto \mathcal{Q}\left(\lambda, \gamma, \zeta\right) \in 2^{C([0,T];W) \times C([0,T];Z)} \setminus \{\emptyset\}, \tag{1.1}$$

where $\mathcal{Q}\left(\lambda, \gamma, \zeta\right)$ is the set of mild solutions to Problem 1.1 corresponding to the triple (λ, γ, ζ).

This work makes the following specific contributions.

(1) We formulate a new class of perturbed differential system in the setting of real Banach spaces that consists of a Caputo–type fractional order evolution equation and a time-dependent constrained hemivariational inequality (Problem 1.1).

(2) Using appropriate conditions, a stability result in the sense of upper semicontinuity for the mild solution set mapping of the differential system is established, where the mapping in the evolution equation, as well as both the mappings and the constraint set in the hemivariational inequality, are perturbed by different parameters.

(3) An existence result for a general optimal control problem governed by the fractional differential hemivariational inequality formulated in Problem 1.1 is also derived, aiming to minimize the effect of the perturbations.

This paper is organized as follows. In Section 2, we introduce the basic concepts and mathematical framework necessary for the analysis of Problem 1.1. Section 3 presents the assumptions and conditions imposed on the data, as well as the existence of solutions for Problem 1.1. The stability results for the mild solution set mapping of the system are derived in Section 4. Finally, in Section 5, we introduce and investigate the existence of solutions of an optimal control problem governed by Problem 1.1.

2. Preliminaries

In this section, we revisit fundamental concepts and established results that will be essential for examining Problem 1.1 in the upcoming sections.

Let W be a Banach space and W^* be its topological dual space. The duality pairing of W and W^* and the norm on W are denoted by $\langle \cdot, \cdot \rangle_W$ and $\| \cdot \|_W$, respectively. Given $0 < T < \infty$ and a subset $K \subset W$, we let

$$C([0,T];K) := \big\{ w \in C([0,T];W) : w(t) \in K \text{ for a.e. } t \in [0,T] \big\}.$$

If $K = \mathbb{R}$, we shortly write $C(0,T)$. The notation $\mathcal{L}(W,Z)$ stands for the space of all linear continuous operators from a Banach space W to a Banach space Z.

Definition 2.1. [7], [21] *A function $\psi \colon W \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be proper if $\psi \not\equiv +\infty$. The function ψ is upper semicontinuous at $w_0 \in W$ if for each $\{w_n\} \subset W$ such that $w_n \to w_0$ the inequality $\limsup_{n\to\infty} \psi(w_n) \leqslant \psi(w_0)$ holds. The function ψ is lower semicontinuous at $w_0 \in W$ if for each $\{w_n\} \subset W$ such that $w_n \to w_0$ the inequality $\psi(w_0) \leqslant \liminf_{n\to\infty} \psi(w_n)$ holds. The function ψ is upper semicontinuous (respectively, lower semicontinuous) on W if ψ is upper semicontinuous (respectively, lower semicontinuous) at each $w_0 \in W$. The function ψ is convex if*

$$\psi(\lambda w + (1-\lambda)v) \leqslant \lambda \psi(w) + (1-\lambda)\psi(v)$$

for all $w, v \in W$ and $\lambda \in [0,1]$.

Definition 2.2. [7], [21] *A function $\psi : W \to \mathbb{R}$ is called locally Lipschitz if, for each $w \in W$, there exists a neighborhood $\mathcal{N}_w$ of w and a constant $l_w > 0$ such that*

$$|\psi(v_1) - \psi(v_2)| \leqslant l_w \|v_1 - v_2\|_W \ \ \textit{for all} \ \ v_1, v_2 \in \mathcal{N}_w.$$

Given a locally Lipschitz function $\psi \colon W \to \mathbb{R}$, we denote by $\psi^\circ(w;v)$ the Clarke's generalized directional derivative of ψ at the point $w \in W$ in the direction $v \in W$, which is defined by

$$\psi^\circ(w;v) := \limsup_{x\to w,\ t\to 0^+} \frac{\psi(x+tv) - \psi(w)}{t}.$$

The generalized gradient of ψ at $w \in W$ in the sense of Clarke, denoted by $\partial\psi(w)$, is a subset of W^* given by

$$\partial\psi(w) = \big\{ \eta^* \in W^* \mid \psi^\circ(w;v) \geqslant \langle \eta^*, v\rangle_W \textit{ for all } v \in W \big\}.$$

The following proposition presents some fundamental results regarding the generalized directional derivative and the generalized gradient in the sense of Clarke for a locally Lipschitz function.

Proposition 2.1. [21, Prop. 3.23] *Let $\psi \colon W \to \mathbb{R}$ be a locally Lipschitz function. Then*

(i) *For every $w \in W$, the function $W \ni v \mapsto \psi^\circ(w;v) \in \mathbb{R}$ is positively homogeneous, i.e., $\psi^\circ(w;\lambda v) = \lambda\psi^\circ(w;v)$ for all $\lambda \geqslant 0, v \in W$ and subadditive, i.e.,*

$$\psi^\circ(w; v_1 + v_2) \leqslant \psi^\circ(w; v_1) + \psi^\circ(w; v_2)$$

for all $v_1, v_2 \in W$.

(ii) *For every $v \in W$ we have $\psi^\circ(w;v) = \max\{\langle \eta, v\rangle_W \mid \eta \in \partial\psi(w)\}$.*

(iii) *The function $W \times W \ni (w,v) \mapsto \psi^\circ(w;v) \in \mathbb{R}$ is upper semicontinuous.*

(iv) *For every $w \in W$ the generalized gradient $\partial\psi(w) \subset W^*$ is a nonempty, weakly* compact (which is bounded by the Lipschitz constant $l_w > 0$ of ψ near w), convex set of W^*.*

Next, we recall the definition of Caputo fractional derivative operator.

Definition 2.3. [14], [39] *For a function $w \in C^n([0,T], W)$, the Caputo fractional order derivative of w of order $q > 0$, is defined as follows:*

$$ {}_{t_0}^{C}\mathbf{D}_{\tau}^{q} w(\tau) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{\tau} \frac{w^{(n)}(\sigma)}{(\tau-\sigma)^{q+1-n}} d\sigma, $$

where $n = \min\{k \in \mathbb{N} : k > q\}$, and $w^{(n)}(\tau)$ is the nth-order derivative of $w(\tau)$, and $\Gamma(\cdot)$ is the Gamma function defined by

$$ \Gamma(q) = \int_{0}^{+\infty} t^{q-1} e^{-t}\, dt. $$

More specifically, we observe that

(i) *when q is equal to a positive integer n, we define*

$$ {}_{t_0}^{C}\mathbf{D}_{\tau}^{q} w(\tau) = w^{(n)}(\tau); $$

(ii) *when $0 < q < 1$, we have*

$$ {}_{t_0}^{C}\mathbf{D}_{\tau}^{q} w(\tau) = \frac{1}{\Gamma(1-q)} \int_{0}^{\tau} \frac{w'(\sigma)}{(\tau-\sigma)^{q}} d\sigma. $$

The symbol "$\rightharpoonup$" will be employed to denote the weak convergence. We are going to review some properties of set–valued mappings.

Definition 2.4. [2] *Let W and Z be two topological spaces and $Q \colon W \to 2^Z$ be a set–valued mapping. Then Q is said to be*

(a) *convex– (resp., closed–, bounded–, compact–, weak compact–) valued if $Q(w)$ is convex (resp., closed, bounded, compact, weak compact) for each $w \in W$;*

(b) *upper semicontinuous at $w_0 \in W$ if, for each open set U of $Q(w_0)$, there exists a neighborhood V of w_0 such that $U \supset Q(w)$ for all $w \in V$;*

(c) *closed at $w_0 \in W$ if for each $\{w_k\} \subset W$, $w_k \to w_0$ and $\{z_k\} \subset Z$, $z_k \to z_0$ such that $z_k \in Q(w_k)$, we have $z_0 \in Q(w_0)$.*

If Q satisfies a certain property at every point of a set $K \subset W$, then Q is said to satisfy that property on K. If $K = W$, we omit the phrase "on W" in the statement.

Lemma 2.1. [3] *Let Z be a Banach space and K be a nonempty subset of another Banach space. Assume that $Q \colon K \to 2^Z$ is a set–valued mapping with nonempty, weakly compact and convex values. Then Q is weakly upper semicontinuous if and only if for each sequence $\{w_k\} \subset K$, which converges to $w_0 \in K$, and for each sequence $\{z_k\} \subset Q(w_k)$, there exists $z_0 \in Q(w_0)$ such that $z_k \rightharpoonup z_0$ up to a subsequence.*

3. Assumptions and existence results

This section presents the assumptions and conditions necessary for the analysis in the subsequent sections. Moreover, the existence of solutions for Problem 1.1 is also established.

Inspired by [33, Def. 2.5], we proceed to define the solutions to Problem 1.1 corresponding to the triple (λ, γ, ζ) in the mild sense.

Definition 3.1. *Given $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$, a pair of functions $(w, z) \in C([0, T]; W) \times C([0, T]; Z)$ is said to be a mild solution of Problem 1.1 if*

$$w(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^{\tau} (\tau - \sigma)^{q-1}\mathcal{S}_q(\tau - \sigma)H(\sigma, w(\sigma), z(\sigma), \lambda)d\sigma, \tag{3.1}$$

for all $\tau \in [0, T]$, where $z(\tau)$ solves the problem $(\mathrm{HVI})_w^{\gamma\zeta}$ for a.e. $\tau \in [0, T]$.

Here $\mathcal{R}_q(\tau)$ and $\mathcal{S}_q(\tau)$ are given by

$$\mathcal{R}_q(\tau) = \int_0^{\infty} \varphi_q(\sigma)\mathcal{T}(\tau^q\sigma)\, d\sigma \text{ and } \mathcal{S}_q(\tau) = q\int_0^{\infty} \sigma\varphi_q(\sigma)\mathcal{T}(\tau^q\sigma)\, d\sigma$$

with

$$\begin{aligned} \varphi_q(\sigma) &= \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q\left(\sigma^{-\frac{1}{q}}\right), \\ \varpi_q(\sigma) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(nq\pi), \sigma \in (0, \infty). \end{aligned}$$

Remark 3.1. *It is well–known that $\{\mathcal{R}_q(\tau)\}_{\tau\geqslant 0}$ and $\{\mathcal{S}_q(\tau)\}_{\tau\geqslant 0}$ are families of bounded linear operators on W, where φ_q is a probability density function defined on $(0, \infty)$, that is,*

$$\varphi_q(\sigma) \geqslant 0, \quad \sigma \in (0, \infty) \quad \textit{and} \quad \int_0^{\infty} \varphi_q(\sigma)\, d\sigma = 1.$$

These operator families were introduced to establish the definition of mild solutions for a class of nonlocal Cauchy problems involving Caputo–type fractional order derivatives. For further details, we refer the reader to Zhou – Jiao [38].

Remark 3.2. *The mild solution given in (3.1) takes the form of a fractional variation of constants (Duhamel–type) formula for the Caputo–type fractional equation in Problem 1.1. This representation consists of*

- *the homogeneous part $\mathcal{R}_q(\tau)w_0$, corresponding to the case $H \equiv 0$;*
- *the integral term associated with the perturbation induced by the nonlinear function H.*

In particular, if $H \equiv 0$, then $w(\tau) = \mathcal{R}_q(\tau)w_0$ is the solution of the homogeneous linear fractional Cauchy problem

$$\begin{cases} {}_0^C\mathbf{D}_{\tau}^q w(\tau) = Aw(\tau), & a.e.\ \tau \in [0, T], \\ w(0) = w_0. \end{cases}$$

In this case, $\{\mathcal{R}_q(\tau)\}_{\tau\geqslant 0}$ is the fractional resolvent (solution operator) family generated by A.

We now recall some auxiliary lemmas concerning the operators $\mathcal{R}_q(\,\cdot\,)$ and $\mathcal{S}_q(\,\cdot\,)$, which will play a key role in our compactness arguments.

Lemma 3.1. [38, Lms. 3.2-3.4] *The operators $\mathcal{R}_q(\tau)$ and $\mathcal{S}_q(\tau)$ possess the following properties:*

(i) *For each fixed* $\tau \geqslant 0$, *operator* $\mathcal{R}_q(\tau)$ *and* $\mathcal{S}_q(\tau)$ *are linear and bounded and satisfy*

$$\|\mathcal{R}_q(\tau)w\|_W \leqslant b_{\mathcal{T}}\|w\|_W \text{ and } \|\mathcal{S}_q(\tau)w\|_W \leqslant \frac{qb_{\mathcal{T}}}{\Gamma(1+q)}\|w\|_W \quad \text{for all} \quad w \in W, \quad \tau \geqslant 0$$

where $b_{\mathcal{T}} = \sup_{\tau\in[0,+\infty)} \|\mathcal{T}(\tau)\| < \infty$.

(ii) $\{\mathcal{R}_q(\tau)\}_{\tau\geqslant 0}$ *and* $\{\mathcal{S}_q(\tau)\}_{\tau\geqslant 0}$ *are strongly continuous.*

(iii) *For each* $\tau \geqslant 0$, *if* $\mathcal{T}(\tau)$ *is compact, then* $\mathcal{R}_q(\tau)$ *and* $\mathcal{S}_q(\tau)$ *are also compact.*

Lemma 3.2. [31, Lm. 3.5] *Let* A *be an infinitesimal generator of a compact* C_0*-semigroup* $\{\mathcal{T}(\tau)\}_{\tau\geqslant 0}$. *Then, for* $1 > q > \frac{1}{p}$ *with* $p > 1$, *the operator* $\Sigma : \mathcal{L}^p([0,T],W) \to C([0,T],W)$, *given by*

$$(\Sigma h)(\tau) = \int_0^\tau (\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h(\sigma)d\sigma,$$

is compact.

We proceed to assumptions and conditions on the data of Problem 1.1.

a$(\mathcal{T})$: $\{\mathcal{T}(\tau)\}_{\tau\geqslant 0}$ is compact.

a(P): The set–valued map $P : \mathfrak{G} \to 2^Z$ has nonempty, closed and convex values, for each $\gamma \in \mathfrak{G}$, and for each $\gamma_n \in \mathfrak{G}$ such that $\gamma_n \to \gamma$ in $\mathfrak{G}$, one has $P(\gamma_n) \xrightarrow{M} P(\gamma)$, i.e.,

(i) For any $z_n \in P(\gamma_n)$ with $z_n \to z$ in Z, up to a subsequence, $z \in P(\gamma)$;

(ii) For any $z \in P(\gamma)$, there is $z_n \in P(\gamma_n)$ with $z_n \to z$ in Z.

a(H): The mapping $H : [0,T] \times W \times Z \times \Lambda \to W$ satisfies

(i) $H(\tau, w, E, \lambda)$ is convex in Z for all $(\tau, w, \lambda) \in [0,T] \times W \times \Lambda$, where $E \subset P(\mathfrak{G})$ is convex.

(ii) There exist $b_H \in \mathcal{L}^p([0,T];\mathbb{R}_+)$ such that
$$\|H(\tau,w,z,\lambda)\|_W \leqslant b_H(\tau)(1 + \|w\|_W + \|z\|_Z)$$
for all $(\tau, w, z, \lambda) \in [0,T] \times W \times Z \times \Lambda$.

(iii) For all $(w, z, \lambda) \in W \times Z \times \Lambda$ the mapping $H(\,\cdot\,, w, z, \lambda) : [0,T] \to W$ is measurable.

(iv) For each $\tau \in [0,T]$, $(w_n, z_n, \lambda_n) \to (w, z, \lambda)$ in $W \times Z \times \Lambda$ yields
$$H(\tau, w_n, z_n, \lambda_n) \to H(\tau, w, z, \lambda) \quad \text{in} \quad W.$$

a$(\mathcal{B})$: Given mapping $\mathcal{B} : \Xi \times Z \to Z^*$ satisfies

(i) $\mathcal{B}$ is continuous and for all $\zeta \in \Xi$, $\mathcal{B}(\zeta, \cdot)$ is strongly monotone, i.e., there exists a constant $c_{\mathcal{B}} > 0$ such that
$$\langle \mathcal{B}(\zeta, z_1) - \mathcal{B}(\zeta, z_2), z_1 - z_2\rangle_Z \geqslant c_{\mathcal{B}}\|z_1 - z_2\|_Z^2 \quad \text{for all} \quad z_1, z_2 \in Z.$$

(ii) For all $z, u \in P(\mathfrak{G})$ and $\zeta \in \Xi$, one has
$$\liminf_{\lambda\to 0_+}\langle \mathcal{B}(\zeta, z + \lambda(u-z)), u - z\rangle_Z \leqslant \langle \mathcal{B}(\zeta, z), u - z\rangle_Z;$$

(iii) There exists a function $\phi_1 : \mathbb{R}_+ \to \mathbb{R}$ and $\phi_1(z) \to +\infty$ as $z \to \infty$ such that
$$\langle \mathcal{B}(\zeta, z), z - u\rangle_Z \geqslant \phi_1(\|z\|_Z)\|z\|_Z \quad \text{for all} \quad u \in P(\mathfrak{G}).$$

a(J): The functional $J : \Xi \times Z \to \mathbb{R}$ is such that

(i) $J(\zeta, \cdot)$ is locally Lipschitz for all $\zeta \in \Xi$.

(ii) There exists a constant $c_J > 0$ such that for each $\zeta \in \Xi$,

$$\langle \eta_1 - \eta_2, z_1 - z_2 \rangle_Z \geqslant -c_J \|z_1 - z_2\|_Z^2 ,$$

for all $z_1, z_2 \in Z, \eta_1 \in \partial_2 J(\zeta, z_1), \eta_2 \in \partial_2 J(\zeta, z_2)$.

(iii) For all $z, v \in Z$, the function $\zeta \mapsto J^\circ(\zeta, z; v)$ is upper semicontinuous.

a(N): The linear operator $N : Z \to Z$ is bounded and compact.

a(Ψ): The function $\Psi : [0, T] \times W \times Z \to Z^*$ is such that

(i) For all $w \in W$ and $z \in Z, \Psi(\cdot, w, z)$ is continuous.

(ii) For all $\tau \in [0, T]$, $\Psi(\tau, \cdot, \cdot)$ is Lipschitz continuous, i.e., there exist constants $l_1, l_2 > 0$ such that

$$\|\Psi(\tau, w_1, z_1) - \Psi(\tau, w_2, z_2)\|_{Z^*} \leqslant l_1 \|w_1 - w_2\| + l_2 \|z_1 - z_2\| .$$

(iii) For all $u \in P(\mathfrak{G})$, one has

$$\liminf_{\lambda \to 0_+} \langle \Psi(\tau, w, z + \lambda(u - z)), z - u \rangle_Z \leqslant \langle \Psi(\tau, w, z), z - u \rangle_Z .$$

(iv) There exists a bounded function $\rho : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ and a function $\phi_2 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle \Psi(\tau, w, z), u - z \rangle_Z + \inf_{\eta \in \partial_2 J(\zeta, Nz)} \langle \eta, N(u - z) \rangle_Z \geqslant \phi_2(\|z\|_Z)\|z\|_Z + (1 + \|z\|_Z)\rho(\tau, \|w\|_W),$$

for all $u \in P(\mathfrak{G})$.

a$(\mathit{const.})$: The inequalities hold

$$c_{\mathcal{B}} \geqslant l_2, \qquad \frac{q b_{\mathcal{T}}}{\Gamma(q+1)} \left(\frac{p-1}{pq-1} \right)^{\frac{p-1}{p}} T^{q - \frac{1}{p}} \left(T^{\frac{1}{p}} + \|b_H\|_{\mathcal{L}^p([0,T], \mathbb{R})} \right) < 1.$$

To derive several analytical properties of Problem 1.1 in the simplified setting of Problem 1.2, we also consider the *non–parametric* counterpart of the above assumptions obtained by removing the dependence on the parameter spaces $\mathfrak{G}$, Ξ and Λ. Accordingly, we impose the following non–parametric assumptions of Problem 1.2.

$\widetilde{\mathbf{a}}(P)$: P is a nonempty, closed and convex subset of Z.

$\widetilde{\mathbf{a}}(H)$: The mapping $H : [0, T] \times W \times Z \to W$ satisfies

(i) $H(\tau, w, E)$ is convex in Z for all $(\tau, w) \in [0, T] \times W$, where $E \subset P$ is convex.

(ii) There exist $b_H \in \mathcal{L}^p([0, T]; \mathbb{R}_+)$ such that

$$\|H(\tau, w, z)\|_W \leqslant b_H(\tau)(1 + \|w\|_W + \|z\|_Z)$$

for all $(\tau, w, z) \in [0, T] \times W \times Z$.

(iii) For all $(w, z) \in W \times Z$, $H(\cdot, w, z) : [0, T] \to W$ is measurable.

(iv) For each $\tau \in [0, T]$, $(w_n, z_n) \to (w, z)$ in $W \times Z$ yields

$$H(\tau, w_n, z_n) \to H(\tau, w, z) \quad \text{in} \quad W.$$

$\widetilde{\mathbf{a}}(\mathcal{B})$: Given mapping $\mathcal{B}: Z \to Z^*$ satisfies

(i) $\mathcal{B}$ is continuous and strongly monotone, i.e., there exists a constant $c_{\mathcal{B}} > 0$ such that

$$\langle \mathcal{B}(z_1) - \mathcal{B}(z_2), z_1 - z_2 \rangle_Z \geqslant c_{\mathcal{B}} \|z_1 - z_2\|_Z^2 \quad \text{for all} \quad z_1, z_2 \in Z.$$

(ii) For all $z, u \in P$, one has

$$\liminf_{\lambda \to 0_+} \langle \mathcal{B}(z + \lambda(u - z)), u - z \rangle_Z \leqslant \langle \mathcal{B}(z), u - z \rangle_Z.$$

(iii) There exists a function $\phi_1 : \mathbb{R}_+ \to \mathbb{R}$ and $\phi_1(z) \to +\infty$ as $z \to \infty$ such that

$$\langle \mathcal{B}(z), z - u \rangle_Z \geqslant \phi_1(\|z\|_Z)\|z\|_Z, \quad \text{for all} \quad u \in P.$$

$\widetilde{\mathbf{a}}(J)$: The functional $J : Z \to \mathbb{R}$ is such that

(i) J is locally Lipschitz for all $z \in Z$;

(ii) There exists a constant $c_J > 0$ such that

$$\langle \eta_1 - \eta_2, z_1 - z_2 \rangle_Z \geqslant -c_J \|z_1 - z_2\|_Z^2$$

for all $z_1, z_2 \in Z, \eta_1 \in \partial J(z_1), \eta_2 \in \partial J(z_2)$

$\widetilde{\mathbf{a}}(\Psi)$: The function $\Psi : [0, T] \times W \times Z \to Z^*$ is such that

(i) For all $w \in W$ and $z \in Z, \Psi(\,\cdot\,, w, z)$ is continuous.

(ii) For all $\tau \in [0, T]$, $\Psi(\tau, \,\cdot\,, \,\cdot\,)$ is Lipschitz continuous, i.e., there exist constants $l_1, l_2 > 0$ such that

$$\|\Psi(\tau, w_1, z_1) - \Psi(\tau, w_2, z_2)\|_{Z^*} \leqslant l_1 \|w_1 - w_2\| + l_2 \|z_1 - z_2\|.$$

(iii) For all $u \in P$, one has

$$\liminf_{\lambda \to 0_+} \langle \Psi(\tau, w, z + \lambda(u - z)), z - u \rangle_Z \leqslant \langle \Psi(\tau, w, z), z - u \rangle_Z.$$

(iv) There exists a bounded function $\rho : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ and a function $\phi_2 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle \Psi(\tau, w, z), u - z \rangle_Z + \inf_{\eta \in \partial J(Nz)} \langle \eta, N(u - z) \rangle_Z \\ \geqslant \phi_2(\|z\|_Z)\|z\|_Z + (1 + \|z\|_Z)\rho(\tau, \|w\|_W)$$

for all $u \in P$.

Remark 3.3. *(i) It is worth noting that, once the parameters $(\gamma, \zeta, \lambda) \in \mathfrak{G} \times \Xi \times \Lambda$ are fixed, the parametric assumptions $\mathbf{a}(P)$, $\mathbf{a}(H)$, $\mathbf{a}(\mathcal{B})$, $\mathbf{a}(J)$, and $\mathbf{a}(\Psi)$ directly imply their non–parametric counterparts $\widetilde{\mathbf{a}}(P)$, $\widetilde{\mathbf{a}}(H)$, $\widetilde{\mathbf{a}}(\mathcal{B})$, $\widetilde{\mathbf{a}}(J)$, and $\widetilde{\mathbf{a}}(\Psi)$, respectively.*

(ii) The collection of hypotheses $\mathbf{a}(\mathcal{T})$, $\widetilde{\mathbf{a}}(P)$, $\widetilde{\mathbf{a}}(H)$, $\widetilde{\mathbf{a}}(\mathcal{B})$, $\widetilde{\mathbf{a}}(J)$, $\mathbf{a}(N)$, $\widetilde{\mathbf{a}}(\Psi)$, and $\mathbf{a}(\text{const.})$ corresponds to the framework investigated by Wu and Chen [33] for a simplified version of Problem 1.2 (specifically, in the absence of fuzzy mappings and delay terms in the differential inclusion).

By Remark 3.3, using [33, Lm. 3.1] and arguing as in the proof of [33, Lm. 3.2], we obtain the following result for the problem $(\text{HVI})_w^{\gamma\zeta}$.

Lemma 3.3. *Given $(\gamma, \zeta) \in \mathfrak{G} \times \Xi$ and $w \in W$, suppose that the conditions $\mathbf{a}(P)$, $\mathbf{a}(\mathcal{B})$, $\mathbf{a}(J)$, $\mathbf{a}(\Psi)$, $\mathbf{a}(N)$ and $\mathbf{a}(const.)$ are satisfied. Furthermore, assume the conditions*

$\mathbf{a}$(*mon.*): *For each $w \in W$ and $\tau \in [0,T]$, the mapping*

$$z \mapsto \mathcal{B}(\zeta, z) - \Psi(\tau, w, z) + N^*\partial_2 J(\zeta, Nz)$$

is monotone.

Then $\mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta)$ is nonempty and compact in Z.

Remark 3.4. *The condition* $\mathbf{a}$(mon.) *in Lemma 3.3 corresponds to the following non-parametric condition, which was considered by Wu – Chen [33].*

$\widetilde{\mathbf{a}}$(mon.): *For each $w \in W$ and $\tau \in [0,T]$, the mapping*

$$z \ \mapsto \ \mathcal{B}(z) - \Psi(\tau, w, z) + N^*\partial J(Nz)$$

is monotone.

By Remark 3.3(ii) and Remark 3.4, the following result follows directly from [33, Thm. 3.1] for Problem 1.2 in the absence of the fuzzy mapping and the delay term in the differential inclusion.

Corollary 3.1. *Under Assumptions $\mathbf{a}(\mathcal{T})$, $\widetilde{\mathbf{a}}(P)$, $\widetilde{\mathbf{a}}(H)$, $\widetilde{\mathbf{a}}(\mathcal{B})$, $\widetilde{\mathbf{a}}(J)$, $\mathbf{a}(N)$, $\widetilde{\mathbf{a}}(\Psi)$, $\mathbf{a}$(const.) and $\widetilde{\mathbf{a}}$(mon.), Problem 1.2 has at least one mild solution.*

By Remark 1.1(i), if the parameters λ, γ, and ζ are fixed, then Problem 1.1 reduces to the corresponding Problem 1.2. Hence, by Remark 3.3, Remark 3.4, and Corollary 3.1, we immediately obtain an existence result for mild solutions of Problem 1.1.

Theorem 3.1. *Given $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$, let Assumptions $\mathbf{a}(\mathcal{T})$, $\mathbf{a}(H)$, $\mathbf{a}(P)$, $\mathbf{a}(\mathcal{B})$, $\mathbf{a}(\Psi)$, $\mathbf{a}(J)$, $\mathbf{a}(N)$, $\mathbf{a}$(const.) and $\mathbf{a}$(mon.) be satisfied. Then Problem 1.1 has at least one mild solution.*

Remark 3.5. (i) *The result of Theorem 3.1 means that the solution set $\mathcal{Q}(\lambda, \gamma, \zeta) \neq \emptyset$ for all $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$.*

(ii) *Also, by [33, Thm. 3.1] we obtain that the set of mild trajectory functions of Problem 1.1 is compact $C([0,T]; W)$. Therefore, combining the compactness of the solution set $\mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta)$ of the problem $(\mathrm{HVI})_w^{\gamma\zeta}$ (by Lemma 3.3), we are able to conclude that the set $\mathcal{Q}(\lambda, \gamma, \zeta)$ is compact in $C([0,T]; W) \times C([0,T]; Z)$ for all $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$.*

To illustrate that all the assumptions in Theorem 3.1 can be satisfied, we provide the following numerical example.

Example. Let $W = Z = \mathbb{R}$ and $\mathfrak{G} = \Xi = \Lambda = [0,1]$, and choose $p = 2$, $q = 0.6$, $T = 0.4$, and $w_0 = 0.015$. Let

$$P\colon \mathfrak{G} \to 2^Z, \qquad A : W \to W, \qquad H : [0,T] \times W \times Z \times \Lambda \to W, \qquad \mathcal{B} : \Xi \times Z \to Z^*,$$
$$\Psi : [0,T] \times W \times Z \to Z^*, \qquad J : \Xi \times Z \to \mathbb{R}, \qquad N : Z \to Z$$

be defined by

$$P(\gamma) := [\gamma, 1+\gamma], \qquad Aw := -2w, \qquad H(\tau, w, z, \lambda) := \tfrac{1}{100}\tau z + \tfrac{1}{50}\sin w + \tfrac{1}{50}e^{-\lambda},$$
$$\mathcal{B}(\zeta, z) := (2 + \cos(\zeta^2))z, \qquad \Psi(\tau, w, z) := \tfrac{1}{2}\tau z - \sin w - 1, \qquad J(\zeta, y) := \tfrac{1}{2}y^2, \qquad N := \mathrm{Id}_{\mathbb{R}}.$$

We now verify each assumption of Theorem 3.1 explicitly.

$\mathbf{a}(\mathcal{T})$**:** Since the state space $W = \mathbb{R}$ is finite-dimensional, every bounded linear operator on W is compact. The linear operator $A : W \to W$ defined by $Aw = -2w$ generates the C_0–semigroup given by

$$\mathcal{T}(\tau)w = e^{-2\tau}w \qquad (\text{for } \tau \geqslant 0).$$

Consequently, $\mathcal{T}(\tau)$ is a compact operator for every $\tau \geqslant 0$. Furthermore, we have

$$b_{\mathcal{T}} := \sup_{\tau\geqslant 0} \|\mathcal{T}(\tau)\| = \sup_{\tau\geqslant 0} e^{-2\tau} = 1.$$

The assumption $\mathbf{a}(\mathcal{T})$ is satisfied.

$\mathbf{a}(P)$**:** For each $\gamma \in \mathfrak{G} = [0,1]$, the set $P(\gamma) = [\gamma, 1+\gamma]$ is clearly nonempty, closed, and convex. To verify the Mosco convergence, let $\gamma_n \to \gamma$ in $\mathfrak{G}$.

(i) Suppose $z_n \in P(\gamma_n)$ such that $z_n \to z$. Since $\gamma_n \leqslant z_n \leqslant 1+\gamma_n$ for all n, taking the limit yields $\gamma \leqslant z \leqslant 1+\gamma$, implying $z \in P(\gamma)$.

(ii) For any $z \in P(\gamma)$, we construct the sequence $z_n := z + (\gamma_n - \gamma)$. It is easy to check that $\gamma_n \leqslant z_n \leqslant 1+\gamma_n$, so $z_n \in P(\gamma_n)$. Moreover, clearly $z_n \to z$.

Thus, $P(\gamma_n) \xrightarrow{M} P(\gamma)$, and assumption $\mathbf{a}(P)$ holds.

$\mathbf{a}(H)$**:** Let $H(\tau, w, z, \lambda) = \frac{1}{100}\tau z + \frac{1}{50}\sin w + \frac{1}{50}e^{-\lambda}$.

(i) For fixed $(\tau, w, \lambda) \in [0,T] \times W \times \Lambda$, the map $z \mapsto H(\tau, w, z, \lambda)$ is affine; hence it is convex on Z (and therefore on any convex subset $E \subset P(\mathfrak{G})$).

(ii) Using $|\sin w| \leqslant 1$ and $0 < e^{-\lambda} \leqslant 1$ for $\lambda \in [0,1]$, we get

$$|H(\tau, w, z, \lambda)| \leqslant \tfrac{1}{100}\tau|z| + \tfrac{1}{25} \leqslant \Big(\tfrac{1}{25} + \tfrac{1}{100}\tau\Big)\big(1 + |w| + |z|\big).$$

Thus $\mathbf{a}(H)(ii)$ holds with

$$b_H(\tau) := \tfrac{1}{25} + \tfrac{1}{100}\tau \in \mathcal{L}^2([0,T]; \mathbb{R}_+).$$

(iii) For each $(w, z, \lambda) \in W \times Z \times \Lambda$, the function $\tau \mapsto H(\tau, w, z, \lambda)$ is continuous on $[0,T]$, hence measurable.

(iv) Since H is a continuous function on $[0,T] \times W \times Z \times \Lambda$, for each $\tau \in [0,T]$, the convergence $(w_n, z_n, \lambda_n) \to (w, z, \lambda)$ implies $H(\tau, w_n, z_n, \lambda_n) \to H(\tau, w, z, \lambda)$.

$\mathbf{a}(\mathcal{B})$**:** Let $\mathcal{B}(\zeta, z) = (2 + \cos(\zeta^2))z$. Since $\cos(\zeta^2) \geqslant \cos 1$ for $\zeta \in [0,1]$, set $c_{\mathcal{B}} := 2 + \cos 1 > 0$. Then for all $z_1, z_2 \in \mathbb{R}$,

$$\langle \mathcal{B}(\zeta, z_1) - \mathcal{B}(\zeta, z_2), z_1 - z_2 \rangle = (2 + \cos(\zeta^2))|z_1 - z_2|^2 \geqslant c_{\mathcal{B}}|z_1 - z_2|^2,$$

so $\mathbf{a}(\mathcal{B})(i)$ holds, and (ii) follows from continuity. Moreover, one has

$$P(\mathfrak{G}) = \bigcup_{\gamma\in[0,1]} [\gamma, 1+\gamma] = [0,2].$$

Hence, for $u \in P(\mathfrak{G})$, we have

$$\langle \mathcal{B}(\zeta, z), z - u \rangle = (2 + \cos(\zeta^2))z(z-u) \geqslant c_{\mathcal{B}}|z|^2 - 2c_{\mathcal{B}}|z|,$$

which verifies $\mathbf{a}(\mathcal{B})(iii)$ with $\phi_1(r) := c_{\mathcal{B}}r - 2c_{\mathcal{B}}$.

$\mathbf{a}(J)$**:** Let $J(\zeta, y) = \frac{1}{2}y^2$.

(i) For each ζ, $J(\zeta, \cdot\,)$ is C^1, hence locally Lipschitz.

(ii) Since $\partial_2 J(\zeta, y) = \{y\}$, for $z_i \in Z = \mathbb{R}$ and $\eta_i \in \partial_2 J(\zeta, z_i)$, we have $\eta_i = z_i$, and thus

$$\langle \eta_1 - \eta_2, z_1 - z_2 \rangle = (z_1 - z_2)^2 \geqslant -c_J|z_1 - z_2|^2$$

holds with $c_J = 0$.

(iii) The generalized directional derivative is $J^\circ(\zeta, z; v) = zv$, which is independent of ζ; hence $\zeta \mapsto J^\circ(\zeta, z; v)$ is upper semicontinuous.

a(N)**:** $N = \mathrm{Id}_{\mathbb{R}}$ is bounded and compact.

a(Ψ)**:** Let $\Psi(\tau, w, z) = \frac{1}{2}\tau z - \sin w - 1$.

(i) For fixed $(w, z) \in \mathbb{R}^2$, $\tau \mapsto \Psi(\tau, w, z)$ is affine, hence continuous on $[0, T]$.

(ii) For all $(w_i, z_i) \in \mathbb{R}^2$ and $\tau \in [0, T]$,
$$\begin{aligned}|\Psi(\tau, w_1, z_1) - \Psi(\tau, w_2, z_2)| &\leqslant \tfrac{1}{2}\tau|z_1 - z_2| + |\sin w_1 - \sin w_2| \\ &\leqslant 0.2|z_1 - z_2| + |w_1 - w_2|,\end{aligned}$$
so **a**$(\Psi)(ii)$ holds with $l_1 = 1$ and $l_2 = 0.2$.

(iii) Continuity of $z \mapsto \Psi(\tau, w, z)$ yields the stated lim inf inequality.

(iv) Since $N = \mathrm{Id}_{\mathbb{R}}$ and $\partial_2 J(\zeta, Nz) = \{z\}$, for $u \in P(\mathfrak{G}) = [0, 2]$ we have
$$\begin{aligned}\langle \Psi(\tau, w, z), u - z\rangle + \inf_{\eta \in \partial_2 J(\zeta, Nz)} \langle \eta, N(u - z)\rangle &= \big(\tfrac{1}{2}\tau z - \sin w - 1\big)(u - z) + z(u - z) \\ &= \big((1 + \tfrac{\tau}{2})z - \sin w - 1\big)(u - z).\end{aligned}$$
We estimate the term by establishing a lower bound. Note that
$$\begin{aligned}\Big|\big((1 + \tfrac{\tau}{2})z - \sin w - 1\big)(u - z)\Big| &\leqslant (1.2|z| + |\sin w| + 1)(|u| + |z|) \\ &\leqslant (1.2|z| + 2)(2 + |z|) \\ &= 1.2|z|^2 + 4.4|z| + 4.\end{aligned}$$
Thus,
$$\big((1 + \tfrac{\tau}{2})z - \sin w - 1\big)(u - z) \geqslant -(1.2|z|^2 + 4.4|z| + 4).$$
Choose
$$\phi_2(r) := -1.2\, r - 0.4, \qquad \rho(\tau, r) := -4.$$
Then
$$\begin{aligned}\phi_2(|z|)\,|z| + (1 + |z|)\rho(\tau, |w|) =&(-1.2|z| - 0.4)|z| - 4(1 + |z|) \\ =& -(1.2|z|^2 + 4.4|z| + 4).\end{aligned}$$
This implies
$$\langle \Psi(\tau, w, z), u - z\rangle + \inf_{\eta \in \partial_2 J(\zeta, Nz)} \langle \eta, N(u - z)\rangle \geqslant \phi_2(|z|)|z| + (1 + |z|)\rho(\tau, |w|),$$
so **a**$(\Psi)(iv)$ holds.

a(const.)**:** First, $c_{\mathcal{B}} = 2 + \cos 1 > 0.2 = l_2$, hence $c_{\mathcal{B}} \geqslant l_2$.
Second, with $b_{\mathcal{T}} = 1$, $p = 2$, $q = 0.6$, $T = 0.4$, we compute the $\mathcal{L}^2$-norm of $b_H(\tau) = \frac{1}{25} + \frac{1}{100}\tau$:
$$\|b_H\|_{\mathcal{L}^2([0,T])} = \left(\int_0^{0.4} \left(\tfrac{1}{25} + \tfrac{1}{100}\tau\right)^2 d\tau\right)^{1/2} = \sqrt{\frac{331}{468750}}.$$
Therefore, we have
$$\begin{aligned}&\frac{q b_{\mathcal{T}}}{\Gamma(q + 1)} \left(\frac{p - 1}{pq - 1}\right)^{\frac{p-1}{p}} T^{q - \frac{1}{p}} \left(T^{\frac{1}{p}} + \|b_H\|_{\mathcal{L}^p}\right) \\ &\qquad = \frac{0.6}{\Gamma(1.6)} \left(\frac{1}{1.2 - 1}\right)^{\frac{1}{2}} (0.4)^{0.1} \left(\sqrt{0.4} + \sqrt{\frac{331}{468750}}\right) \approx 0.9029 < 1.\end{aligned}$$
Hence condition **a**(const.) is strictly satisfied.

a(mon.)**:** For fixed $(\tau, w, \zeta) \in [0,T] \times W \times \Xi$, define the operator $\mathcal{F}\colon Z \to Z^*$ by

$$\begin{aligned}\mathcal{F}(z) &:= \mathcal{B}(\zeta, z) - \Psi(\tau, w, z) + N^*\partial_2 J(\zeta, Nz)\\ &= (2+\cos(\zeta^2))z - \left(\tfrac{1}{2}\tau z - \sin w - 1\right) + z\\ &= \left(3+\cos(\zeta^2) - \tfrac{\tau}{2}\right)z + (\sin w + 1).\end{aligned}$$

The slope coefficient satisfies

$$3+\cos(\zeta^2) - \tfrac{\tau}{2} \geqslant 3 + \cos 1 - 0.2 > 0.$$

Hence the operator is strongly monotone, verifying **a**(mon.).

Therefore, all assumptions of Theorem 3.1 are satisfied, and Problem 1.1 possesses a mild solution on $[0,T]$.

In particular, given $(\gamma, \zeta) \in \mathfrak{G} \times \Xi$ and $w : [0,T] \to W$, for a.e. $\tau \in [0,T]$, the problem $(\mathrm{HVI})_w^{\gamma\zeta}$ reads: find $z(\tau) \in [\gamma, 1+\gamma]$ such that

$$\left(\left(3+\cos(\zeta^2) - \tfrac{\tau}{2}\right)z(\tau) + \sin(w(\tau)) + 1\right)(v - z(\tau)) \geqslant 0, \quad \text{for all} \quad v \in [\gamma, 1+\gamma].$$

Observe that for all $(\gamma, \zeta, w, \tau) \in \mathfrak{G} \times \Xi \times W \times [0,T]$,

$$\left(3+\cos(\zeta^2) - \tfrac{\tau}{2}\right)z + \sin w + 1 \geqslant (2.8 + \cos 1)z \geqslant 0 \quad \text{for all } z \in [\gamma, 1+\gamma] \subset [0,2].$$

Thus, the unique solution of the problem $(\mathrm{HVI})_w^{\gamma\zeta}$ is

$$z(\tau) = \gamma, \qquad \text{for all} \quad \tau \in [0,T],$$

and consequently, $\mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta) = \{\gamma\}$, which is nonempty and compact.

We also present a numerical simulation of Problem 1.1 governed by the following

$$\begin{cases} {}_0^C\mathbf{D}_\tau^q w(\tau) = -2w(\tau) + \frac{1}{100}\tau z(\tau) + \frac{1}{50}\sin w(\tau) + \frac{1}{50}e^{-\lambda} \quad \text{a.e.} \quad \tau \in [0,T],\\ z(\tau) \in [\gamma, 1+\gamma] \quad \text{for a.e.} \quad \tau \in [0,T] \quad \text{such that}\\ \left(\left(3+\cos(\zeta^2) - \frac{\tau}{2}\right)z(\tau) + \sin(w(\tau)) + 1\right)(v - z(\tau)) \geqslant 0\\ \text{for all} \quad v \in [\gamma, 1+\gamma] \quad \text{a.e.} \quad \tau \in [0,T],\\ w(0) = w_0 = 0.015, \end{cases} \tag{3.2}$$

To visualize the results, we fix the parameters $\lambda = \zeta = 0.35$. The resulting trajectories for the mild solution w and the control z of system (3.2) are shown on Fig. 1 and Fig. 2 corresponding to the distinct values $\gamma \in \{0.01, 0.25, 0.75, 1\}$. These numerical results were obtained using the `fde12.m` solver in MATLAB, based on Garrappa's modification of the standard Predictor-Corrector method [9], with a discretization step size of $h = 2^{-7}$.

4. Stability analysis

Recall the mild solution set mapping $\mathcal{Q}$ of Problem 1.1 given in (1.1). This section establishes the closedness and upper semicontinuity of the mild solution set mapping $\mathcal{Q}$ for Problem 1.1, where the mapping in the evolution equation, along with both the mappings and the constraint set in the hemivariational inequality are perturbed by various parameters.

For each $w \in C([0,T]; W)$ and $\lambda \in \Lambda$, we let

$$\mho_z^{\gamma\zeta}(\tau, w(\tau), \lambda) = \{H(\tau, w(\tau), z(\tau), \lambda) : z(\tau) \in \mathbf{S}_w(P_\gamma, \mathcal{B}_\zeta, \Psi, J_\zeta), \text{ for a.e. } \tau \in [0,T]\}. \tag{4.1}$$

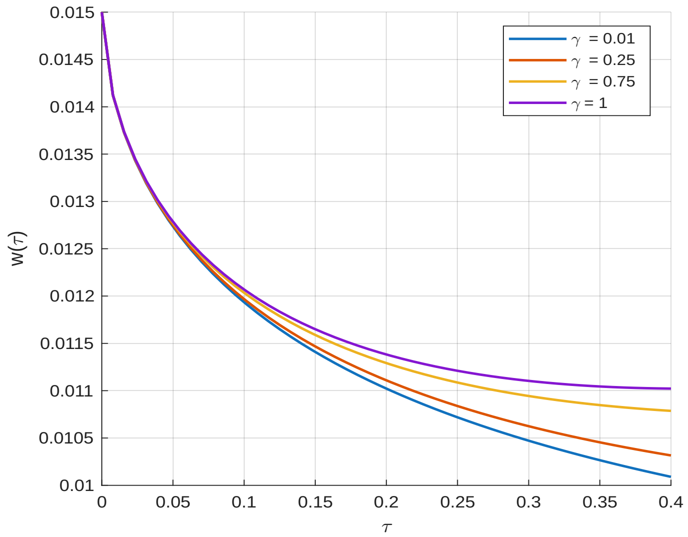

FIGURE 1. Transient behavior of the mild trajectory w for the system (3.2).

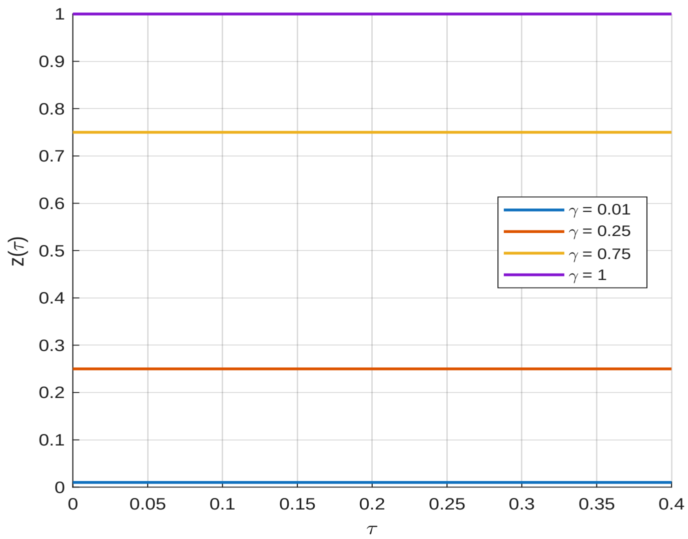

FIGURE 2. Transient behavior of the variational control trajectory z for the system (3.2).

Then the solution of Problem 1.1 given in Definition 3.1 can be rewritten as

$$w(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^{\tau} (\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda}(\sigma)d\sigma$$

for all $\tau \in [0,T]$, where $h^{\lambda} \in \mathcal{N}_H(w,\lambda)$ with $\mathcal{N}_H(w,\lambda)$ given by

$$\mathcal{N}_H(w,\lambda) = \left\{h^{\lambda} \in \mathcal{L}^p([0,T];W) : h^{\lambda}(\tau) \in \mho_z^{\gamma\zeta}(\tau,w(\tau),\lambda), \text{ for a.e. } \tau \in [0,T]\right\}. \tag{4.2}$$

Remark 4.1. *Let Assumption* $\mathbf{a}(H)$ *hold. Following the lines of the proof of Lemma 3.5 in* [33], *one can show that* $\mathcal{N}_H$ *defined by* (4.2) *is weakly upper semicontinuous with nonempty convex values. Moreover,*

$$\begin{aligned}\|\mho_z^{\gamma\zeta}(\tau,w(\tau),\lambda)\|_W &= \sup\{\|h^{\lambda}(\tau)\|_W : h^{\lambda}(\tau) \in \mho_z^{\gamma\zeta}(\tau,w(\tau),\lambda) \textit{ for a.e. } \tau\in[0,T]\} \\ &\leqslant \|H(\tau,w(\tau),z(\tau),\lambda)\|_W \leqslant b_H(\tau)(1+\|w\|_W+\|z\|_Z).\end{aligned} \tag{4.3}$$

In addition, if $(w,z)\in C([0,T];W)\times C([0,T];Z)$ *is the mild solution of Problem 1.1, then we obtain the boundedness of the solution set* $\mathbf{S}_w(P_\gamma,\mathcal{B}_\zeta,\Psi,J_\zeta)$ *(by Lemma 3.3) and the boundedness of* $w(\tau)$ *in* $C([0,T];W)$ [33, Lm. 3.4]. *Hence, it follows from* (4.2) *and* (4.3) *that there exists* $\mathbf{B}>0$ *such that*

$$\|h^\lambda(\tau)\|\leqslant b_H(\tau)\mathbf{B}$$

for a.e. $\tau\in[0,T]$ *and for all* $h^\lambda\in\mathcal{N}_H(w,\lambda)$. *Thus,* $\mathcal{N}_H(w,\lambda)$ *is bounded.*

Now we consider $\{h_n^\lambda\}\subset\mathcal{N}_H(w,\lambda)$ *with* $h_n^\lambda\to h_0^\lambda$ *in* $\mathcal{L}^p([0,T];W)$. *By the Riesz – Fischer Theorem* [27], *we can find a subsequence of* $\{h_n^\lambda\}$, *denoted again by* $\{h_n^\lambda\}$, *such that* $h_n^\lambda(\tau)\to h_0^\lambda(\tau)$ *for a.e.* $\tau\in[0,T]$. *In view of the closedness of* $\mho_z^{\gamma\zeta}(\tau,w(\tau),\lambda)$, *we obtain that* $h_0^\lambda(\tau)\in\mho_z^{\gamma\zeta}(\tau,w(\tau),\lambda)$. *Therefore* $h_0^\lambda(\tau)\in\mathcal{N}_H(w,\lambda)$ *and so* $\mathcal{N}_H$ *has closed values.*

Therefore, we conclude that $\mathcal{N}_H$ *defined by* (4.2) *is weakly upper semicontinuous with nonempty, weakly compact and convex values.*

The closedness of the mild solution set mapping $\mathcal{Q}$ for Problem 1.1 is established in the following result.

Theorem 4.1. *Let Assumptions* $\mathbf{a}(\mathcal{T})$, $\mathbf{a}(H)$, $\mathbf{a}(P)$, $\mathbf{a}(\mathcal{B})$, $\mathbf{a}(\Psi)$, $\mathbf{a}(J)$, $\mathbf{a}(N)$, $\mathbf{a}(const.)$ *and* $\mathbf{a}(mon.)$ *be satisfied. Then the set–valued mapping* $\Lambda\times\mathfrak{G}\times\Xi\ni(\lambda,\gamma,\zeta)\mapsto\mathcal{Q}(\lambda,\gamma,\zeta)$ *is closed.*

Proof. For given $(\lambda,\gamma,\zeta)\in\Lambda\times\mathfrak{G}\times\Xi$, it follows from Remark 3.5(i) that the set $\mathcal{Q}(\lambda,\gamma,\zeta)$ is nonempty. Let us prove that the set–valued mapping $(\lambda,\gamma,\zeta)\mapsto\mathcal{Q}(\lambda,\gamma,\zeta)$ is closed. Indeed, let $\{(\lambda_k,\gamma_k,\zeta_k)\}\subset\Lambda\times\mathfrak{G}\times\Xi$ and $\{(w_k,z_k)\}\subset C([0,T];W)\times C([0,T];Z)$ be sequences such that

$$\begin{array}{lll}(\lambda_k,\gamma_k,\zeta_k)\to(\lambda^*,\gamma^*,\zeta^*)\quad\text{in}\quad\Lambda\times\mathfrak{G}\times\Xi & \text{as}\quad k\to\infty,\\ (w_k,z_k)\in\mathcal{Q}\left(\lambda_k,\gamma_k,\zeta_k\right) & \text{for each}\quad k\in\mathbb{N},\\ (w_k,z_k)\to(w^*,z^*)\quad\text{in}\quad C([0,T];W)\times C([0,T];Z) & \text{as}\quad k\to\infty.\end{array}\tag{4.4}$$

Since $(w_k,z_k)\in\mathcal{Q}\left(\lambda_k,\gamma_k,\zeta_k\right)$, there exists $h^{\lambda_k}\in\mathcal{N}_H(w_k,\lambda_k)$ such that

$$w_k(\tau)=\mathcal{R}_q(\tau)w_0+\int_0^\tau(\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda_k}(\sigma)d\sigma,\tag{4.5}$$

for all $\tau\in[0,T]$ and $z_k(\tau)\in P\left(\gamma_k\right)$,

$$\langle\mathcal{B}(\zeta_k,z_k(\tau))-\Psi(\tau,w_k(\tau),z_k(\tau)),v-z_k(\tau)\rangle_{Z^*\times Z}+J^\circ(\zeta_k,Nz_k(\tau);N(v-z_k(\tau)))\geqslant0\tag{4.6}$$

for all $v\in P(\gamma_k)$, for a.e. $\tau\in[0,T]$.

Thanks to Remark 4.1, we obtain that the sequence $\{h^{\lambda_k}\}\subset\mathcal{L}^p([0,T];W)$ is bounded. By the reflexivity of $\mathcal{L}^p([0,T];W)$, without loss of generality, we can assume, passing to a subsequence if necessary, that

$$h^{\lambda_k}\rightharpoonup h^{\lambda^*}\ \text{in}\ \mathcal{L}^p([0,T];W)\quad\text{as}\quad k\to\infty.\tag{4.7}$$

Since $\mathcal{N}_H$ is weakly upper semicontinuous with nonempty, weakly compact and convex values and $w_k\to w^*$ in $C([0,T];W)$ and $\lambda_k\to\lambda^*$ as $k\to\infty$, it follows from Lemma 2.1 that $h^{\lambda^*}\in\mathcal{N}_H(w^*,\lambda^*)$.

By (4.7) and Lemma 3.2 we have

$$\int_0^\tau(\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda_k}(\sigma)d\sigma\to\int_0^\tau(\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda^*}(\sigma)d\sigma,\quad\text{as}\quad k\to\infty.\tag{4.8}$$

Combining (4.5) and (4.8), one has

$$w_k(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda_k}(\sigma)d\sigma$$
$$\to \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda^*}(\sigma)d\sigma,$$

for all $\tau \in [0,T]$. Furthermore, the convergence $w_k \to w^*$ in $C([0,T];W)$ implies

$$w^*(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau-\sigma)^{q-1}\mathcal{S}_q(\tau-\sigma)h^{\lambda^*}(\sigma)d\sigma \tag{4.9}$$

for all $\tau \in [0,T]$, where $h^{\lambda^*} \in \mathcal{N}_H(w^*,\lambda^*)$.

On the other hand, since $\gamma_k \to \gamma^*$ and $z_k(\tau) \to z^*(\tau)$ for a.e. $\tau \in [0,T]$, it follows from assumption $\mathbf{a}(P)$(i) that $z^*(\tau) \in P(\gamma^*)$ for a.e. $\tau \in [0,T]$. Moreover, using assumption $\mathbf{a}(P)$(ii), we see that for each $v \in P(\gamma^*)$, there exists a sequence $\{v_k\} \subset Z$ with $v_k \in P(\gamma_k)$ such that $v_k \to v$ as $k \to \infty$. Hence, by (4.6) we obtain

$$\langle \mathcal{B}(\zeta_k, z_k(\tau)) - \Psi(\tau, w_k(\tau), z_k(\tau)), v_k - z_k(\tau)\rangle_{Z^*\times Z} + J^\circ(\zeta_k, Nz_k(\tau); N(v_k - z_k(\tau))) \geqslant 0$$

for a.e. $\tau \in [0,T]$. Passing to the upper limit as $k \to \infty$ in the above inequality with using (4.4), conditions $\mathbf{a}(\mathcal{B})$(i), $\mathbf{a}(\Psi)$(ii) and $\mathbf{a}(J)$(iii), we find

$$\begin{aligned}
0 \leqslant & \limsup_{k\to\infty} \big(\langle \mathcal{B}(\zeta_k, z_k(\tau)) - \Psi(\tau, w_k(\tau), z_k(\tau)), v_k - z_k(\tau)\rangle_{Z^*\times Z} \\
& \qquad + J^\circ(\zeta_k, Nz_k(\tau); N(v_k - z_k(\tau)))\big) \\
\leqslant & \lim_{k\to\infty} \langle \mathcal{B}(\zeta_k, z_k(\tau)) - \Psi(\tau, w_k(\tau), z_k(\tau)), v_k - z_k(\tau)\rangle_{Z^*\times Z} \\
& \quad + \limsup_{k\to\infty} J^\circ(\zeta_k, Nz_k(\tau); N(v_k - z_k(\tau))) \\
= & \langle \mathcal{B}(\zeta^*, z^*(\tau)) - \Psi(\tau, w^*(\tau), z^*(\tau)), v - z^*(\tau)\rangle_{Z^*\times Z} + J^\circ(\zeta^*, Nz^*(\tau); N(v - z^*(\tau))),
\end{aligned}$$

for all $v \in P(\gamma^*)$ and a.e. $\tau \in [0,T]$. Hence, $z^*(\tau)$ solves the problem $(\mathrm{HVI})_{w^*}^{\gamma^*\zeta^*}$ for a.e. $\tau \in [0,T]$. Combining this with (4.9), we conclude that $(w^*, z^*) \in \mathcal{Q}(\lambda^*, \gamma^*, \zeta^*)$. Therefore, $\mathcal{Q}$ is closed. This completes the proof. □

We now consider the upper semicontinuity of the set–valued mapping $\mathcal{Q}$ with respect to perturbed parameters on the data of Problem 1.1.

Theorem 4.2. *Let $(\lambda^*, \gamma^*, \zeta^*) \in \Lambda \times \mathfrak{G} \times \Xi$ be fixed and Assumptions $\mathbf{a}(\mathcal{T})$, $\mathbf{a}(H)$, $\mathbf{a}(P)$, $\mathbf{a}(\mathcal{B})$, $\mathbf{a}(\Psi)$, $\mathbf{a}(J)$, $\mathbf{a}(N)$, $\mathbf{a}$(const.) and $\mathbf{a}$(mon.) be satisfied. Then the set–valued mapping $(\lambda, \gamma, \zeta) \mapsto \mathcal{Q}(\lambda, \gamma, \zeta)$ is upper semicontinuous at $(\lambda^*, \gamma^*, \zeta^*)$.*

Proof. We argue by contradiction and suppose that the set–valued mapping $\mathcal{Q}$ is not upper semicontinuous at $(\lambda^*, \gamma^*, \zeta^*) \in \Lambda \times \mathfrak{G} \times \Xi$. Then, there exist two sequences

$$\{(\lambda_k, \gamma_k, \zeta_k)\} \subset \Lambda \times \mathfrak{G} \times \Xi, \qquad \{(w_k, z_k)\} \subset C([0,T];W) \times C([0,T];Z)$$

and an open set $\mathcal{U}$ in $C([0,T];W) \times C([0,T];Z)$ with

$$(\lambda_k, \gamma_k, \zeta_k) \to (\lambda^*, \gamma^*, \zeta^*) \quad \text{in} \quad \Lambda \times \mathfrak{G} \times \Xi \quad \text{as} \quad k \to \infty,$$
$$(w_k, z_k) \in \mathcal{Q}(\lambda_k, \gamma_k, \zeta_k) \quad \text{for each} \quad k \in \mathbb{N}, \qquad \mathcal{Q}(\lambda^*, \gamma^*, \zeta^*) \subset \mathcal{U}$$

such that $(w_k, z_k) \notin \mathcal{U}$ for each $k \in \mathbb{N}$. Since $(w_k, z_k) \in \mathcal{Q}(\lambda_k, \gamma_k, \zeta_k)$, there exists $h^{\lambda_k} \in \mathcal{N}_H(w_k, \lambda_k)$ such that $z_k(\tau) \in P(\gamma_k)$,

$$\langle \mathcal{B}(\zeta_k, z_k(\tau)) - \Psi(\tau, w_k(\tau), z_k(\tau)), v - z_k(\tau)\rangle_{Z^*\times Z} + J^\circ(\zeta_k, Nz_k(\tau); N(v - z_k(\tau))) \geqslant 0, \quad (4.10)$$

for all $v \in P(\gamma_k)$, for a.e. $\tau \in [0, T]$ and

$$w_k(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau - \sigma)^{q-1}\mathcal{S}_q(\tau - \sigma)h^{\lambda_k}(\sigma)d\sigma,$$

for a.e. $\tau \in [0, T]$.

Repeating the argument used in the proof of Theorem 4.1, we obtain (4.8) noting that $h^{\lambda_k} \rightharpoonup h^{\lambda^*}$ in $\mathcal{L}^p([0, T]; W)$ as $k \to \infty$. We hence have

$$\begin{aligned} w_k(\tau) &= \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau - \sigma)^{q-1}\mathcal{S}_q(\tau - \sigma)h^{\lambda_k}(\sigma)d\sigma \\ &\to \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau - \sigma)^{q-1}\mathcal{S}_q(\tau - \sigma)h^{\lambda^*}(\sigma)d\sigma, \end{aligned}$$

as $k \to \infty$ and for all $\tau \in [0, T]$. We define $w^*\colon [0, T] \to W$ as

$$w^*(\tau) = \mathcal{R}_q(\tau)w_0 + \int_0^\tau (\tau - \sigma)^{q-1}\mathcal{S}_q(\tau - \sigma)h^{\lambda^*}(\sigma)d\sigma.$$

It is clear that w^* is continuous on $[0, T]$. Since $w_k \in C([0, T]; W)$ for all $k \in \mathbb{N}$, we deduce that $w_k \to w^*$ in $C([0, T]; W)$. By the weakly upper semicontinuity of $\mathcal{N}_H$ and $w_k \to w^*$ in $C([0, T]; W)$ and $\lambda_k \to \lambda^*$ as $k \to \infty$, using Lemma 2.1 again leads to $h^{\lambda^*} \in \mathcal{N}_H(w^*, \lambda^*)$.

Since $\{z_k(\tau)\} \subset \mathbf{S}_{w_k}(P_{\gamma_k}, \mathcal{B}_{\zeta_k}, \Psi, J_{\zeta_k})$, $\{z_k\}$ is bounded in $C([0, T]; Z)$. Hence, there exists a subsequence of $\{z_k\}$, denoted again by $\{z_k\}$, such that $z_k \to z^*$ in $C([0, T]; Z)$. Using the assumption $\mathbf{a}(P)$(i) and note that P has closed values, we conclude that $z^*(\tau) \in P(\gamma^*)$.

Next, we show that $z^*(\tau) \in \mathbf{S}_{w^*}(P_{\gamma^*}, \mathcal{B}_{\zeta^*}, \Psi, J_{\zeta^*})$ for a.e. $\tau \in [0, T]$. Indeed, by the similar argument in the proof of Theorem 4.1 based on (4.10), we obtain that

$$\langle \mathcal{B}(\zeta^*, z^*(\tau)) - \Psi(\tau, w^*(\tau), z^*(\tau)), v - z^*(\tau)\rangle_{Z^*\times Z} + J^\circ(\zeta^*, Nz^*(\tau); N(v - z^*(\tau))) \geqslant 0$$

for a.e. $\tau \in [0, T]$ and all $v \in P(\gamma^*)$. Since $z^*(\tau) \in P(\gamma^*)$, we have $z^*(\tau) \in \mathbf{S}_{w^*}(P_{\gamma^*}, \mathcal{B}_{\zeta^*}, \Psi, J_{\zeta^*})$ for a.e. $\tau \in [0, T]$. Furthermore, it follows from Theorem 4.1 that the set $\mathcal{Q}(\lambda^*, \gamma^*, \zeta^*)$ is closed, and hence we get that $(w^*, z^*) \in \mathcal{Q}(\lambda^*, \gamma^*, \zeta^*) \subset \mathcal{U}$. However, based on initial assumptions, it is known that $(w_k, z_k) \notin \mathcal{U}$ for all $k \in \mathbb{N}$, which is a contradiction. This completes the proof. □

5. Optimal control problem

In this section we investigate an optimal control problem governed by the fractional differential hemivariational inequality formulated in Problem 1.1 as follows.

Problem 5.1. *Find* $(\lambda^*, \gamma^*, \zeta^*) \in \Lambda \times \mathfrak{G} \times \Xi$ *such that*

$$(\lambda^*, \gamma^*, \zeta^*) \in \arg\min\{\Theta(\lambda, \gamma, \zeta) : (\lambda, \gamma, \zeta) \in \mathcal{K}(\Lambda) \times \mathcal{K}(\mathfrak{G}) \times \mathcal{K}(\Xi)\}$$

where the cost function $\Theta : \Lambda \times \mathfrak{G} \times \Xi \to \mathbb{R}$ *is defined by*

$$\Theta(\lambda, \gamma, \zeta) := \inf_{(w,z)\in\mathcal{Q}(\lambda,\gamma,\zeta)} \left(\Upsilon(w_0, w(T), \lambda) + \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda)d\tau + \kappa\Omega(\gamma, \zeta) \right), \quad (5.1)$$

$\kappa > 0$ is a given regularization parameter, $\mathcal{K}(\Lambda), \mathcal{K}(\mathfrak{G})$ and $\mathcal{K}(\Xi)$ are compact sets of $\Lambda, \mathfrak{G}$ and Ξ, respectively and $\mathcal{Q}(\lambda, \gamma, \zeta)$ is the solution set of FDHVI in Problem 1.1 corresponding to (λ, γ, ζ). Furthermore, functions $\Upsilon : W \times W \times \Lambda \to \mathbb{R}$, $\varrho : [0,T] \times W \times Z \times \Lambda \to \mathbb{R}$, and $\Omega \colon \mathfrak{G} \times \Xi \to \mathbb{R}$ are supposed to satisfy the following assumptions.

a(Υ): *$\Upsilon : W \times W \times \Lambda \to \mathbb{R}$ is such that $(w, \lambda) \mapsto \Upsilon(w_0, w, \lambda)$ is a lower semicontinuous function which is bounded from below.*

a(ϱ): *$\varrho : [0,T] \times W \times Z \times \Lambda \to \mathbb{R}$ is such that*

(i) for all $(w, z, \lambda) \in W \times Z \times \Lambda$, the function $\tau \mapsto \varrho(\tau, w, z, \lambda)$ is measurable on $[0,T]$;

(ii) the function $(w, z, \lambda) \mapsto \varrho(\tau, w, z, \lambda)$ is lower semicontinuous, for a.e. $\tau \in [0,T]$;

(iii) for any bounded subsets $D_1 \subset \mathcal{L}^1([0,T]; W)$ and $D_2 \subset \mathcal{L}^1([0,T]; Z)$, there exists a function $\omega(D_1, D_2) \in \mathcal{L}^1([0,T], \mathbb{R})$ such that $\varrho(\tau, w(\tau), z(\tau), \Lambda) \geqslant \omega(D_1, D_2)(\tau)$ for a.e. $\tau \in [0,T]$ and all $(w, z) \in D_1 \times D_2$.

a(Ω): *The function $\Omega : \mathfrak{G} \times \Xi \to \mathbb{R}$ is lower semicontinuous.*

Remark 5.1. *(i) The construction of the cost function Θ in (5.1) is inspired by the work of Li – Liu [15]. Indeed, if $\Omega \equiv 0$, then (5.1) reduces to*

$$\Theta(\lambda) = \inf_{(w,z) \in \mathcal{Q}(\lambda)} \left(\Upsilon\left(w_0, w(T), \lambda\right) + \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda) d\tau \right).$$

This cost function was previously considered by Li – Liu [15] in the context of optimal control problems for a class of differential hemivariational inequalities involving a single perturbed parameter in the differential system. Here, $\Upsilon : W \times W \times \Lambda \to \mathbb{R}$ denotes the function accounting for the initial and terminal costs together with the perturbed parameter, while $\varrho : [0,T] \times W \times Z \times \Lambda \to \mathbb{R}$ represents the cost integrand and $\mathcal{Q}(\lambda)$ is the solution set of the differential hemivariational inequality corresponding to λ.

(ii) The component $\kappa\Omega(\gamma, \zeta)$ appearing in (5.1) reflects the optimal objectives associated with the perturbed parameters γ and ζ in problem $(\mathrm{HVI})_w^{\gamma\zeta}$. As an illustrative example, one may define the mapping $\Omega \colon \mathfrak{G} \times \Xi \to \mathbb{R}$ as a quadratic function of the form

$$\Omega(\gamma, \zeta) = \frac{1}{2}\left(d_{\mathfrak{G}}^2(\gamma, \gamma^*) + d_{\Xi}^2(\zeta, \zeta^*)\right),$$

where $(\gamma^*, \zeta^*) \in \mathfrak{G} \times \Xi$ is fixed.

The following result establishes the existence of a solution to Problem 5.1.

Theorem 5.1. *Assume that the hypotheses* **a**$(\mathcal{T})$, **a**(H), **a**(P), **a**$(\mathcal{B})$, **a**(Ψ), **a**(J), **a**(N), **a**$(const.)$, **a**$(mon.)$, **a**(Υ), **a**(ϱ), *and* **a**(Ω) *hold. Then Problem 5.1 admits a solution.*

Proof. First, we show that the cost function $\Theta : \Lambda \times \mathfrak{G} \times \Xi \to \mathbb{R}$ defined by (5.1) is well-defined. Let $(\lambda, \gamma, \zeta) \in \Lambda \times \mathfrak{G} \times \Xi$ be fixed. We claim that the function

$$\mathcal{L}^1([0,T]; W) \times \mathcal{L}^1([0,T]; Z) \ni (w, z) \mapsto \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda) d\tau \tag{5.2}$$

is lower semicontinuous. Indeed, let $\{(w_n, z_n)\}$ be a sequence in $\mathcal{L}^1([0,T]; W) \times \mathcal{L}^1([0,T]; Z)$ such that

$$(w_n, z_n) \to (w, z) \text{ in } \mathcal{L}^1([0,T]; W) \times \mathcal{L}^1([0,T]; Z) \text{ as } n \to \infty.$$

Applying the converse Lebesgue — dominated convergence theorem (see, e.g., [21, Thm. 2.39]), we can assume that

$$w_n(\tau) \to w(\tau) \quad \text{in} \quad W \quad \text{and} \quad z_n(\tau) \to z(\tau) \quad \text{in} \quad Z \quad \text{as} \quad n \to \infty \tag{5.3}$$

for a.e. $\tau \in [0, T]$. By the lower semicontinuity of ϱ in assumption $\mathbf{a}(\varrho)$(ii) and convergence (5.3), we have

$$\liminf_{n\to\infty} \varrho\,(\tau, w_n(\tau), z_n(\tau), \lambda) \geqslant \varrho(\tau, w(\tau), z(\tau), \lambda)$$

for a.e. $\tau \in [0, T]$. Owing to condition $\mathbf{a}(\varrho)$(iii), we can employ the Fatou's lemma to get

$$\begin{aligned}\liminf_{n\to\infty} \int_0^T \varrho\,(\tau, w_n(\tau), z_n(\tau), \lambda)\, d\tau &\geqslant \int_0^T \liminf_{n\to\infty} \varrho\,(\tau, w_n(\tau), z_n(\tau), \lambda)\, d\tau \\ &\geqslant \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda) d\tau.\end{aligned}$$

Thus, the function given by (5.2) is lower semicontinuous.

Next, since $W \times \Lambda \ni (w, \lambda) \mapsto \Upsilon(w_0, w, \lambda)$ is a lower semicontinuous function which is bounded from below, it can be seen easily that the function

$$C([0,T]; W) \times C([0,T]; Z) \ni (w, z) \mapsto \Upsilon\,(w_0, w(T), \lambda) + \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda) d\tau$$

is lower semicontinuous and bounded from below. Moreover, we obtain

$$\mathcal{Q}(\lambda, \gamma, \zeta) \subset C([0,T]; W) \times C([0,T]; Z),$$

and so there exists a minimizing sequence $\{(w_n, z_n)\} \subset \mathcal{Q}(\lambda, \gamma, \zeta)$ such that

$$\begin{aligned}&\lim_{n\to\infty} \left(\Upsilon\,(w_0, w_n(T), \lambda) + \int_0^T \varrho(\tau, w_n(\tau), z_n(\tau), \lambda) d\tau + \kappa\Omega(\gamma, \zeta) \right) \\ &\quad = \inf_{(w,z)\in\mathcal{Q}(\lambda,\gamma,\zeta)} \left(\Upsilon\,(w_0, w(T), \lambda) + \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda) d\tau + \kappa\Omega(\gamma, \zeta) \right).\end{aligned} \tag{5.4}$$

It is well–known that the set $\mathcal{Q}(\lambda, \gamma, \zeta)$ is compact in $C([0,T]; W) \times C([0,T]; Z)$, see Remark 3.5(ii). Thus, without loss of generality, we can assume that $(w_n, z_n) \to (w^*, z^*)$ in $C([0,T]; W) \times C([0,T]; Z)$ as $n \to \infty$ with some $(w^*, z^*) \in \mathcal{Q}(\lambda, \gamma, \zeta)$. This leads to

$$w_n(\tau) \to w^*(\tau) \quad \text{in} \quad W \quad \text{and} \quad z_n(\tau) \to z^*(\tau) \quad \text{in} \quad Z \quad \text{as} \quad n \to \infty \tag{5.5}$$

for all $\tau \in [0,T]$. Using Assumptions $\mathbf{a}(\Upsilon)$, $\mathbf{a}(\varrho)$, convergences (5.5), and by the Fatou's lemma we find

$$\begin{aligned}\liminf_{n\to\infty}&\left(\Upsilon(w_0,w_n(T),\lambda)+\int_0^T\varrho(\tau,w_n(\tau),z_n(\tau),\lambda)d\tau+\kappa\Omega(\gamma,\zeta)\right)\\&\geqslant\liminf_{n\to\infty}\Upsilon(w_0,w_n(T),\lambda)+\int_0^T\liminf_{n\to\infty}\varrho(\tau,w_n(\tau),z_n(\tau),\lambda)d\tau+\kappa\Omega(\gamma,\zeta)\\&\geqslant\Upsilon(w_0,w^*(T),\lambda)+\int_0^T\varrho(\tau,w^*(\tau),z^*(\tau),\lambda)\,d\tau+\kappa\Omega(\gamma,\zeta)\\&\geqslant\inf_{(w,z)\in\mathcal{Q}(\lambda,\gamma,\zeta)}\left(\Upsilon(w_0,w(T),\lambda)+\int_0^T\varrho(\tau,w(\tau),z(\tau),\lambda)d\tau+\kappa\Omega(\gamma,\zeta)\right).\end{aligned}\tag{5.6}$$

Combining (5.4) and (5.6), we find

$$\Theta(\lambda,\gamma,\zeta)=\Upsilon(w_0,w^*(T),\lambda)+\int_0^T\varrho(\tau,w^*(\tau),z^*(\tau),\lambda)d\tau+\kappa\Omega(\gamma,\zeta)$$

for some $(w^*,z^*)\in\mathcal{Q}(\lambda,\gamma,\zeta)$, i.e., Θ is well–defined.

We proceed to verifying that Problem 5.1 possesses a solution. Indeed, let $\{(\lambda_n,\gamma_n,\zeta_n)\}\subset\mathcal{K}(\Lambda)\times\mathcal{K}(\mathfrak{G})\times\mathcal{K}(\Xi)$ be a minimizing sequence for Problem 5.1, that is,

$$\inf_{(\lambda,\gamma,\zeta)\in\mathcal{K}(\Lambda)\times\mathcal{K}(\mathfrak{G})\times\mathcal{K}(\Xi)}\Theta(\lambda,\gamma,\zeta)=\lim_{n\to\infty}\Theta(\lambda_n,\gamma_n,\zeta_n)\tag{5.7}$$

Since $\mathcal{K}(\Lambda)\times\mathcal{K}(\mathfrak{G})\times\mathcal{K}(\Xi)$ is compact, the sequence $\{(\lambda_n,\gamma_n,\zeta_n)\}$ is relatively compact in $\Lambda\times\mathfrak{G}\times\Xi$. This ensures that $\{(\lambda_n,\gamma_n,\zeta_n)\}$ has a subsequence, which is still denoted in the same way, such that $(\lambda_n,\gamma_n,\zeta_n)\to(\lambda^*,\gamma^*,\zeta^*)$ in $\Lambda\times\mathfrak{G}\times\Xi$, as $n\to\infty$ for some $(\lambda^*,\gamma^*,\zeta^*)\in\Lambda\times\mathfrak{G}\times\Xi$. Let $\{(w_n,z_n)\}\subset C([0,T];W)\times C([0,T];Z)$ be a sequence satisfying $(w_n,z_n)\in\mathcal{Q}(\lambda_n,\gamma_n,\zeta_n)$ for all $n\in\mathbb{N}$, and

$$\Theta(\lambda_n,\gamma_n,\zeta_n)=\Upsilon(w_0,w_n(T),\lambda_n)+\int_0^T\varrho(\tau,w_n(\tau),z_n(\tau),\lambda_n)d\tau+\kappa\Omega(\gamma_n,\zeta_n)\tag{5.8}$$

Recall that the sequence $\{(\lambda_n,\gamma_n,\zeta_n)\}$ is relatively compact in $\Lambda\times\mathfrak{G}\times\Xi$. Hence, it follows from the compactness of $\mathcal{Q}(\lambda_n,\gamma_n,\zeta_n)$ that the sequence $\{(w_n,z_n)\}$ is also relatively compact in $C([0,T];W)\times C([0,T];Z)$. Thus, we can find a subsequence of $\{(w_n,z_n)\}$, denoted again by $\{(w_n,z_n)\}$, and $(w^*,z^*)\in C([0,T];W)\times C([0,T];Z)$ such that $(w_n,z_n)\to(w^*,z^*)$ in $C([0,T];W)\times C([0,T];Z)$, as $n\to\infty$, and hence,

$$w_n(\tau)\to w^*(\tau)\quad\text{in}\quad W\quad\text{and}\quad z_n(\tau)\to z^*(\tau)\quad\text{in}\quad Z\quad\text{as}\quad n\to\infty\tag{5.9}$$

for all $\tau\in[0,T]$. It follows from Theorem 4.1 that the set–valued mapping $(\lambda,\gamma,\zeta)\mapsto\mathcal{Q}(\lambda,\gamma,\zeta)$ is closed. Hence, we have $(w^*,z^*)\in\mathcal{Q}(\lambda^*,\gamma^*,\zeta^*)$.

Passing to the lower limit, as $n\to\infty$ in (5.8), and using convergences (5.9) and the Fatou's lemma, we obtain

$$\liminf_{n\to\infty}\Theta(\lambda_n,\gamma_n,\zeta_n)=\liminf_{n\to\infty}\left(\Upsilon(w_0,w_n(T),\lambda_n)+\int_0^T\varrho(\tau,w_n(\tau),z_n(\tau),\lambda_n)d\tau+\kappa\Omega(\gamma_n,\zeta_n)\right)$$

$$\geqslant \liminf_{n\to\infty} \Upsilon\left(w_0, w_n(T), \lambda_n\right) + \int_0^T \liminf_{n\to\infty} \varrho(\tau, w_n(\tau), z_n(\tau), \lambda_n)\, d\tau$$

$$+ \liminf_{n\to\infty} \kappa\Omega(\gamma_n, \zeta_n)$$

$$\geqslant \Upsilon\left(w_0, w^*(T), \lambda^*\right) + \int_0^T \varrho(\tau, w^*(\tau), z^*(\tau), \lambda^*) d\tau + \kappa\Omega(\gamma^*, \zeta^*),$$

where the lower semicontinuity of Υ, ϱ and Ω is used. Combining with (5.7) and the fact $(w^*, z^*) \in \mathcal{Q}(\lambda^*, \gamma^*, \zeta^*)$, we arrive at

$$\begin{aligned}
\inf_{(\lambda,\gamma,\zeta)\in\mathcal{K}(\Lambda)\times\mathcal{K}(\mathfrak{G})\times\mathcal{K}(\Xi)} \Theta(\lambda, \gamma, \zeta) &= \lim_{n\to\infty} \Theta(\lambda_n, \gamma_n, \zeta_n) = \liminf_{n\to\infty} \Theta(\lambda_n, \gamma_n, \zeta_n) \\
&\geqslant \Upsilon\left(w_0, w^*(T), \lambda^*\right) + \int_0^T \varrho(\tau, w^*(\tau), z^*(\tau), \lambda^*) d\tau + \kappa\Omega(\gamma^*, \zeta^*) \\
&\geqslant \inf_{(w,z)\in\mathcal{Q}(\lambda^*,\gamma^*,\zeta^*)} \left(\Upsilon\left(w_0, w(T), \lambda^*\right) + \int_0^T \varrho(\tau, w(\tau), z(\tau), \lambda^*) d\tau + \kappa\Omega(\gamma^*, \zeta^*) \right) \\
&= \Theta(\lambda^*, \gamma^*, \zeta^*) \geqslant \inf_{(\lambda,\gamma,\zeta)\in\mathcal{K}(\Lambda)\times\mathcal{K}(\mathfrak{G})\times\mathcal{K}(\Xi)} \Theta(\lambda, \gamma, \zeta).
\end{aligned}$$

This implies that $(\lambda^*, \gamma^*, \zeta^*) \in \Lambda\times\mathfrak{G}\times\Xi$ is a solution to Problem 5.1. The proof is complete. □

6. Conclusions

In this paper, we have introduced a new theoretical framework for a class of perturbed differential systems in Banach spaces, consisting of a Caputo–type fractional order evolution equation coupled with a time–dependent constrained hemivariational inequality. A key feature of our analysis is that the mapping in the fractional order differential equation, as well as both the mappings and the constraint set in the hemivariational inequality, are perturbed by different parameters. Within this framework, we established a stability result in the sense of upper semicontinuity for the mild solution set mapping under perturbations (Section 4), and derived an existence result for an associated general optimal control problem (Section 5).

For future work, we intend to explore applications of the perturbed framework of Caputo–type fractional differential hemivariational inequalities developed in this paper to concrete models, in particular to contact problems and parabolic–elliptic systems governed by Caputo–type fractional order derivatives.

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