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KERNEL DETERMINATION PROBLEM IN MULTI–DIMENSIONAL TIME FRACTIONAL DIFFUSION EQUATION IN BOUNDED DOMAIN

J.J. JUMAEV, D.K. DURDIEV

Abstract. In this paper, we investigate the inverse problem on identification of the convolution kernel in an integral term of a multi–dimensional time–fractional diffusion equation, where the spatial operator is a uniformly elliptic operator in divergence form. The corresponding direct problem is formulated as an initial–boundary value problem for a multi–dimensional time–fractional integro–differential diffusion equation. We establish global existence and uniqueness theorems for the solution to the direct problem by using the Fourier spectral method. To determine the unknown kernel, we impose an additional integral–type overdetermination condition on the solution of the direct problem. The inverse problem is then transformed into an equivalent auxiliary problem. By applying fractional integration and differentiation techniques the inverse problem is reduced to a nonlinear Volterra integral equation of the second kind with convolution structure. Using the fixed point principle, we prove local existence and global uniqueness of the solution to the inverse problem. In addition, a stability estimate for the solution to the inverse problem is obtained.

Keywords: fractional diffusion equation, integro-differential equation, inverse problem, spectral problem, fixed point theorem, Gronwall inequality, existence, uniqueness

Mathematics Subject Classification: 58J35, 58J50, 35C15

1. Introduction and statement of problem

Let Ω be a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. In the domain $Q_T := \{(x,t) : x \in \Omega, 0 < t \leqslant T\}$ we consider the integro-differential fractional diffusion equation

$$\partial_t^\alpha u - Lu = \int_0^t k(t-\tau)u(x,\tau)\,d\tau + g(x,t), \qquad 0<\alpha\leqslant 1, \qquad (x,t)\in Q_T, \tag{1.1}$$

with the initial value

$$u|_{t=0} = \varphi(x), \qquad x\in\overline{\Omega}, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad x\in\partial\Omega, \quad t\in[0,T], \tag{1.3}$$

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where ∂_t^α is the regularized fractional derivative (Caputo derivative) with respect to the variable t [24, Sect. 2.4]

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, \qquad \partial_t^1 u(x,t) = \frac{\partial}{\partial t} u(x,t),$$

and

$$L = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) - c(x)$$

is a uniformly elliptic operator (with $N \geqslant 1$), the coefficients of which satisfy the conditions

$$a_{ij} \in C^1(\overline{\Omega}), \qquad a_{ij} = a_{ji}, \qquad c \in C(\overline{\Omega}), \qquad c \geqslant 0,$$
$$\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geqslant \rho \sum_{i=1}^N \xi_i^2, \qquad (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^N, \quad x \in \overline{\Omega},$$

where $\rho > 0$ is a fixed constant independent of ξ and x, and $\varphi(x)$, $g(x,t)$ are given functions.

If the function $k(t)$ is known, then the initial-boundary value problem (1.1)–(1.3) is called the *direct problem*.

Nowadays, fractional order calculus is sufficiently explored in engineering, because it represents more accurately some natural behavior related to different areas of engineering. It is also used in bioengineering [1], viscoelasticity [2], [3], electronics [4], robotics [5], [6], control theory [7], signal processing [8], [9], and other applications.

Let $0 < T < \infty$ and $C[0,T]$ be the space of continuous functions f on $[0,T]$ with the usual norm

$$\|f\| = \max_{t\in[0,T]} |f(t)|.$$

We denote by $C^m[0,T]$, $m \in \mathbb{N}_0 := \mathbb{N} \cup 0$ the space of functions f, which are m times continuously differentiable on $[0,T]$ with the norm

$$\|f\|_{C^m} = \sum_{k=0}^m \|f^{(k)}\| = \sum_{k=0}^m \max_{t\in[0,T]} |f^k(t)|.$$

We introduce the weighted space $C_\gamma[0,T]$, $\gamma \in (0,1)$ of functions f defined on $(0,T]$ such that $t^\gamma f(t) \in C[0,T]$ wit the norm

$$\|f\|_\gamma = \|t^\gamma f(t)\|, \qquad C_0[0,T] = C[0,T].$$

For $n \in \mathbb{N}$ we denote by $C_\gamma^n[0,T]$ the Banach space of functions $f(t)$, which are $(n-1)$ times continuously differentiable on $[0,T]$ and have the derivative $f^{(n)}(t)$ on $(0,T]$ such that $f^{(n)}(t) \in C_\gamma[0,T]$:

$$C_\gamma^n[0,T] = \left\{ f : \ \|f\|_{C_\gamma^n[0,T]} = \sum_{k=0}^{n-1} \|f^{(k)}\| + \|f^{(n)}\|_\gamma \right\}, \qquad C_\gamma^0[0,T] = C_\gamma[0,T].$$

We denote by $C\left(\overline{Q}_T\right)$ the class of continuous in $\overline{Q}_T$ functions. Let $C^{m,\alpha}\left(Q_T\right)$ be the class of functions, which are m times continuously differentiable with respect to x in Q_T and possess a continuous Caputo derivative of order α.

The space $H^m(\Omega)$, called the Sobolev space, consists of functions defined on the domain Ω, the weak derivatives of which up to order m are locally summable functions. More precisely,

$$H^m(\Omega) = \left\{ f \in L^2(\Omega) : D^\theta f \in L^2(\Omega), \text{ for all multi-indices } \theta, \ |\theta| \leqslant m \right\},$$

where $D^\theta f$ stands for the weak derivative of f of order θ.

Definition 1.1. *A classical solution of the initial–boundary value problem (1.1)-(1.3) is a function $u(x,t) \in C^{2,\alpha}(Q_T) \cap C(\overline{Q}_T)$, which satisfies (1.1)-(1.3) in the usual sense.*

Recently, the inverse problems for integer or fractional partial differential equations have received great attention from many researchers due to their wide range of applications [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. In [21], the uniqueness was proven for the inverse problem on determination of the time–independent smooth coefficient in the time–fractional diffusion equation by measurements of the solution on a certain subset at a fixed time. In [22] and [23] the authors studied inverse problems on finding space–dependent and time–dependent terms, respectively, in the classical and fractional diffusion equations. The existence, uniqueness, and stability theorems were proved.

The inverse problem consists in determining the unknown coefficient $k(t)$, $t > 0$, by the available additional data on the solution to the direct problem

$$\int\limits_{\Omega} \eta(x)u(x,t)dx = h(t), \qquad 0 \leqslant t \leqslant T, \tag{1.4}$$

where $\eta(x)$, $h(t)$ are the given functions.

We introduce the Banach space

$$X := \left\{ u \in C(\overline{Q}_T) \cap C^{2,\alpha}(Q_T) : \ \frac{\partial}{\partial t}(\partial^{\alpha}_{0+,t}u), \ \frac{\partial}{\partial t}(Lu), \ u_t \in C_{\gamma}(\overline{Q_T}) \right\}, \quad \gamma = 1-\alpha.$$

Definition 1.2. *The solution to the inverse problem (1.1)–(1.4) is a pair of functions $u(x,t)$ and $k(t)$ from the classes X and $C_{\gamma}[0,T]$, which satisfy (1.1)-(1.4).*

Integral condition (1.4) arises naturally and can be used as supplementary information in determining the source term. Such type of conditions can model various physical phenomena in the context of chemical engineering [25], [26], thermoelasticity [27], [8], heat conduction and diffusion process [29], [31] fluid flow in porous media [32]. In the heat conduction process, integral condition (1.4) arises in problems related to particle diffusion in turbulent plasma, and also in heat propagation in a thin rod, in which the law of variation $h(t)$ of the total quantity of heat in the rod is given [30]. A problem in the design of periodic contact in thermoelasticity formulated as an inverse problem with a prescribed contact pattern, which was considered in [31], is reduced to a heat conduction problem with energy specification.

The paper is organized as follows. In Section 2 we introduce the eigenvalues and eigenfunctions of the uniformly elliptic operator and discuss some of their properties. In Section 3 we address an auxiliary problem related to the original problem and prove the unique solvability of this auxiliary problem. In Section 4 we investigate the existence and uniqueness of the solution to the inverse problem, as well as the continuous dependence of the solution on the given data.

2. Study of direct problem

The operator L has the following properties [38, Thm. 6.5.1]:

1. Every eigenvalue of L is real;
2. The eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ taken in the ascending order counting their multiplicities satisfy
$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \text{and} \quad \lim_{m\to\infty} \lambda_m = \infty;$$
3. The eigenfunctions of L form an orthonormal basis $\{v_m\}_{m=1}^{\infty}$ in $L^2(\Omega)$.

The solution to problem (1.1)–(1.3) is sought as the Fourier series

$$u(x,t) = \sum_{m=1}^{\infty} u_m(t)v_m(x), \tag{2.1}$$

where $u_m(t)$ are Fourier coefficients defined by the formula

$$u_m(t) = \int\limits_{\Omega} u(x,t)v_m(x)\,dx. \tag{2.2}$$

Substituting (2.1) into equations (1.1)–(1.2), we obtain the problem for the Fourier coefficients $u_m(t)$

$$\partial_t^\alpha u_m(t) + \lambda_m u_m(t) - \int\limits_0^t k(t-\tau)u_m(\tau)\,d\tau = g_m(t), \tag{2.3}$$

$$u_m(0) = \varphi_m, \tag{2.4}$$

where φ_m, $g_m(t)$ are the Fourier coefficients of functions $\varphi(x)$, $g(x,t)$

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m v_m(x), \qquad g(x,t) = \sum_{m=1}^{\infty} g_m(t)v_m(x).$$

Problem (2.3) and (2.4) is equivalent to the following integral equation [24, Sect. 3.5]:

$$u_m(t) = \varphi_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right) + \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,u_m)ds, \tag{2.5}$$

where

$$G_m(t;k,u_m) := g_m(t) + \int\limits_0^t k(t-\tau)u_m(\tau)d\tau,$$

and $E_{\alpha,\beta}(z)$ is the Mittag — Leffler function defined by the series [24, Sect. 1.8]

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

with $\alpha, z, \beta \in \mathbb{C}$, and α has positive real part. We recall some important properties of the Mittag — Leffler function, which will be employed in what follows.

Proposition 2.1. [24, Sect. 1.8]. *Given arbitrary $0 < \alpha < 2$ and $\beta \in \mathbb{R}$, suppose κ satisfies $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \qquad \kappa \leqslant \arg(z)| \leqslant \pi.$$

Proposition 2.2. [24, Sect. 1.8]. *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leqslant E_{\alpha,\alpha}(-\eta) \leqslant \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonically decreasing function in $\eta > 0$.*

Equation (2.5) is an integral equation with a weakly singular kernel of Volterra type of the second kind with respect to the unknown function $u_m(t)$. It follows from the theory of integral equations that the solution to this equation is unique and can be obtained by the method of successive approximations. Now we shall prove two lemmas, which will be used in the proof of the main theorem on the solvability of inverse problem.

Lemma 2.1. *The estimates hold*

$$|u_m(t)| \leqslant \left(|\varphi_m| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g_m\|\right) E_\alpha(\|k\|T^\alpha), \tag{2.6}$$

$$|\partial_t^\alpha u_m(t)| \leqslant (|\lambda_m| + T\|k\|)\left(|\varphi_m| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|g_m\|\right) E_\alpha(\|k\|T^\alpha) + \|g_m\|. \tag{2.7}$$

Proof. Proposition 2.1 and formula (2.5) imply

$$\begin{aligned}|u_m(t)| \leqslant &|\varphi_m| + \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) \left(g_m(s) + \int_0^s k(s-\tau) u_m(\tau) d\tau \right)(s) ds \right| \\ \leqslant &|\varphi_m| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|g_m\| + \frac{\|k\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^\alpha |u_m(\tau)| d\tau.\end{aligned}$$

By the Grönwall type integral inequality [36], [37, Ch. 3], we arrive at (2.6). Substituting (2.6) into (2.3), we obtain (2.7). The proof is complete. □

The main result of this section is the following theorem.

Theorem 2.1. *Let* $k \in C[0,T]$ *and*

$$a_{ij} \in C^{\left[\frac{N}{2}\right]+2}(\bar{\Omega}), \qquad c \in C^{\left[\frac{N}{2}\right]+1}(\bar{\Omega}), \qquad \varphi \in H^{\left[\frac{N}{2}\right]+3}(\Omega), \qquad g \in C\left([0,T]; H^{\left[\frac{N}{2}\right]+3}(\Omega)\right),$$

$$\{\varphi, g(\,\cdot\,,t)\}, L\{\varphi, g(\,\cdot\,,t)\}, \ldots, L^{\left[\frac{N}{4}+1\right]}\{\varphi, g(\,\cdot\,,t)\} \in H_0^1(\Omega).$$

Then there exists a unique classical solution to problem (1.1)–(1.3).

Proof. Formally applying the operators L and ∂_t^α the series (2.1), we get

$$\partial_t^\alpha u(x,t) = \sum_{m=1}^\infty \partial_t^\alpha u_m(t) v_m(x), \tag{2.8}$$

$$Lu(x,t) = \sum_{m=1}^\infty u_m(t) L v_m(x) = -\sum_{m=1}^\infty \lambda_m u_m(t) v_m(x). \tag{2.9}$$

Let us prove the convergence of series (2.1), (2.8), (2.9).

It follows from (2.1) and (2.8) that

$$\begin{aligned}|u(x,t)| \leqslant &\sum_{m=1}^\infty |v_m(x)| \left(|\varphi_m| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|g_m\| \right) E_\alpha(\|k\| T^\alpha) \\ \leqslant &E_\alpha(\|k\|T^\alpha) \left(\sum_{m=1}^\infty |\varphi_m||v_m(x)| + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{m=1}^\infty |g_m(t)||v_m(x)| \right).\end{aligned}$$

Proceeding similarly for series (2.8) and (2.9), we have

$$\begin{aligned}|\partial_t^\alpha u(x,t)| \leqslant &E_\alpha(\|k\|T^\alpha) \left(\sum_{m=1}^\infty \lambda_m |\varphi_m||v_m(x)| \right. \\ &+ T\|k\| \sum_{m=1}^\infty \left(|\varphi_m||v_m(x)| + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{m=1}^\infty \|g_m\||v_m(x)| \right) \\ &\left. + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{m=1}^\infty \lambda_m |g_m(t)||v_m(x)| \right) + \sum_{m=1}^\infty |g_m(t)||v_m(x)|,\end{aligned}$$

and

$$|Lu(x,t)| \leqslant E_\alpha(\|k\|T^\alpha) \left(\sum_{m=1}^\infty \lambda_m |\varphi_m||v_m(x)| + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{m=1}^\infty \lambda_m |g_m(t)||v_m(x)| \right).$$

If we ensure the convergence of the series

$$\sum_{m=1}^{\infty} \lambda_m \omega_m |v_m(x)|$$

with $\omega_m = \max\{|\varphi_m|, \|g_m\|\}$, this will ensure the convergence of series (2.1), (2.8), (2.9). We shall only prove the convergence of

$$\sum_{m=1}^{\infty} \lambda_m |\varphi_m| |v_m(x)|;$$

the other series can be treated in the same way. By the Cauchy — Schwarz — Bunyakovsky inequality we have

$$\sum_{m=1}^{\infty} \lambda_m |\varphi_m| |v_m(x)| = \sum_{m=1}^{\infty} \frac{|v_m(x)|}{\lambda_m^{\frac{\left[\frac{N}{2}\right]+1}{2}}} |\varphi_m| \lambda_m^{\frac{\left[\frac{N}{2}\right]+3}{2}} \leqslant \left(\sum_{m=1}^{\infty} \frac{|v_m(x)|^2}{\lambda_m^{\left[\frac{N}{2}\right]+1}} \right)^{\frac{1}{2}} \cdot \left(\sum_{m=1}^{\infty} |\varphi_m|^2 \lambda_m^{\left[\frac{N}{2}\right]+3} \right)^{\frac{1}{2}}.$$

The first series in the right side converges by [33, Lm. 1], and the second series converges by [33, Lm. 5]. Generally, under the assumptions of the theorem series (2.1), (2.8), (2.9) converge uniformly and absolutely in $\overline{Q}_T$. Thus, the function $u(x,t)$ defined by series (2.1) is a solution to problem (1.1)–(1.3) in $\overline{Q}_T$.

We proceed to proving the uniqueness of this solution. For $\varphi(x) \equiv 0$ and $g(x,t) \equiv 0$ we obtain the identities $\varphi_m \equiv 0$, $\quad g_m(t) \equiv 0$. Then formula (2.5) implies that $u_m \equiv 0$ since u_m is a solution to the homogeneous equation

$$u_m(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) \int_0^s k(s-\tau) u_m(\tau) d\tau \, ds.$$

Letting $u_m \equiv 0$ in equation (2.2), we obtain

$$\int_\Omega u(x,t) v_m(x) dx = 0.$$

Since the system v_m is complete in $L_2(\Omega)$, function $u(x,t) = 0$ almost everywhere in Ω and for each $t \in [0,T]$. Since $u(x,t) \in C\left(\overline{Q}_T\right)$, we conclude that $u(x,t) \equiv 0$ in $\overline{Q}_T$. The proof is complete. □

3. Auxiliary problem

Let $u(x,t), k(t)$ be a classical solution to problem (1.1)-(1.4) and φ, g, h be sufficiently smooth functions.

Definition 3.1. [24, Sect. 2.1] *The Riemann — Liouville time fractional derivative of order $0 < \alpha < 1$ of the integrable function u is defined as*

$$D_{0+,t}^{\alpha} u(x,t) = \frac{\partial}{\partial t} I_{0+,t}^{1-\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(x,\tau) d\tau.$$

The next statements provides an auxiliary equivalent problem.

Lemma 3.1. *Problem (1.1)–(1.4) is equivalent to the auxiliary problem on determining the functions $\vartheta(x,t)$, $k(t)$ by the equations*

$$D^{\alpha}_{0+,t}\vartheta - L\vartheta = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)\,d\tau + g_t(x,t), \tag{3.1}$$

$$I^{1-\alpha}_{0+,t}\vartheta\Big|_{t=0} = L\varphi(x), \tag{3.2}$$

$$\vartheta(x,t) = 0, \qquad (x,t)\in\partial Q_T, \tag{3.3}$$

$$\int_\Omega \eta(x)\vartheta(x,t)\,dx = h'(t), \qquad 0\leqslant t\leqslant T, \tag{3.4}$$

where $\vartheta(x,t) = u_t(x,t)$.

Proof. Introducing the new function $\vartheta(x,t) = u_t(x,t)$, we differentiate identities (1.1)–(1.4) with respect to t using the identities

$$\frac{\partial}{\partial t}\partial_t^\alpha u(x,t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}\,d\tau = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{\vartheta(x,\tau)}{(t-\tau)^\alpha}d\tau = D^\alpha_{0+,t}\vartheta(x,t).$$

As a result, we arrive at problem (3.1)–(3.4), which consists in finding the functions $\vartheta(x,t)$ and $k(t)$. Initial condition (3.2) is obtained from identity (1.1) by letting $t=0$. The proof follows the same lines as in [23], and we omit it here. The proof is complete. □

We introduce the weighted spaces

$$C^{2,\alpha}_\gamma(Q_T) = \big\{\vartheta(x,t):\ \vartheta(\cdot,t)\in C^2(\Omega),\ t\in[0,T],\ \vartheta(x,\cdot)\in C_\gamma[0,T],\ D^\alpha_{0+,t}\vartheta(x,\cdot)\in C_\gamma[0,T],\ x\in\Omega\big\},$$

$$C_\gamma(\overline{Q}_T) = \big\{\vartheta(x,t):\ \vartheta(\cdot,t)\in C(\overline{\Omega}),\ t\in[0,T],\ \vartheta(x,\cdot)\in C_\gamma[0,T],\ x\in\overline{\Omega}\big\}, \qquad \gamma = 1-\alpha.$$

Definition 3.2. *A solution to inverse problem (3.1)–(3.4) is a pair of functions $\vartheta(x,t)$ and $k(t)$ in the classes $C^{2,\alpha}_\gamma(Q_T)\cap C_\gamma(\overline{Q}_T)$ and $C_\gamma[0,T]$, which satisfies (3.1)–(3.4).*

To investigate problem (3.1)–(3.4), we proceed as in Section 2. We seek the solution to problem (3.1)–(3.3) as

$$\vartheta(x,t) = \sum_{m=1}^\infty \vartheta_m(t)v_m(x), \tag{3.5}$$

where $\vartheta_m(t)$ are the Fourier coefficients defined by the formula

$$\vartheta_m(t) = \int_\Omega \vartheta(x,t)v_m(x)\,dx.$$

Substituting (3.5) into equations (3.1)–(3.3), we obtain the problem for the Fourier coefficients $\vartheta_m(t)$

$$D^\alpha_{0+,t}\vartheta_m(t) + \lambda_m\vartheta_m(t) - k(t)\varphi_m - \int_0^t k(t-\tau)\vartheta_m(\tau)d\tau = g'_m(t), \tag{3.6}$$

$$I^{1-\alpha}_{0+,t}\vartheta_m(0) = -\lambda_m\varphi_m, \tag{3.7}$$

where ψ_m, $g'_m(t)$ are the Fourier coefficients of functions $\psi(x)$, $g_t(x,t)$

$$L\varphi(x) = -\sum_{m=1}^\infty \lambda_m\varphi_m v_m(x), \qquad g_t(x,t) = \sum_{m=1}^\infty g'_m(t)v_m(x).$$

Problem (3.6) and (3.7) is equivalent to the integral equation [24, Sect. 3.1]

$$\vartheta_m(t) = -\lambda_m t^{\alpha-1}\varphi_m E_{\alpha,\alpha}\left(-\lambda_m t^\alpha\right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,\vartheta_m)\, ds, \qquad (3.8)$$

where

$$G_m(t;k,\vartheta_m) := g'_m(t) + k(t)\varphi_m + \int_0^t k(t-\tau)\vartheta_m(\tau)\, d\tau.$$

We prove the existence of a solution to integral equation (3.8) in the space $C_\gamma[0,T]$ by using the method of successive approximation.

For this equation, we consider the sequence of functions on $[0,T]$

$$t^\gamma(\vartheta_m)_n(t) = t^\gamma \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) \int_0^\tau k(\alpha)(\vartheta_m)_{n-1}(\tau-\alpha) d\alpha\, d\tau,$$

where $n \in \mathbb{N}$, $\gamma = 1-\alpha$, $(\vartheta_m)_0(t) = F_m(t)$ for $[0,T]$,

$$F_m(t) = -\lambda_m t^{\alpha-1}\varphi_m E_{\alpha,\alpha}\left(-\lambda_m t^\alpha\right) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) \left[g'_m(s) + k(s)\varphi_m\right] ds.$$

Using Propositions 2.1 and 2.2, we find

$$\begin{aligned}
|t^\gamma(\vartheta_m)_0(t)| =& |t^\gamma F_m(t)| \leqslant |\lambda_m\varphi_m| t^{\alpha+\gamma-1} E_{\alpha,\alpha}(-\lambda_m^2 t^\alpha) \\
&+ t^\gamma \left|\varphi_m \int_0^t (t-\tau)^{\alpha-1}\tau^{-\gamma} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) \tau^\gamma k(\tau) d\tau\right| \\
&+ t^\gamma \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) g'_m(\tau) d\tau\right| \\
\leqslant& \frac{\lambda_m|\varphi_m| t^{\alpha+\gamma-1}}{\Gamma(\alpha)} + t^\gamma|\varphi_m|\|k\|_\gamma I^\alpha_{0+}(t^{-\gamma}) + \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+1)}\|g'_m\| \\
=& \frac{\lambda_m|\varphi_m| t^{\alpha+\gamma-1}}{\Gamma(\alpha)} + \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} t^\alpha|\varphi_m|\|k\|_\gamma + \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+1)}\|g'_m\| \\
\leqslant& C_0(\lambda_m|\varphi_m| + \|g'_m\|) = F^0_m, \qquad t \in [0,T]
\end{aligned}$$

where the constant C_0 depends on $\|k\|_\gamma$, $\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}$, $\Gamma(\alpha)$, T.

Proposition 3.1. [24, Sect. 2.1]. *Let*

$$\alpha > 0, \qquad \beta > 0, \qquad p \geqslant 1 \qquad \alpha + \beta < \frac{1}{p}.$$

If $f \in L_p(\mathbb{R}_+)$, *then*

$$I^\alpha_{0+,x} I^\beta_{0+,x} f = I^{\alpha+\beta}_{0+,x} f.$$

Proposition 3.2. [24, Sect. 2.1]. *If* $\alpha \geqslant 0$ *and* $\beta \in \mathbb{C}$ *has a positive real part, then*

$$I^\alpha_{0+,x} x^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\beta+\alpha-1}.$$

In accordance with Proposition 3.1 and Proposition 3.2, we obtain

$$\begin{aligned}|t^\gamma(\vartheta_m)_1(t)| \leqslant & t^\gamma \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) \\ & \cdot \int_0^\tau \beta^{-\gamma}(\tau-\beta)^{-\gamma}\left|\beta^\gamma k(\beta)\right|\left|(\tau-\beta)^\gamma(\vartheta_m)_0(\tau-\beta)\right| d\beta d\tau \\ \leqslant & t^\gamma \|k\|_\gamma F_m^0 \Gamma(1-\gamma) I_{0+,t}^\alpha (I_{0+,t}^{1-\gamma}(t^{-\gamma})) = \|k\|_\gamma F_m^0 \frac{\Gamma^2(1-\gamma)}{\Gamma(2+\alpha-2\gamma)} t^{1+\alpha-\gamma}.\end{aligned}$$

In the same way we find

$$\begin{aligned}|t^\gamma(\vartheta_m)_2(t)| \leqslant & t^\gamma \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) \\ & \cdot \int_0^\tau \beta^{-\gamma}(\tau-\beta)^{-\gamma}\left|\beta^\gamma k(\beta)\right|\left|(\tau-\beta)^\gamma(\vartheta_m)_1(\tau-\beta)\right| d\beta d\tau \\ \leqslant & t^\gamma \|k\|_\gamma^2 F_m^0 \frac{\Gamma^2(1-\gamma)}{\Gamma(2+\alpha-2\gamma)} I_{0+,t}^{2(1+\alpha-\gamma)}(t^{-\gamma}) = \|k\|_\gamma^2 F_m^0 \frac{\Gamma^3(1-\gamma)}{\Gamma(3+2\alpha-3\gamma)} t^{2(1+\alpha-\gamma)}.\end{aligned}$$

For an arbitrary $n = 0, 1, 2, \dots$ we have

$$\begin{aligned}|t^\gamma(\vartheta_m)_n(t)| \leqslant & t^\gamma \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m^2(t-\tau)^\alpha\right) \\ & \cdot \int_0^\tau \beta^{-\gamma}(\tau-\beta)^{-\gamma}\left|\beta^\gamma k(\beta)\right|\left|(\tau-\beta)^\gamma(\vartheta_m)_{n-1}(\tau-\beta)\right| d\beta d\tau \\ \leqslant & \Gamma(1-\gamma) F_m^0 \frac{(\|k\|_\gamma \Gamma(1-\gamma) t^{1+\alpha-\gamma})^n}{\Gamma((1+\alpha-\gamma)n+1-\gamma)}.\end{aligned}$$

These estimates imply that the series

$$t^\gamma \vartheta_m(t) = \sum_{n=1}^\infty t^\gamma(\vartheta_m)_n(t)$$

converges uniformly in $[0, T]$ because it can be majorized in $[0, T]$ by the convergent scalar series

$$\Gamma(1-\gamma) F_m^0 \sum_{n=1}^\infty \frac{(\|k\|_\gamma \Gamma(1-\gamma) t^{1+\alpha-\gamma})^n}{\Gamma((1+\alpha-\gamma)n+1-\gamma)}.$$

Hence, we the solution of the integral equation (2.9) satisfies the estimate

$$\begin{aligned}|t^\gamma \vartheta_m(t)| &\leqslant \Gamma(1-\gamma) F_m^0 \sum_{n=1}^\infty \frac{(\|k\|_\gamma \Gamma(1-\gamma) t^{1+\alpha-\gamma})^n}{\Gamma((1+\alpha-\gamma)n+1-\gamma)} \\ &= \Gamma(1-\gamma) F_m^0 E_{1+\alpha-\gamma,1-\gamma}(\|k\|_\gamma \Gamma(1-\gamma) T^{1+\alpha-\gamma}) \\ &\leqslant C_1(\lambda_m|\varphi_m| + \|g_m'\|), \\ C_1 &= \Gamma(1-\gamma) C_0 E_{1+\alpha-\gamma,1-\gamma}(\|k\|_\gamma \Gamma(1-\gamma) T^{1+\alpha-\gamma}),\end{aligned} \tag{3.9}$$

where $E_{1+\alpha-\gamma,1-\gamma}$ is the Mittag — Leffler function of a nonnegative real argument.

Using identity (3.7), we obtain the estimate for $D^{\alpha}_{0+}\vartheta_m(t)$

$$\begin{aligned}|t^{\gamma}D^{\alpha}_{0+}\vartheta_m(t)| \leqslant & \Big(\lambda_m + \frac{\|k\|_{\gamma}\Gamma^2(1-\gamma)T^{1-\gamma}}{\Gamma(2-2\gamma)}\Big)\Gamma(1-\gamma)\\ & \cdot F^0_m E_{1+\alpha-\gamma,1-\gamma}(\|k\|_{\gamma}\Gamma(1-\gamma)T^{1+\alpha-\gamma})\\ & + |\varphi_m|\|k\|_{\gamma} + \|g'_m\| \leqslant C_2(\lambda^2_m|\varphi_m| + \lambda_m\|g'_m\|),\end{aligned} \tag{3.10}$$

where the constant C_2 depends on C_0, C_1, γ, T.

Lemma 3.2. *Let $\vartheta^1_m(t)$, $\vartheta^2_m(t)$ be solutions to (3.8) corresponding to functions $k^1(t)$, $k^2(t)$, respectively. For $t \in [0,T]$ and fixed $m \in \mathbb{N}$ the estimate holds*

$$\begin{aligned}|t^{\gamma}(\vartheta^1_m(t) - \vartheta^2_m(t))| \leqslant & \Gamma^2(1-\gamma)\|k^1 - k^2\|_{\gamma}\Big(\frac{T^{\alpha}|\varphi_m|}{\Gamma(1+\alpha-\gamma)} + \|\vartheta^1_m\|_{\gamma}\frac{T^{1+\alpha-\gamma}\Gamma(1-\gamma)}{\Gamma(2-2\gamma)}\Big)\\ & \cdot E_{1+\alpha-\gamma,1-\gamma}(\|k^2\|_{\gamma}\Gamma(1-\gamma)T^{1+\alpha-\gamma})\end{aligned} \tag{3.11}$$

where

$$\|\vartheta^1_m\|_{\gamma} \leqslant \overline{C}_1(\lambda_m|\varphi_m| + \|g'_m\|), \quad \overline{C}_1 = \Gamma(1-\gamma)C_0E_{1+\alpha-\gamma,1-\gamma}(\|k^1\|_{\gamma}\Gamma(1-\gamma)T^{1+\alpha-\gamma}).$$

Proof. To prove estimate (3.6) for $t \in [0,T]$, we see that

$$\begin{aligned}|\vartheta^1_m(t) - \vartheta^2_m(t)| \leqslant \int\limits_0^t ds\,(t-s)^{\alpha-1}\Big(&|k^1(s) - k^2(s)||\varphi_m|\\ &+ \int\limits_0^s \big(|k^1(s-\tau) - k^2(s-\tau)||\vartheta^1_m(\tau)|\\ &+ |k^2(s-\tau)||\vartheta^1_m(\tau) - \vartheta^1_m(\tau)|\big)\,d\tau\Big).\end{aligned} \tag{3.12}$$

Applying the successive approximation method to this equation by means of (3.12), we arrive at (3.11). The proof is complete. □

The main result of this section is the following theorem.

Theorem 3.1. *Let $k \in C_{\gamma}[0,T]$, the functions a and c satisfies the assumptions of Theorem 2.1 hold and*

$$\varphi \in H^{[\frac{N}{2}]+5}(\Omega), \qquad g \in C^1\left([0,T]; H^{[\frac{N}{2}]+3}(\Omega)\right);$$

$$\{\varphi, g(\,\cdot\,,t)\}, L\{\varphi,\ g(\,\cdot\,,t)\}, \quad \ldots, \quad L^{[\frac{N}{4}+1]}\{\varphi,\ g(\,\cdot\,,t)\} \in H^1_0(\Omega).$$

Then there exists a unique classical solution to problem (3.1)–(3.3).

Proof. Formally applying the operators L and $D^{\alpha}_{0+,t}$ to series (3.5), we get

$$t^{\gamma}D^{\alpha}_{0+,t}\vartheta(x,t) = \sum_{m=1}^{\infty} t^{\gamma}D^{\alpha}_{0+,t}\vartheta_m(t)v_m(x), \tag{3.13}$$

$$t^{\gamma}L\vartheta(x,t) = \sum_{m=1}^{\infty} t^{\gamma}\vartheta_m(t)Lv_m(x) = -\sum_{m=1}^{\infty} t^{\gamma}\lambda_m\vartheta_m(t)v_m(x). \tag{3.14}$$

Let us prove the convergence of series (3.5), (3.13), (3.14).

It follows from (3.9) and (3.10) that

$$|t^{\gamma}\vartheta(x,t)| \leqslant \sum_{m=1}^{\infty}|v_m(x)|C_1(\lambda_m|\varphi_m| + \|g'_m\|) = C_1(\sum_{m=1}^{\infty}|v_m(x)|\lambda_m|\varphi_m| + \sum_{m=1}^{\infty}|v_m(x)|\|g'_m\|).$$

Proceeding similarly for series (2.8), we have

$$\left|t^{\gamma} D_{0+,t}^{\alpha} \vartheta(x,t)\right| \leqslant C_2\Big(\sum_{m=1}^{\infty} |v_m(x)|\lambda_m^2|\varphi_m| + \sum_{m=1}^{\infty} \lambda_m |v_m(x)| \|g_m'\|\Big),$$

and

$$|t^{\gamma} L\vartheta(x,t)| \leqslant C_2\Big(\sum_{m=1}^{\infty} |v_m(x)|\lambda_m^2|\varphi_m| + \sum_{m=1}^{\infty} \lambda_m |v_m(x)| \|g_m'\|\Big).$$

If we ensure the convergence of the series

$$\sum_{m=1}^{\infty} \lambda_m^2 \omega_m |v_m(x)|$$

with $\omega_m = \max\{|\varphi_m|, \|g_m\|\}$, this guarantees the convergence of series (3.5), (3.13), (3.14). We shall only prove the convergence of

$$\sum_{m=1}^{\infty} \lambda_m^2 |\varphi_m||v_m(x)|;$$

the other series can be treated in the same way.

By the Cauchy — Schwarz — Bunyakovsky inequality we have

$$\sum_{m=1}^{\infty} \lambda_m^2|\varphi_m||v_m(x)| = \sum_{m=1}^{\infty} \frac{|v_m(x)|}{\lambda_m^{\frac{[\frac{N}{2}]+1}{2}}} |\varphi_m| \lambda_m^{\frac{[\frac{N}{2}]+5}{2}} \leqslant \left(\sum_{m=1}^{\infty} \frac{|v_m(x)|^2}{\lambda_m^{[\frac{N}{2}]+1}}\right)^{\frac{1}{2}} \cdot \left(\sum_{m=1}^{\infty} |\varphi_m|^2 \lambda_m^{[\frac{N}{2}]+5}\right)^{\frac{1}{2}}.$$

The first series on the right side of this relation converges by [33, Lm. 1], and the second series converges by [33, Lm. 5]. Generally, under the assumptions of the theorem, series (3.5), (3.13), (3.14) converge uniformly and absolutely in $\overline{Q}_T$. Thus, the function $\vartheta(x,t)$ defined by series (3.5) is a solution to problem (3.1)–(3.3) in $\overline{Q}_T$. The proof is complete. □

4. Solvability theorem for inverse problem

Let a pair of functions $\vartheta(x,t), k(t)$ be a solution of problem (3.1)–(3.4). We multiply both sides of equation (3.1) by function $\eta(x)$ and integrate over Ω

$$\begin{aligned}\int_{\Omega} \eta(x) D_{0+,t}^{\alpha} \vartheta(x,t) dx - \int_{\Omega} \eta(x) L\vartheta(x,t)\, dx =& k(t)\int_{\Omega} \eta(x)\varphi(x)\, dx \\ &+ \int_0^t k(t-\tau)\int_{\Omega} \eta(x)\vartheta(x,\tau)\, dx d\tau \\ &+ \int_{\Omega} \eta(x) g_t(x,t)\, dx\end{aligned} \tag{4.1}$$

for all $0 < t \leqslant T$.

By means of (3.3) and (3.4), we can rewrite (4.1) as

$$\begin{aligned}D_{0+,t}^{\alpha} h'(t) - \int_{\Omega} \eta(x) L\vartheta(x,t)\, dx =& k(t)\int_{\Omega} \eta(x)\varphi(x)\, dx \\ &+ \int_0^t k(t-\tau) h'(\tau)\, d\tau + \int_{\Omega} \eta(x) g_t(x,t)\, dx.\end{aligned} \tag{4.2}$$

This yields the integral equation for $k(t)$

$$k(t) = \frac{1}{\varphi_0}\left(D^{\alpha}_{0+,t}h'(t) - \int\limits_{\Omega} \eta(x)L\vartheta(x,t)\,dx - \int\limits_0^t k(t-\tau)h'(\tau)\,d\tau - \int\limits_{\Omega} \eta(x)g_t(x,t)\,dx\right), \quad (4.3)$$

where

$$\varphi_0 = \int\limits_{\Omega} \eta(x)\varphi(x)\,dx \neq 0.$$

We introduce the nonlinear operator $A:\, C_\gamma[0,T] \to C_\gamma[0,T]$, $\gamma = 1-\alpha$ by the rule

$$(A(k))(t) = k_0(t) - \{\varphi_0\}^{-1}\left(\sum_{m=1}^{\infty} \lambda_m \vartheta_m(t;k)\int\limits_{\Omega}\eta(x)v_m(x)\,dx + \int\limits_0^t k(t-\tau)h'(\tau)\,d\tau\right). \quad (4.4)$$

Here $\vartheta_m(t;k)$ denotes the solution of integral equation (3.8), which depends on the function $k(t)$, i.e., $\vartheta_m = \vartheta_m(t;k)$. The function $k_0(t)$ is given by

$$k_0(t) = \frac{1}{\varphi_0}\left(D^{\alpha}_{0+,t}h'(t) - \int\limits_{\Omega}\eta(x)g_t(x,t)\,dx\right).$$

By (4.4), equation (4.3) can be rewritten in operator form

$$k = A(k). \quad (4.5)$$

We fix a number $\rho > 0$ and consider the set

$$R_T\,(k_0,\rho) := \left\{k(t):\ k \in C_\gamma[0,T],\ \|k-k_0\|_\gamma \leqslant \rho\right\}, \qquad \gamma = 1-\alpha.$$

Now we are in position to formulate our main result.

Theorem 4.1. *Let $h \in C^2[0,T]$, $\varphi_0 \neq 0, \eta \in L^1(\Omega)$ and*

$$a_{ij} \in C^{[\frac{N}{2}]+4}(\overline{\Omega}), \quad c \in C^{[\frac{N}{2}]+3}(\overline{\Omega}), \qquad \varphi \in H^{[\frac{N}{2}]+5}(\Omega), \qquad g \in C\left([0,T], H^{[\frac{N}{2}]+3}(\Omega)\right),$$

$$\{\varphi,\ g(\,\cdot\,,t)\}, L\{\varphi,\ g(\,\cdot\,,t)\}, \quad \ldots, \quad L^{[\frac{N}{4}+1]}\{\varphi,\ g(\,\cdot\,,t)\} \in H^1_0(\Omega), \qquad L^{[\frac{N}{4}+2]}\varphi \in H^1_0(\Omega).$$

Then there exists a unique solution to operator equation (4.5).

Proof. Let us show that for a sufficiently small $T > 0$ the operator A is a contraction mapping on $R_T(k_0,\rho)$. Indeed, after some calculations we have estimates

$$\begin{aligned} \left|t^\gamma((A(k))(t) - k_0(t))\right| \leqslant & \left|\{\varphi_0\}\right|^{-1}\left|\sum_{m=1}^{\infty}\lambda_m t^\gamma \vartheta_m(t;k)\int\limits_{\Omega}\eta(x)v_m(x)\,dx \right. \\ & \left. + t^\gamma\int\limits_0^t k(t-\tau)h'(\tau)\,d\tau\right| \\ \leqslant & \left|\{\varphi_0\}\right|^{-1}\|\eta\|_{L_1(\Omega)}\Gamma(1-\gamma)E_{1+\alpha-\gamma,1-\gamma}(\|k\|_\gamma\Gamma(1-\gamma)T^{1+\alpha-\gamma}) \\ & \cdot\sum_{m=1}^{\infty}|v_m(x)|\lambda_m\left\{\frac{\lambda_m|\varphi_m|T^{\alpha+\gamma-1}}{\Gamma(\alpha)} + \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}T^\alpha|\varphi_m|\right\}\|k\|_\gamma \\ & + \left|\{\varphi_0\}\right|^{-1}\|h\|_{C^1[0,T]}\|k\|_\gamma T. \end{aligned} \quad (4.6)$$

$$
\begin{aligned}
&\left|t^\gamma\big((A(k^1))(t)-(A(k^2))(t)\big)\right| \\
&\qquad \leqslant \{|\varphi_0|\}^{-1}\Bigg(\sum_{m=1}^{\infty}\lambda_m\left|t^\gamma(\vartheta_m^1(t;k^1)-\vartheta_m^2(t;k^2))\right|\int\limits_{\Omega}|\eta(x)||\upsilon_m(x)|\,dx \\
&\qquad\qquad + t^\gamma\int\limits_0^t |k^1(t-\tau)-k^2(t-\tau)||h'(\tau)|\,d\tau\Bigg) \\
&\qquad \leqslant \left|\{\varphi_0\}\right|^{-1}\Bigg(\Gamma^2(1-\gamma)\sum_{m=1}^{\infty}\lambda_m\left(\frac{T^\alpha|\varphi_m|}{\Gamma(1+\alpha-\gamma)}+\|\vartheta_m^1\|_\gamma\frac{T^{1+\alpha-\gamma}\Gamma(1-\gamma)}{\Gamma(2-2\gamma)}\right) \\
&\qquad\qquad \cdot E_{1+\alpha-\gamma,1-\gamma}(\|k^2\|_\gamma\Gamma(1-\gamma)T^{1+\alpha-\gamma})|\eta(x)|_{L^1(\Omega)}|\upsilon_m(x)| \\
&\qquad\qquad + \frac{T}{1-\gamma}\|h\|_{C^1[0,T]}\Bigg)\|k^1-k^2\|_\gamma.
\end{aligned}
\tag{4.7}
$$

Hence, the series in (4.6) and (4.7) are convergent. We also note that the expressions in the right sides of these inequalities are monotonically increasing functions of T, $\|k\|_\gamma$, $\|k^1\|_\gamma$, $\|k^2\|_\gamma$. The belonging $k,k^1,k^2\in R_T\,(k_0,\rho)$ implies the inequality

$$\left\|(k,k^1,k^2)\right\|_\gamma\leqslant\rho+\|k_0\|_\gamma\,.$$

In view of these facts, for $(x,t)\in Q_T$ we obtain

$$\|(A(k))(t)-k_0(t)\|_\gamma\leqslant m_1(T),\qquad \left\|(A(k^1))(t)-(A(k^2))(t)\right\|_\gamma=m_2(T)\left\|k^1-k^2\right\|_\gamma,$$

where $m_1(T)$ denotes the right hand side of inequality (4.6) and $m_2(T)$ is the coefficient at $\|k^1-k^2\|_\gamma$ in the right hand side of (4.7) with the only difference that k, k^1, k^2 are to be replaced by $\rho+\|k_0\|_\gamma$ in both cases.

We observe that $m_i(T)$, $i=1,2$, are positive monotonically increasing functions of T, and $m_i(0)=0$. This yields that the equations $m_1(T)=\rho$ and $m_2(T)=1$ have unique positive roots. We denote these roots by T_1 and T_2, respectively. Then, it is clear that if we choose $T^*<\min\{T_1,T_2\}$, then the operator A is a contraction inn the ball $R_T\,(k_0,\rho)$. By the Banach fixed point theorem, the operator A has a unique fixed point in the ball $R_T\,(k_0,\rho)$; i.e., there exists a unique solution to equation (4.5). The proof is complete. □

Based on the known function $k(t)$, the solution to direct problem (3.1)–(3.3) is found by the formula (3.8), where $\vartheta_m(t)$ is the solution to the integral equation (3.8).

Thus, by the existence and uniqueness of the solution $(\vartheta(x,t),k(t))$ to the auxiliary problem (3.1)–(3.4), where $\vartheta\in C_\gamma^{2,\alpha}(Q_T)\cap C_\gamma(\overline{Q}_T)$, $k\in C_\gamma[0,T]$, and in view of the relation $\vartheta(x,t)=u_t(x,t)$, we see that the solution to original problem (1.1)–(1.4) belongs to the class

$$u(x,t)\in X,\qquad k(t)\in C_\gamma[0,T].$$

Let T be a positive fixed number. We consider the set $\Psi(\omega_0)$, $\omega_0>0$ is some fixed number, of the given functions (g,φ,h), which obey the assumptions of Theorem 4.1 and

$$\max\left\{\|g\|_{C^1\left(H^{\left[\frac{N}{2}\right]+3}(\Omega);[0,T]\right)},\ \|\varphi\|_{H^{\left[\frac{N}{2}\right]+5}(\Omega)},\ \|h\|_{C^2[0,T]}\right\}\leqslant\omega_0.$$

By $K(\omega_1)$ we denote the class of functions $k\in C_\gamma\,[0,T]$ satisfying the inequality $\|k\|\leqslant\omega_1$ with some fixed positive number ω_1.

The next theorems characterize the conditional stability and global uniqueness for the solution to the inverse problem. They can be proved quite similar to similar theorems in [14].

Theorem 4.2. *Let* $(g, \varphi, h) \in \Psi(\omega_0)$, $\left(\tilde{g}, \tilde{\varphi}, \tilde{h}\right) \in \Psi(\omega_0)$ *and* $\left(k, \tilde{k}\right) \in K(\omega_1)$. *Then, the solution to inverse problem* (3.1)-(3.4) *satisfies the estimate*

$$\|k - \tilde{k}\|_\gamma \leqslant d\kappa,$$

where

$$\kappa := \|g - \tilde{g}\|_{C^1\left(H^{\left[\frac{N}{2}\right]+3}(\Omega);[0,T]\right)} + \|\varphi - \tilde{\varphi}\|_{H^{\left[\frac{N}{2}\right]+5}(\Omega)} + \|h - \tilde{h}\|_{C^2[0,T]},$$

and the constant d depends only on T, α, N, and the norms of a_{ij}, c in the corresponding spaces given in Theorem 4.1 for these functions.

The uniqueness result is immediately implied by Theorem 4.2.

Theorem 4.3. *Let the functions k, g, φ, h and $\tilde{k}$, $\tilde{g}$, $\tilde{\varphi}$, $\tilde{h}$ be the same as in Theorem 4.2. Moreover, if $g = \tilde{g}$, $\varphi = \tilde{\varphi}$, $h = \tilde{h}$ for $t \in [0, T]$, then $k(t) = \tilde{k}(t)$ for $t \in [0, T]$.*

Conclusion

In this paper, we studied the inverse problem on determining the convolution kernel of the integral term in the initial boundary value problem for the multi–dimensional time–fractional diffusion equation with the uniformly elliptic operator of the divergent form. The inverse problem is transformed to the equivalent auxiliary problem and the direct problem is investigated using the Fourier spectral method. Fractional integral–differentiating techniques reduced the inverse problem to a nonlinear Volterra integral equation of the second kind of convolution type. Fixed point argument proved the local existence and global uniqueness results. The stability estimate for the solution to the inverse problem are also obtained.

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Jonibek Jamolovich Jumaev,
V.I. Romanovskiy Institute of Mathematics
Academy of Sciences of Uzbekistan,
University str 9,
Tashkent, 100174 , Uzbekistan.
Bukhara State University,
M.Ikbol 11,
Bukhara, 200114, Uzbekistan.
E-mail: jonibekjj@mail.ru

Durdimurod Kalandarovich Durdiev,
V.I. Romanovskiy Institute of Mathematics
Academy of Sciences of Uzbekistan,
University str 9,
Tashkent, 100174 , Uzbekistan.
Bukhara State University,
M.Ikbol 11,
Bukhara, 200114, Uzbekistan
North–Caucasus Center for Mathematical Research,
Vladikavkaz Scientific Centre, RAS,
Williams str. 1,
363110, Vladikavkaz, Russian Federation
E-mail: durdievdd@gmail.com, durdiev65@mail.ru