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# PRIMARY SUBMODULES DUAL TO INVARIANT SUBSPACES OF $\Omega$–ULTRADIFFERENTIABLE FUNCTIONS

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**Abstract.** In this work we consider locally convex spaces of entire functions dual to general spaces of $\Omega$–ultradifferentiable functions. These spaces are also topological modules over the polynomial ring, which makes the problem on local description of their closed subspaces invariant under multiplication by the independent variable, in short, submodules, meaningful. More precisely, we study primary submodules, that is, those generated by a single function. It is shown that among principal ones there are submodules that do not admit a local description in the weak sense. Each result on submodules, in turn, leads to an equivalent dual statement on the (non)admissibility of spectral synthesis in the weak sense for subspaces of $\Omega$–ultradifferentiable functions invariant under the differentiation operator.

**Keywords:** entire function, zero set, submodule, local description, ultradistribution, Fourier–Laplace transform.

**Mathematics Subject Classification:** 30D15, 42A38, 46F12

## 1. Introduction

In this work we study some questions arising within the framework of the well–known problem on local description of ideals and submodules. For a locally convex space of entire functions $\mathcal{P}$ having the structure of a topological module over the polynomial ring $\mathbb{C}[z]$, this problem consists in determining whether it is possible to recover a closed submodule $\mathcal{J} \subset \mathcal{P}$ by the set of common zeros of the functions in this submodule.

We consider closed submodules in the modules $\mathcal{P}_\Omega(a;b)$, which arise as analytic realizations of the strong dual spaces for spaces of $\Omega$–ultradifferentiable functions (in short, $\Omega$–UDF) in the general theory of $\Omega$–UDF and $\Omega$–ultradistributions proposed by Abanin [1], [2].

Precise formulations of the definitions and some facts from the theory of $\Omega$–UDF will be given in the beginning of the next section; now we only note that the spaces considered in this work are determined by an increasing sequence of *non–quasianalytic weights* $\Omega = \{\omega_n\}$. Unless otherwise is stated, all considered submodules consideration are closed in $\mathcal{P}_\Omega(a;b)$, so for brevity we shall omit the word "closed".

The conditions imposed on the weight sequence $\Omega$ imply the following properties of the elements of the space $\mathcal{P}_\Omega(a;b)$: every $\varphi \in \mathcal{P}_\Omega(a;b)$ belongs to the Cartwright class, and therefore is an entire function of exponential type and completely regular growth of order 1 in the entire plane, and the indicator diagram of the function $\varphi$ is a segment of the imaginary axis $\mathrm{i}[c_\varphi; d_\varphi]$.

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A submodule $\mathcal{J} \subset \mathcal{P}_\Omega(a;b)$ determines uniquely its *indicator interval* $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ and *zero set* $\mathcal{Z}_{\mathcal{J}}$. Namely, $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ is the closure in $\overline{\mathbb{R}}$ of the set $\bigcup\limits_{\varphi\in\mathcal{J}} [c_\varphi; d_\varphi]$, and $\mathcal{Z}_{\mathcal{J}}$ is the sequence of multiple points in $\mathbb{C}$ given by $\bigcap\limits_{\varphi\in\mathcal{J}\setminus\{0\}} \mathcal{Z}_\varphi$, where the symbol $\mathcal{Z}_\varphi$ denotes the *zero set of the function* $\varphi$, that is, the set of all points in $\mathbb{C}$ at which $\varphi$ vanishes with some positive multiplicity: $\mathcal{Z}_\varphi = \{\lambda_j\}$, $0 \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \dots$.

A submodule $\mathcal{J}$ is *localizable* if it contains *all* functions $\psi \in \mathcal{P}_\Omega(a;b)$ such that $\mathcal{Z}_\psi \supset \mathcal{Z}_{\mathcal{J}}$.

A submodule $\mathcal{J}$ is *weakly localizable* if it contains *all* functions $\psi \in \mathcal{P}_\Omega(a;b)$ such that $[c_\psi; d_\psi] \subset [c_{\mathcal{J}}; d_{\mathcal{J}}]$ and $\mathcal{Z}_\psi \supset \mathcal{Z}_{\mathcal{J}}$.

According to these definitions, a localizable submodule is completely determined by its zero set, while a weakly localizable submodule is determined by its two characteristics, the zero set and the indicator interval.

Using the properties of the weights $\omega_n \in \Omega$, it is easy to verify that both localizability and weak localizability of a submodule $\mathcal{J}$ imply its *stability* under division by a binomial: if $\psi \in \mathcal{J}$ and the multiplicity of vanishing of the function $\psi$ at some point $\lambda \in \mathbb{C}$ exceeds the multiplicity of this point as an element of the set $\mathcal{Z}_{\mathcal{J}}$, then $\dfrac{\psi}{(z-\lambda)} \in \mathcal{J}$.

In [10] we established the following facts:

1) *a stable submodule $\mathcal{J} \subset \mathcal{P}_\Omega(a;b)$ is localizable if and only if its indicator interval* $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ *equals* $[a;b]$ [10, Corollary 4.1]*;*
2) *in the case* $[c_{\mathcal{J}}; d_{\mathcal{J}}] \subsetneq [a;b]$ *we have*
   a) *the condition* $r(\mathrm{i}\mathcal{Z}_{\mathcal{J}}) < \dfrac{d_{\mathcal{J}} - c_{\mathcal{J}}}{2}$, *where* $r(\mathrm{i}\mathcal{Z}_{\mathcal{J}})$ *is the completeness radius of the sequence* $(\mathrm{i}\mathcal{Z}_{\mathcal{J}})$ *is sufficient for the weak localizability of the submodule* $\mathcal{J}$ [10, Thm. 4.2]*;*
   b) *if* $r(\mathrm{i}\mathcal{Z}_{\mathcal{J}}) > \dfrac{d_{\mathcal{J}} - c_{\mathcal{J}}}{2}$, *then* $\mathcal{J} = \{0\}$.

The question that has not yet received a definitive answer concerns the conditions for weak localizability of a stable submodule $\mathcal{J}$ with a critical relation between the characteristics

$$r(\mathrm{i}\mathcal{Z}_{\mathcal{J}}) = \frac{d_{\mathcal{J}} - c_{\mathcal{J}}}{2}. \tag{1.1}$$

Let us describe in more detail the current state–of–art. In order to do this, we recall the definition of the principal submodule $\mathcal{J}_\varphi$ generated by a function

$$\mathcal{J}_\varphi := \overline{\{p\varphi : \ p \in \mathbb{C}[z]\}}.$$

For a sufficiently wide class of weight sequences $\Omega$ one can find weakly localizable stable submodules satisfying (1.1) (see [6, Prop. 1.11]). Also, it is easy to verify that for each non–quasianalytic sequence $\Omega = \{\omega_n\}_{n=1}^\infty$ admissible for the $\Omega$–UDF theory, under the condition $0 \in (a;b)$, the examples of such submodules is the principal submodule generated in $\mathcal{P}_\Omega(a;b)$ by the polynomial $\psi_0$, and if we suppose that the weights $\omega_n$ are *canonical* (for the definition see, e.g., in [1, Ch. 1]), then the principal submodule generated by the function $\psi_0(z) = p(z)\sin cz$, $p \in \mathbb{C}[z]$, is weakly localizable for each $c > 0$ such that $[-c;c] \subset (a;b)$. In the first of the indicated cases the indicator interval of the submodule $\mathcal{J}_{\psi_0}$ consists of a single point 0, in the second it is equal to $[-c;c]$.

On the other hand, for the particular case when the weight sequence $\Omega$ has the form $\{n\omega\}_{n=1}^\infty$, the first of the authors of the present work established the existence of stable but not weakly localizable submodules.

**Theorem 1.1** ([6]). *Let $\Omega = \{n\omega\}_{n=1}^{\infty}$, $\omega$ be the weight function, and the generator of the submodule $\mathcal{J}_\varphi \subset \mathcal{P}_\Omega(-a;a)$ reads $\varphi = \dfrac{\Phi}{f}$, where $\Phi$ is a sine–type function, and $f$ is an entire function of minimal type of order* 1.

*If the orders of the function $f$ on the rays $\arg z = 0$ and $\arg z = \pi$, defined by the identities*

$$\rho_0 = \limsup_{r\to+\infty} \frac{\ln\ln|f(r)|}{\ln r}, \qquad \rho_\pi = \limsup_{r\to+\infty} \frac{\ln\ln|f(-r)|}{\ln r},$$

*respectively satisfy one of the relations*

$$\rho_\omega \leqslant \rho_0 < \frac{1}{4} < \frac{1}{2} \leqslant \rho_\pi \quad \text{or} \quad \rho_\omega \leqslant \rho_\pi < \frac{1}{4} < \frac{1}{2} \leqslant \rho_0,$$

*then the primary submodule $\mathcal{J}_\varphi := \overline{\{p\varphi:\ p \in \mathbb{C}[z]\}}$ is not weakly localizable.*

We recall that the *order* of a weight function $\omega$ is the quantity $\rho_\omega = \overline{\lim\limits_{x\to\infty}} \dfrac{\ln\omega(x)}{\ln x}$.

An example of a function for which the conditions of Theorem 1.1 are satisfied is the function $\dfrac{\sqrt{z}\sin\pi z}{\sin\pi\sqrt{z}}$ for $a > \pi$.

The main result of the present work, Theorem 2.1, is a statement similar to Theorem 1.1, but for a much more general situation, the module $\mathcal{P}_{\Omega,a}$ generated by an arbitrary increasing non–quasianalytic regular weight sequence $\Omega$. On the whole, the results obtained here complement and generalize the studies carried out earlier in [6], [10], [11].

The article is organized as follows. In Section 2 we present the preliminary information necessary for the precise formulation of the results and for the further exposition, as well as the actual formulations of the main Theorem 2.1 and its corollaries. In Section 3 we carry out auxiliary analytic work, the construction of a special entire function used in the proof of the main result. Section 4 contains the proofs of Theorem 2.1 and its two corollaries, and Section 5 provides the statements dual to Theorem 2.1 and Corollaries 2.1, 2.2 on differentiation–invariant subspaces of $\Omega$–UDF.

## 2. Preliminary information and formulations of main results

*A weight function* or *weight* is a Lebesgue measurable, locally bounded on $\mathbb{R}$ function $\omega : \mathbb{R} \to [0;\infty)$ such that

$$\int\limits_{\mathbb{R}} e^{\omega(t)}\,dt < \infty, \qquad \int\limits_{1}^{\infty} \frac{\overline{\omega}(t)}{t^2}\,dt < \infty, \qquad \overline{\omega}(t) := \sup\{\omega(s):\ |s| \leqslant t\};$$

the second of the above inequalities is the *non–quasianalyticity condition* for the weight $\omega$.

For given weight $\omega$ and segment $[c;d] \subset \mathbb{R}$ we define

$$H_\omega(x+\mathrm{i}y) = \frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{\omega(x+\xi y)}{1+\xi^2}\,d\xi,$$

$$\|\varphi\|_{\omega,[c;d]} := \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{\exp(d(\operatorname{Im} z)^+ + c(-\operatorname{Im} z)^+ + H_\omega(-z))},$$

$$\mathcal{P}_{\omega,[c;d]} = \left\{\varphi \in H(\mathbb{C}) : \|\varphi\|_{\omega,[c;d]} < \infty\right\}.$$

We let

$$\mathcal{P}_\Omega(a;b) := \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} \mathcal{P}_{\omega_n,[c_k;d_k]},$$

where $\{[c_k; d_k]\}_{k=1}^{\infty}$ is a sequence exhausting the interval $(a; b)$, $\Omega = \{\omega_n\}_{n=1}^{\infty}$ is a regular non–quasianalytic weight sequence such that $\omega_n(x) + \ln(1 + |x|) \leqslant \omega_{n+1}(x) + C_n$, $x \in \mathbb{R}$, $C_n$, $n \in \mathbb{N}$, are positive constants, see [1, Sect. 2.3].

In what follows, without loss of generality, instead of $(a; b)$ we shall consider the symmetric interval $(-a; a)$ and denote the corresponding space $\mathcal{P}_\Omega(-a; a)$ by the symbol $\mathcal{P}_{\Omega,a}$.

Let $\mathcal{D}_{\Omega,a}$ and $\mathcal{E}_{\Omega,a}$ be the spaces of *test* $\Omega$–UDF and of *all* $\Omega$–UDF on the interval $(-a; a)$, respectively (for precise definitions and properties see [1, Sects. 2.1, 4.1], [2]). The strong dual of $\mathcal{E}_{\Omega,a}$ is the space of all $\Omega$–ultradistributions with compact supports in the interval $(-a; a)$. Moreover, $\mathcal{D}_{\Omega,a} \subset \mathcal{E}'_{\Omega,a}$ in the sense that every function $g \in \mathcal{D}_{\Omega,a}$ generates a functional $S_g \in \mathcal{E}'_{\Omega,a}$, acting by the rule

$$S_g(f) = \int\limits_{(-a;a)} f(t)g(t)\, dt, \qquad f \in \mathcal{E}_{\Omega,a}.$$

The Fourier — Laplace transform $\mathcal{F}$ for an $\Omega$–ultradistribution with compact support $S \in \mathcal{E}'_{\Omega,a}$, as usual, is defined by the formula

$$\mathcal{F}(S)(z) = S(e^{-\mathrm{i}tz}).$$

For the spaces of test $\Omega$–UDF $\mathcal{D}_{\Omega,a}$ and $\Omega$–ultradistributions with compact supports $\mathcal{E}_{\Omega,a}$ in [1] analogues of the Paley — Wiener and Paley — Wiener — Schwartz theorems were proved. Namely, according to Theorem 2.4.2 in [1], the operator $\mathcal{F}$ establishes a linear topological isomorphism between $\mathcal{D}_{\Omega,a}$ and $\mathcal{H}_{\Omega,a}$, where

$$\mathcal{H}_{\Omega,a} = \bigcup_{n=1}^{\infty} \mathcal{H}_n,$$
$$\mathcal{H}_n := \left\{ \varphi \in H(\mathbb{C}) : \ \|\varphi\|_n := \sup_{z\in\mathbb{C}} \frac{|\varphi(z)|}{\exp(c_n|\operatorname{Im} z| - H_{\omega_n}(z))} \right\}, \quad 0 < c_n \nearrow a.$$

According to Theorem 5.4.2 in the same monograph, the Fourier — Laplace transform is a linear topological isomorphism between the spaces $(\mathcal{E}_{\Omega,a})'$ and $\mathcal{P}_{\Omega,a}$.

Hereafter we assume that the weight sequence $\Omega = \{\omega_n\}_{n=1}^{\infty}$ defining the module $\mathcal{P}_{\Omega,a}$ satisfies the following additional condition:

$$\omega_n(x) \leqslant \nu(|x|) + C_n, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{2.1}$$

where $C_n$, $n \in \mathbb{N}$, are positive constants, and the function $\nu : [0; \infty) \to [0; \infty)$ has the form

$$\nu(t) = \begin{cases} 0, & t = 0, \\ t^{\rho(t)}, & t > 0, \end{cases}$$

$\rho(t)$ is the refined order [8, Sect. 12, Ch. I] and

$$\lim_{t\to\infty} \rho(t) = \frac{1}{2}, \qquad \int\limits_1^{\infty} \frac{\nu(t)}{t^{\frac{3}{2}}}\, dt < \infty. \tag{2.2}$$

As an appropriate example of $\nu$ the function $\frac{t^{\frac{1}{2}}}{\ln^p t}$, $p > 1$, serves.

We recall the properties of the refined order that we will need in what follows:
the limits are well–defined (in our case $\rho = \frac{1}{2}$)

$$\lim_{t\to\infty} \rho(t) = \rho \geqslant 0, \qquad \lim_{r\to\infty} r\rho'(r) \ln r = 0, \tag{2.3}$$

the function $L(t) := t^{\rho(t)-\rho}$ is *slowly varying*, that is

$$\lim_{t\to\infty} \frac{L(kt)}{L(t)} = 1, \tag{2.4}$$

uniformly on each fixed segmet $0 < k_0 \leqslant k \leqslant k_1 < \infty$.

By the symbol $\mathcal{S}_b$, where $b > 0$ is a fixed number, we denote the class of *sine–type functions having exponential type equal to* $b$, that is

$$\varphi \in S_b \Longleftrightarrow \begin{cases} \varphi \in H(\mathbb{C}), \\ \exists\, C_\varphi \geqslant 0: \ \ln|\varphi(z)| \leqslant b|\operatorname{Im} z| + C_\varphi, \quad z \in \mathbb{C}, \\ \exists\, c_\varphi, d_\varphi \in \mathbb{R}: \ \ln|\varphi(x + \mathrm{i}d_\varphi)| \geqslant c_\varphi, \quad x \in \mathbb{R}. \end{cases}$$

One of the properties of functions $\varphi \in S_b$ is that the zero set of each of them is contained in a horizontal strip, in other words, for some $d_0 > 0$, depending only on $\varphi$, we have

$$\varphi(z) \neq 0, \qquad |\operatorname{Im} z| \geqslant d_0. \tag{2.5}$$

We recall that for $x \in (0; +\infty)$ one denotes $\ln^+ x = \max\{\ln x, 0\}$ and $\ln^- x = \max\{-\ln x, 0\}$.

Now we are in position to formulate the main result on the existence of non–weakly localizable principal submodules in the module $\mathcal{P}_{\Omega,a}$.

**Theorem 2.1.** *Let $a > \pi$ and let the function $\varphi_0 \in \mathcal{P}_{\Omega,a}$ have the form*

$$\varphi_0 = \frac{\Phi}{\widetilde{\varphi}},$$

*where $\Phi \in \mathcal{S}_\pi$, $\widetilde{\varphi}$ is an entire function of minimal type of order* 1.

*Assume that the following conditions are satisfied:*

$$\varlimsup_{x\to+\infty} \frac{\ln|\widetilde{\varphi}(-x)|}{x^{\frac{1}{2}}} > 0, \tag{2.6}$$

$$\int_1^\infty \frac{\ln^+|\widetilde{\varphi}(x)|}{x^{\frac{3}{2}}}\, dx < \infty. \tag{2.7}$$

*Then the primary submodule $\mathcal{J}_{\varphi_0}$ is not weakly localizable.*

This theorem imply two corollaries. The first of them provides an example illustrating the essence of the matter, while the second allows one to extend the class of functions $\varphi \in \mathcal{P}_{\Omega,a}$ from Theorem 2.1 that generate non–weakly localizable principal submodules.

**Corollary 2.1.** *Let $a > \pi$ and let the weight sequence $\Omega$ be the same as in Theorem 2.1. The function*

$$\varphi_0(z) = \frac{\sin \pi z}{\sqrt{z}\sin \pi\sqrt{z}}$$

*is contained in $\mathcal{P}_{\Omega,a}$ and the primary submodule $\mathcal{J}_{\varphi_0}$ is not weakly localizable.*

**Corollary 2.2.** *Let $a > \pi$ and the weight sequence $\Omega$ be the same as in Theorem 2.1, $\Phi \in \mathcal{S}_\pi$, $\widetilde{\varphi}$ be an entire function of minimal type of order 1 obeying conditions (2.6), (2.7), $a' > 0$ and the function $\psi \in \mathcal{P}_{\Omega,a'}$ satisfies at least one of the conditions:*

$$\int_1^\infty \frac{\ln^-|\psi(x)|}{x^{\frac{3}{2}}}\, dx < \infty, \tag{2.8}$$

*or*

$$\varlimsup_{z\to\infty} \frac{\ln|\psi(z)|}{|z|^{\frac{1}{2}}} < \infty, \tag{2.9}$$

*If* $\varphi = \dfrac{\Phi\psi}{\widetilde{\varphi}} \in \mathcal{P}_{\Omega,b}$, $b = a + a'$, *then the primary submodule* $\mathcal{J}_\varphi$ *generated by this function in* $\mathcal{P}_{\Omega,b}$ *is not weakly localizable.*

**Remark 2.1.** *Let us illustrate the above statement by some examples.*

1. *It is easy to deduce from the general theory of growth of entire functions that the assumptions of Theorem 2.1 hold for the functions* $\varphi_1$, $\varphi_2$, *defined as follows*
$$\varphi_1(z) = \frac{\sin \pi z}{\sqrt{z} s_1(z) \sin \pi\sqrt{z}},$$
*where* $s_1$ *is the canonical product over some subset of* $\mathbb{Z} \setminus \{n^2\}_{n=0}^{\infty}$ *having a finite density of some order* $\rho_0 < \frac{1}{2}$,
$$\varphi_2(z) = \frac{\sin \pi z}{s_2(z)},$$
*and* $s_2(z)$ *is the ratio of* $\sqrt{z}\sin\pi\sqrt{z}$ *and the canonical product over a subset of* $\{n^2\}_{n=1}^{\infty}$ *having a finite density of some order* $\rho_0 < \frac{1}{2}$.
2. *Let for some* $n_1 < n_2$ *the relation holds*
$$\ln|x| = o(\omega_{n_2}(x) - \omega_{n_1}(x)), \quad |x| \to \infty. \tag{2.10}$$
*We define the function* $u(e^t)$ *as the greatest convex minorant of the function*
$$\min\{\omega_{n_2}(e^t) - \omega_{n_1}(e^t), \omega_{n_2}(-e^t) - \omega_{n_1}(-e^t)\}.$$
*Under condition (2.10), an example of a function* $\psi$ *satisfying the assumptions of Corollary 2.2 is the entire function* $\psi$ *such that the absolute value of its logarithm approximates the subharmonic function* $u(|z|)$ *with a logarithmic accuracy in the entire complex plane, outside an exceptional set of disks with finite sum of radii. The possibility of such an approximation was established in [9].*
3. *Let the function* $\varphi_0 = \dfrac{\Phi}{\widetilde{\varphi}}$ *from Theorem 2.1 belong to the* Schwarz algebra *(that is, it grows on* $\mathbb{R}$ *no faster than a polynomial) and for some* $n_0 \in \mathbb{N}$ *the condition*
$$\ln|x| = o(\omega_{n_0}(x)), \quad |x| \to \infty, \tag{2.11}$$
*holds. For each* $n \geqslant n_0$ *we denote by* $u_n(e^t)$ *the greatest convex minorant of the function* $f_n(t) := \min\{\omega_n(e^t), \omega_n(-e^t)\}$, *which is different from a linear function due to (2.11). It is easy to verify that under the made assumptions, as the function* $\psi$ *in Corollary 2.2 one may take any entire function* $\psi_n$ *such that* $\ln|\psi_n(z)|$ *approximates the subharmonic function* $u_n(|z|)$ *with a logarithmic accuracy in the entire complex plane, outside an exceptional set of disks with finite sum of radii.*

**Remark 2.2.** *The statements of Corollary 2.1* and, in non–strict form, *of Theorem 2.1* for a *narrower class of weight sequences* $\Omega$ *were recently announced in [7] (Theorem 1 and Remark 1, respectively).*

## 3. Auxiliary statements

In this section we carry out some additional constructions. We assume that $a > \pi$ and the weight sequence $\Omega$ satisfies the same assumptions as in Theorem 2.1. Fix an arbitrary $N_0 \in \mathbb{N}$ such that $c_0 := \pi/N_0 < \frac{(a-\pi)}{2}$.

**Proposition 3.1.** *There exists an entire function* $\psi_0$ *satisfying the conditions*

1) $\mathcal{Z}_{\psi_0} \subset \{N_0 k\}_{k\in\mathbb{Z}'}$, *where, as usually,* $\mathbb{Z}' = \mathbb{Z}\setminus\{0\}$*;*
2) $\psi_0 \in \mathcal{F}(\mathcal{D}_{\Omega,c_1})$ *for each* $c_1 > c_0$*;*
3) *for every sufficiently small* $\delta > 0$ *there exists a constant* $m_\delta > 0$*, such that for each* $\varepsilon > 0$ *we have*

$$\ln|\psi_0(z)| \geqslant c_0|\operatorname{Im} z| - (\pi+\varepsilon)\Big(\sin\Big(\frac{\theta}{2}\Big) + \Big|\cos\Big(\frac{\theta}{2}\Big)\Big|\Big)\nu(r) - m_\delta, \tag{3.1}$$

*where*

$$z = re^{\mathrm{i}\theta} \notin \bigcup_{k\in\mathbb{Z}'}\{|z-N_0k|\leqslant\delta\}, \quad r\geqslant r_\varepsilon>0, \quad \theta\in[0;2\pi]. \tag{3.2}$$

The proof of Proposition 3.1 is based on the following lemma, which in turn relies on the classical Levin — Pfluger theory of growth of entire functions.

**Lemma 3.1.** *Let* $\rho(t)$ *be a refined order,*

$$\lim_{t\to\infty}\rho(t) = \frac{1}{2}, \qquad \nu(t) = t^{\rho(t)}, \qquad N_0\in\mathbb{N}.$$

*Then there exist a number* $d_1 > 0$ *and a sequence*

$$\Lambda = \{\lambda_k\} \subset \{N_0, 2N_0, 3N_0, \dots\},$$

*such that the formula*

$$s(z) = \prod_{k=1}^{\infty}\Big(1-\frac{z}{\lambda_k}\Big) \tag{3.3}$$

*defines an entire function of type* $\pi$ *with refined order* $\rho(t)$*, which satisfies the relation*

$$\lim_{r\to\infty}\frac{\ln|s(re^{\mathrm{i}\theta})|}{\nu(r)} = \pi\sin\Big(\frac{\theta}{2}\Big) \tag{3.4}$$

*holds everywhere in* $\mathbb{C}\setminus\mathcal{C}_0$*, where* $\mathcal{C}_0 = \bigcup D_k$ *is an exceptional set of circles*

$$D_k := \Big\{z : |z-\lambda_k| \leqslant \frac{d_1\lambda_k}{\nu(\lambda_k)}\Big\}, \tag{3.5}$$

*mutually disjoint for all* $k\geqslant k_0$ *and such that each* $D_k$ *contains exactly one element of the set* $\Lambda$*, namely the point* $\lambda_k$*; moreover, the convergence in the limit in* (3.4) *is uniform in* $\theta\in[0;2\pi]$*.*

*Proof.* By the properties of the refined order (see [8, Sect. 12, Ch. I]) the function $\nu$ is strictly increasing for sufficiently large values of the variable, say, for $t\geqslant t_0$. Hence, for such $t$ the inverse function $t = l(s)$ is uniquely defined and it is also strictly increasing for $s\geqslant s_0 := \nu(t_0)$. Without loss of generality, we assume that $s_0 = 1$.

We let $\tilde\lambda_k = l(k)$, $k\in\mathbb{N}$, and denote by $n_{\widetilde\Lambda}(t)$ the number of points in the set $\widetilde\Lambda = \{\tilde\lambda_k\}$ in the circle $|z|\leqslant t$ (equivalently, in the interval $(0;t]$). It is obvious that

$$n_{\widetilde\Lambda}(\tilde\lambda_k) = k = \nu(\tilde\lambda_k), \quad k\in\mathbb{N}, \qquad \nu(t)-1\leqslant n_{\widetilde\Lambda}(t)\leqslant\nu(t), \quad t>0,$$

and hence, the following limit is well–defined

$$\lim_{t\to\infty}\frac{n_{\widetilde\Lambda}(t)}{\nu(t)} = 1.$$

By the representation

$$\tilde\lambda_{k+1}-\tilde\lambda_k = \nu^{-1}(k+1)-\nu^{-1}(k) = \frac{1}{\nu(\tilde\lambda_k^*)(\rho(t)\ln t)'|_{t=\tilde\lambda_k^*}}, \qquad \tilde\lambda_k^*\in[\tilde\lambda_k;\tilde\lambda_{k+1}],$$

and the properties of the refined order we easily deduce that there exist $\tilde{d}_1 > 0$ and $\tilde{k}_0 \in \mathbb{N}$ such that the circles

$$\widetilde{D}_k := \left\{ z : |z - \tilde{\lambda}_k| \leqslant \frac{\tilde{d}_1 \tilde{\lambda}_k}{\nu(\tilde{\lambda}_k)} \right\},$$

$k \geqslant \tilde{k}_0$, are mutually disjoint and each of them contains exactly one point of the set $\widetilde{\Lambda}$.

Now we define the set $\Lambda = \{\lambda_k\}$ as follows: the point $\lambda_k$ is the element of the set $\{N_0, 2N_0, \dots\}$ that is right closest to the corresponding point $\tilde{\lambda}_k \in \widetilde{\Lambda}$. Clearly, then

$$\tilde{\lambda}_k - \lambda_k = O(1), \quad k \to \infty.$$

It is also easy to verify that

$$\nu(t) - n_\Lambda(t) = O(1), \quad t \to \infty,$$

and hence,

$$\lim_{t\to\infty} \frac{n_\Lambda(t)}{\nu(t)} = 1.$$

By the above arguing on the circles $\widetilde{D}_k$ and the definition of the set $\Lambda$ we conclude that for some $k_0 \in \mathbb{N}$ and $d_1 > 0$ the circles $D_k$, defined by formula (3.5), are mutually disjoint for all $k \geqslant k_0$ and each of them contains exactly one point $\lambda_k$ of the set $\Lambda$.

According to Theorem 5 in [8, Sect. 1, Ch. II], the formula

$$s(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) \tag{3.6}$$

defines an entire function of type $\pi$ with refined order $\rho(t)$, such that relation (3.4) holds everywhere in $\mathbb{C} \setminus \mathcal{C}_0$, where $\mathcal{C}_0 = \bigcup\limits_{k\in\mathbb{N}} D_k$, and the convergence in the limit in (3.4) as $r \to \infty$, $z = re^{\mathrm{i}\theta} \notin \mathcal{C}_0$, is uniform in $\theta \in [0; 2\pi]$. The proof is complete. □

*Proof of Proposition 3.1.* We let

$$\psi_0(z) = \frac{\sin c_0 z}{s(z)s(-z)},$$

where $s$ is the function constructed in Lemma 3.1. Then, in view the definition of the exceptional disks (3.5), for each $\varepsilon > 0$ by relation (3.4) we can find $r'_\varepsilon > 0$ such that for all $r \geqslant r'_\varepsilon$ and $\theta \in [0; 2\pi]$ we have

$$\ln|s(re^{\mathrm{i}\theta})s(-r\mathrm{i}\theta)| \leqslant \left(\pi + \frac{\varepsilon}{2}\right)\left(\sin\left(\frac{\theta}{2}\right) + \left|\cos\left(\frac{\theta}{2}\right)\right|\right)\nu(\tilde{r}),$$

where

$$\tilde{r} = r\left(1 + \frac{d_1/(\nu(r))}{1 - d_1/(\nu(r))}\right).$$

Using properties (2.3), (2.4) of the refined order, we thus obtain that the estimate

$$\ln|s(re^{\mathrm{i}\theta})s(-r\mathrm{i}\theta)| \leqslant (\pi + \varepsilon)\left(\sin\left(\frac{\theta}{2}\right) + \left|\cos\left(\frac{\theta}{2}\right)\right|\right)\nu(r)$$

holds for all $r \geqslant r_\varepsilon \geqslant r'_\varepsilon$ and $\theta \in [0; 2\pi]$. By this fact and the well–known lower bounds for the function $\sin c_0 z$ as an element of the class of sine–type functions [14] we conclude that the estimate (3.1) holds for all $z \in \mathbb{C}$ determined by relations (3.2).

Now let us prove that $\psi_0 \in \mathcal{F}(\mathcal{D}_{\Omega,c_1})$ for each fixed $c_1 > c_0$. In order to do this, we note that identity (3.4) implies the lower estimate

$$\ln|s(re^{\mathrm{i}\theta})s(-re^{\mathrm{i}\theta})| \geqslant \left(\pi - \frac{\varepsilon}{2}\right)\left(\sin\left(\frac{\theta}{2}\right) + \left|\cos\left(\frac{\theta}{2}\right)\right|\right)\nu(\hat{r}), \tag{3.7}$$

where

$$\hat{r} = r\left(1 - \frac{d_1/(\nu(r))}{1 - d_1/(\nu(r))}\right).$$

Estimate (3.7) holds for each $\varepsilon > 0$, provided that $r \geqslant r''_\varepsilon > 0$ and $z = re^{\mathrm{i}\theta} \notin \mathcal{C}_0 \bigcup (-\mathcal{C}_0)$.

In view of the inequality

$$\ln|\sin c_0 z| \leqslant c_0|\operatorname{Im} z|, \quad z \in \mathbb{C},$$

and the properties (2.3), (2.4) of the refined order we obtain the estimate

$$\ln|\psi_0(z)| \leqslant c_0|\operatorname{Im}\, z| - (\pi - \varepsilon)\left(\sin\left(\frac{\theta}{2}\right) + \left|\cos\left(\frac{\theta}{2}\right)\right|\right)\nu(|z|) + C_{\psi_0,\varepsilon}, \tag{3.8}$$

for all $z \in \mathbb{C}$ with a positive constant $C_{\psi_0,\varepsilon}$.

It is obvious that for sufficiently small fixed $\varepsilon > 0$ for all $\theta \in [0; 2\pi]$ we have

$$(\pi - \varepsilon)\left(\sin\left(\frac{\theta}{2}\right) + \left|\cos\left(\frac{\theta}{2}\right)\right|\right) \geqslant 1.$$

Therefore, by inequality (3.8), condition (2.1) and the above cited analogue of the Paley — Wiener theorem for test $\Omega$–UDF, we conclude that $\psi_0 \in \mathcal{H}_{\Omega,c_1} = \mathcal{F}(\mathcal{D}_{\Omega,c_1})$. The proof is complete. □

## 4. Proof of main results

**4.1. Proof of Theorem 2.1.** We choose and fix $d_0 > 0$ such that $\Phi(z) \neq 0$ for all $z$ with $|\operatorname{Im}\, z| \geqslant d_0$ (see (2.5)).

Assume that the submodule $\mathcal{J}_{\varphi_0}$ is weakly localizable. Then, in particular, there must exist a *generalized* sequence of polynomials $\{p_\alpha\}$ such that

$$p_\alpha \varphi_0 \to \Phi, \tag{4.1}$$

in the topology of the space $\mathcal{P}_{\Omega,a}$.

We define the $\Omega$–ultradistribution $g_0 \in \mathcal{E}'_{\Omega,a}$ and the function $g_1 \in L^2(-\pi;\pi) \subset \mathcal{E}'_{\Omega,a}$ by the formula

$$\varphi_0 = \mathcal{F}(g_0), \qquad \Phi = \mathcal{F}(g_1).$$

Their existence is guaranteed by the analogue of the Paley — Wiener — Schwartz theorem for $\Omega$–ultradistributions [1, Thm. 5.4.2] and the classical Paley — Wiener theorem, respectively. Then we can write

$$g_\alpha := p_\alpha(\mathrm{i}D)(g_0) \in \mathcal{E}'_{\Omega,a}, \qquad p_\alpha\varphi_0 = \mathcal{F}(g_\alpha),$$

where the symbol $D$ denotes the operator of (generalized) differentiation of $\Omega$–ultradistributions.

Let $f_0 \in \mathcal{D}_{\Omega,c_1}$ be a test $\Omega$–UDF such that $\psi_0 = \mathcal{F}(f_0)$, where $\psi_0$ is the function constructed in Proposition 3.1. As it was mentioned above, the $\Omega$–UDF $f_0$ with compact support in $[-c_0; c_0]$ can be identified with an element in the strong dual of the space $\mathcal{E}_{\Omega,c_1}$, $c_1 > c_0$, which we shall also denote by $f_0$; and hence,

$$f_0(f) = \int\limits_{\mathbb{R}} f_0(t)f(t)\,dt, \quad f \in \mathcal{E}_{\Omega,c_1}.$$

We observe that

$$(p_\alpha\varphi_0 - \Phi)\psi_0 = \mathcal{F}(g_\alpha - g_1)\mathcal{F}(f_0) = \mathcal{F}\left(S_\alpha * f_0\right), \tag{4.2}$$

where $S_\alpha := g_\alpha - g_1$, and $S_\alpha * f_0$ is the convolution of $\Omega$–ultradistributions with compact supports defined by the formula

$$(S_\alpha * f_0)(f) = (S_\alpha)_y\big((f_0)_t(f(y+t))\big), \quad f \in \mathcal{E}_{\Omega,\pi+c_1}, \quad \pi + c_1 \leqslant a.$$

Set $\tilde{f}_0(t) = f_0(-t)$ and denote by $M_\alpha$ the convolution operator generated by the $\Omega$–ultradistribution $S_\alpha$ and acting in $\mathcal{E}_\Omega(\mathbb{R})$ by the formula

$$M_\alpha(f)(h) = S_\alpha(f(\cdot + h)).$$

Then we can rewrite (4.2) as

$$(p_\alpha\varphi_0 - \Phi)\psi_0 = \mathcal{F}\left(M_\alpha(\tilde{f}_0)\right). \tag{4.3}$$

By virtue of the isomorphism between $\mathcal{P}_{\Omega,a}$ and $\mathcal{E}'_{\Omega,a}$, established via the Fourier — Laplace transform, relation (4.1) is equivalent to the fact that the generalized sequence $S_\alpha$ converges to 0 in the strong topology of the dual space $\mathcal{E}'_{\Omega,a}$. Neighbourhoods of zero in this topology are the polars $B^0$ of bounded sets $B \subset \mathcal{E}_{\Omega,a}$. As the set $B$ we take the collection of shifts

$$B := \{\tilde{f}_0(\cdot + \tau) : \ \tau \in [-\pi;\pi]\},$$

their boundedness is implied by the compact support of $\tilde{f}_0$ and Proposition 1.2.2 in monograph [1], and we find the value of the index $\alpha_B$ with the property

$$\sup_{f\in B} |S_\alpha(f)| < 1, \quad \alpha \succ \alpha_B \tag{4.4}$$

The compactness of supports of $\tilde{f}_0$ and $S_\alpha$ yields $M_\alpha(\tilde{f}_0) \in \mathcal{D}_{\Omega,a}$ for all $\alpha$, and by (4.3) and (4.4) we obtain the inequality

$$\sup_{x\in\mathbb{R}} |p_\alpha(x)\varphi_0(x)\psi_0(x)| < \infty$$

for all $\alpha \succ \alpha_B$.

This estimate, together with inequality (3.1) and lower estimates for sine-type functions (see, e.g., [14, 22.1, Lm. 1]), implies the existence of a constant $C_1 > 0$, such that for all $x \in \mathbb{R}$ and $\alpha \succ \alpha_B$ the inequality

$$|p_\alpha(x)| \leqslant C_1 e^{\widetilde{\nu}(x)} \tag{4.5}$$

holds, where

$$\widetilde{\nu}(x) = (\pi+1)\nu(x) + \ln^+ |\widetilde{\varphi}(x)|,$$

and by (2.2) and (2.7),

$$\int_1^\infty \frac{\widetilde{\nu}(x)}{x^{\frac{3}{2}}}\, dx < \infty. \tag{4.6}$$

Letting $q_\alpha(w) = p_\alpha(w^2)$, $\alpha \succ \alpha_B$, we see that

$$|q_\alpha(t)| \leqslant C_2 e^{\widetilde{\nu}(t^2)}, \quad t \in \mathbb{R}, \quad \alpha \succ \alpha_B, \tag{4.7}$$

and we can write (4.6) in the form

$$\int_1^\infty \frac{\widetilde{\nu}(t^2)}{1+t^2}\, dt < \infty.$$

Now we apply the arguing from [15, Ch. VI, Sect. B.2] to obtain upper estimates for the moduli of the polynomials $q_\alpha$ in the entire plane. Using this arguing, by (4.7) we deduce that for each $\varepsilon > 0$ there exists a constant $M_\varepsilon > 0$, independent of $w$ and $\alpha$, such that

$$|q_\alpha(w)| \leqslant M_\varepsilon e^{\varepsilon|w|}, \quad w \in \mathbb{C}, \quad \alpha \succ \alpha_B.$$

This yields

$$|p_\alpha(z)| \leqslant M_\varepsilon e^{\varepsilon|z|^{\frac{1}{2}}}, \quad z \in \mathbb{C}, \quad \alpha \succ \alpha_B. \tag{4.8}$$

In order to arrive at a contradiction with (4.1), note that condition (2.6), the inclusions

$$\mathcal{Z}_{\widetilde{\varphi}} \subset \mathcal{Z}_\Phi \subset \{z : \ |\operatorname{Im} z| < d_0\},$$

properties of sine–type functions and the definition of the number $d_0$ imply the inequalities

$$\begin{aligned} &\varlimsup_{z\to\infty} \frac{\ln|\widetilde{\varphi}(z)|}{|z|^{\frac{1}{2}}} > 0 \qquad \text{if} \qquad z = x + 2\mathrm{i}d_0, \quad x \in \mathbb{R}_-, \\ &|\Phi(z)| \geqslant \text{const} \qquad \text{for all} \quad z = x + 2\mathrm{i}d_0, \quad x \in \mathbb{R}. \end{aligned}$$

In view of the arbitrary choice of $\varepsilon > 0$ in (4.8), we deduce that relation (4.1) cannot hold even in the topology of pointwise convergence in $\mathbb{C}$, which is obviously weaker than the topology of the space $\mathcal{P}_{\Omega,a}$.

The obtained contradiction proves that the submodule $\mathcal{J}_{\varphi_0}$ is not weakly localizable. This completes the proof of Theorem 2.1.

**4.2. Proof of Corollary 2.1.** We proceed to proving Corollary 2.1. It is well–known that $\sin \pi z \in \mathcal{S}_\pi$, and $\widetilde{\varphi} = \sqrt{z}\sin\pi\sqrt{z}$ is an entire function of type $\pi$ of order $\frac{1}{2}$, and for each sufficiently small $\sigma_0 \in (0; \frac{1}{2})$ is satisfies the lower bound (see, for instance, [4, Lm. 1]):

$$|\widetilde{\varphi}(z)| \geqslant c_{\sigma_0} e^{\pi\sqrt{r}\left|\sin\left(\frac{\theta}{2}\right)\right|}, \quad z = re^{\mathrm{i}\theta} \in \mathbb{C} \setminus \bigcup_{k\in\mathbb{N}} \{z : \ |z - k^2| < 3\sigma_0\}.$$

Denoting by $\mathcal{P}_a$ the *Schwarz module* consisting of all entire functions of exponential type less than $a$ and of polynomial growth on the real line, under the assumptions of the corollary we have: $\varphi_0 \in \mathcal{P}_a \subset \mathcal{P}_{\Omega,a}$ for each $a > \pi$, and the function $\widetilde{\varphi}$ satisfies relations (2.6), (2.7). Applying Theorem 2.1, we obtain the desired statement and complete the proof.

**4.3. Proof of Corollary 2.2.** Suppose that condition (2.8) is satisfied. Assume that the submodule $\mathcal{J}_\varphi$ is weakly localizable, and hence there exists a generalized sequence of polynomials $p_\alpha$ such that

$$p_\alpha\varphi \to \Phi\psi \tag{4.9}$$

in the topology of space $\mathcal{P}_{\Omega,b}$.

Arguing in exactly the same way as in the proof of Theorem 2.1, but with the function $\varphi$ instead of $\varphi_0$, we establish the existence of a constant $C_1 > 0$ and a value of the index $\alpha_0$, such that the inequality

$$|p_\alpha(x)| \leqslant C_1 e^{\widetilde{\nu}(x)}, \quad x \in \mathbb{R}, \quad \alpha \succ \alpha_0,$$

holds, where

$$\widetilde{\nu}(x) = (\pi + 1)\nu(x) + \ln^+|\widetilde{\varphi}(x)| + \ln^-|\psi(x)|;$$

and by (2.2), (2.7) and (2.8),

$$\int_1^\infty \frac{\widetilde{\nu}(x)}{x^{\frac{3}{2}}}\, dx < \infty.$$

Proceeding in the same way as in the proof of Theorem 2.1 after formula (4.6), we arrive at a contradiction: relation (4.9) cannot hold even in the topology of pointwise convergence in $\mathbb{C}$.

Now let the generator of the submodule $\mathcal{J}_\varphi$ have the form $\varphi = \dfrac{\Phi\psi}{\widetilde{\varphi}}$, where $\psi \in \mathcal{P}_{\Omega,a'}$ satisfies condition (2.9). Applying the first theorem from [15, III.G.2] to the function of exponential type in the upper half–plane $g(w) := \psi(w^2)$, and taking into consideration the conditions (2.1), (2.2) on the weights $\omega_n$, $\nu$, we deduce that the function $\psi$ also satisfies relation (2.8). Hence, the desired statement follows from the first part of the present proof.

## 5. $D$–invariant subspaces of $\Omega$–UDF not admitting spectral synthesis in the weak sense

Now we formulate the statements on subspaces of $\Omega$–UDF invariant under the differentiation operator, dual to Theorem 2.1 and Corollaries 2.1, 2.2, and related to the studies [6], [10], [11].

Let $D = \frac{d}{dt}$ be the differentiation operator, $W \subset \mathcal{E}_{\Omega,a}$ be a *$D$–invariant subspace*, that is, a closed subspace such that $D(W) \subset W$. By the symbol $\operatorname{Exp} W$ we denote the stock of exponential monomials $t^k e^{-i\lambda t}$ (root elements of the operator $D$) contained in $W$, and $I_W$ stands for the *residual interval* of the subspace $W$ defined as the smallest one among all relatively closed in $(-a; a)$ intervals $I$ with the property

$$W_I := \{g \in \mathcal{E}_{\Omega,a} : \ g|_I = 0\} \subset W.$$

The *problem of spectral synthesis for the operator $D$* consists in describing $D$–invariant subspaces $W$, which are completely determined by their stock of root elements, that is, have the form

$$W = \overline{\operatorname{span}(\operatorname{Exp} W)}. \tag{5.1}$$

The *problem of spectral synthesis in the weak sense for the operator $D$* consists in finding $D$–invariant subspaces $W$, which are represented in the form

$$W = \overline{\operatorname{span}(\operatorname{Exp} W) + W_{I_W}} \tag{5.2}$$

It is easy to see that (5.1) is a special case of (5.2) possible only for $D$–invariant subspaces $W$ with $I_W = (-a; a)$.

The results of our work [10] imply that a *$D$–invariant subspace $W \subset \mathcal{E}_{\Omega,a}$ has the form* (5.1) *if and only if the following conditions hold: $I_W = (-a; a)$ and the spectrum of the restriction of the differentiation operator $D : W \to W$ is discrete.*

Even during the study of the possibility of representation (5.2) for $D$–invariant subspaces in $C^\infty(a; b)$ (see [3], [5], [12], [13]) the considerations were carried out under the condition that the spectrum of the operator $D : W \to W$ is discrete. For spaces of $\Omega$–UDF, the necessity of this requirement was proved by us in [10, Prop. 3.3, 5.1]. A sufficient condition for weak spectral synthesis (5.2) for $D$–invariant subspaces with discrete spectrum in the space $\mathcal{E}_{\Omega,a}$ is the relation $r_W < \frac{|I_W|}{2}$, where $r_W$ is the completeness radius of the system of exponential monomials $\operatorname{Exp} W$, $|I_W|$ is the length of the interval $I_W$ [10, Thm. 2.1]. If $r_W > \frac{|I_W|}{2}$, then $W = \{\mathbf{0}\}$ [10, Props. 3.2, 4.3].

The question of conditions ensuring that $W$ admits spectral synthesis in the weak sense (5.2) in the case $r_W = \frac{|I_W|}{2}$, was not completely studied even for $W \subset C^\infty(a; b)$. In this direction, for $D$–invariant subspaces of $\Omega$–UDF $W \subset \mathcal{E}_{\Omega,a}$ the first of the authors obtained two results: on the existence of $D$–invariant subspaces of $\Omega$–UDF that do not admit weak spectral synthesis in the particular case when $\Omega = \{n\omega\}_{n=1}^\infty$, $\omega$ is a weight function [6, Thm. 1.12] and on the possibility of weak spectral synthesis for a subspace $W$ of a special form, the statement dual to [6, Prop. 1.11].

Theorem 2.1 and Corollaries 2.1, 2.2, in view of the duality between $D$–invariant subspaces in $\mathcal{E}_{\Omega,a}$ and closed submodules in the module $\mathcal{P}_{\Omega,a}$, described in detail by us in [10], lead us to equivalent statements on spectral synthesis.

For a fixed $\Omega$–ultradistribution $S \in \mathcal{E}'_{\Omega,a}$ we let

$$W_S = \{f \in \mathcal{E}_{\Omega,a}: \ S(f^{(k)}) = 0, \ k = 0, 1, \dots\}. \tag{5.3}$$

It is easy to verify that $W_S$ is a $D$–invariant subspace with discrete spectrum, the dual submodule of which is principal with generator $\varphi = \mathcal{F}(S)$. Therefore the admissibility of weak spectral synthesis for $W_S$ is equivalent to the weak localizability of the submodule $\mathcal{J}_\varphi$ [10, Prop. 3.3]. Theorem 2.1 is equivalent to the following statement on $D$–invariant subspaces.

**Theorem 5.1.** *Let the $\Omega$–ultradistribution $S_0 \in \mathcal{E}'_{\Omega,a}$ be such that its Fourier – Laplace transform $\varphi_0 = \mathcal{F}(S_0)$ satisfies the representation*

$$\varphi_0 = \frac{\Phi}{\widetilde{\varphi}},$$

*where $\Phi \in \mathcal{S}_\pi$, $\widetilde{\varphi}$ is an entire function of minimal type of order 1 for which relations (2.6), (2.7) hold.*

*Then the $D$–invariant subspace $W_{S_0}$ is nontrivial, has discrete spectrum, and does not admit spectral synthesis in the weak sense.*

Let us formulate the statements dual to Corollaries 2.1, 2.2.

**Corollary 5.1.** *The $D$–invariant subspace $W_{S_0} \subset \mathcal{E}_{\Omega,a}$, $a > \pi$ generated by the $\Omega$–ultradistribution*

$$S_0 = \mathcal{F}^{-1}\left(\frac{\sin \pi z}{\sqrt{z}\sin \pi\sqrt{z}}\right),$$

*is nontrivial, has discrete spectrum and does not admit weak spectral synthesis.*

**Corollary 5.2.** *Let $a > \pi$, $a' > 0$, $b = a + a'$, $\varphi \in \mathcal{P}_{\Omega,b}$ be a function for which Corollary 2.2 holds. Then the $D$–invariant subspace $W_S \subset \mathcal{E}_{\Omega,b}$ generated by the $\Omega$–ultradistribution $S = \mathcal{F}^{-1}(\varphi)$ according to formula (5.3), is nontrivial, has discrete spectrum and does not admit weak spectral synthesis.*

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