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GEOMETRY OF RICCI TENSOR OF HARMONIC NEARLY TRANS–SASAKIAN MANIFOLDS

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Abstract. In this paper, we study the geometry of the Ricci tensor of a harmonic nearly trans–Sasakian manifold. On the space of associated G-structure we introduce fundamental identities of harmonic nearly trans–Sasakian manifolds. We prove that Ricci–flat harmonic nearly trans–Sasakian manifolds are closely cosymplectic. We obtain conditions, which ensure that harmonic nearly trans–Sasakian manifolds are Einstein and η–Einstein manifolds. We obtain identities for the Ricci tensor of harmonic nearly trans–Sasakian manifolds. We provide local characterizations for the following harmonic nearly trans-Sasakian manifolds: Einstein manifolds; manifolds, the Ricci tensor of which is parallel, η–parallel, the Codazzi tensor, the Killing tensor, and satisfies the three selected identities.

Keywords: harmonic nearly trans–Sasakian manifolds, tensor Ricci, Einstein manifold, closely cosympletic manifold.

Mathematics Subject Classification: 53D15

1. Introduction

In this paper, we continue the study of nearly trans–Sasakian (NTS) manifolds, that is, manifolds equipped with an almost contact metric structure, the linear extension of which belongs to the class $W_1 \oplus W_4$ in the Gray — Hervella classification. In [7], [8], [9], harmonic NTS–manifolds were defined and the local structure of these manifolds was described. To obtain a harmonic NTS–manifold, it suffices to take the Cartesian product of any nearly Kähler manifold M with the real line $\mathbb{R}$ and make a canonical concircular transformation of the exact cosymplectic structure of manifold $M \times \mathbb{R}$. Each harmonic NTS–manifold is (locally) structured in this way.

The paper is organized as follows. In Section 2 we provide basic definitions and formulate known facts from the geometry of harmonic NTS–manifolds. This is important for understanding the further presentation. The study is made by means of the method of associated G-structures. In particular, we provide structural equations, expressions for the components of Ricci tensor, and the scalar curvature of a harmonic NTS–manifold on the space of the associated G-structure. Then we give the first and second fundamental identities, and a new third fundamental identity of harmonic NTS–manifolds. We obtain new results for Einstein and η-Einstein harmonic NTS–manifolds. We prove that a harmonic NTS–manifold has a Φ-invariant Ricci tensor.

In Section 3, we calculate the components of covariant derivative of the Ricci tensor on the space of associated G-structure, obtain certain identities for the Ricci tensor of a harmonic NTS–manifold, and obtain a local characterization of harmonic NTS–manifolds, the Ricci tensor of which satisfies the obtained identities. We also establish classification theorems for harmonic

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NTS–manifolds, the Ricci tensor of which is parallel, η-parallel, the Codazzi tensor, or the Killing tensor.

2. Preliminaries

We recall that an almost contact metric (AC) structure on a manifold M is the set $(\xi, \eta, \Phi, g = \langle\cdot,\cdot\rangle)$ of tensor fields on M, where ξ is a vector field called the characteristic field, η is a differential 1-form called the contact form, Φ is an endomorphism of the module of smooth vector fields of the manifold M called the structure endomorphism, $g = \langle\cdot,\cdot\rangle$ is a Riemann metric. Moreover,

$$1)\ \eta(\xi) = 1; \quad 2)\ \Phi(\xi) = 0; \quad 3)\ \eta \circ \Phi = 0; \quad 4)\ \Phi^2 = -id + \xi \otimes \eta;$$
$$5)\ \langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y); \quad X, Y \in \mathcal{X}(M).$$

A manifold admitting an AC–structure is called an almost contact metric manifold, briefly, AC–manifold.

Definition 2.1 ([7]). *An AC–structure is called the nearly trans–Sasakian (NTS) structure if its linear extension belongs to the class $W_1 \oplus W_4$ of almost Hermitian structures in the Gray – Hervella classification. An AC–manifold equipped with an NTS–structure is called the NTS–manifold.*

Definition 2.2 ([7]). *An NTS–structure with a closed contact form is called the eigen NTS–structure.*

Definition 2.3 ([7], [9]). *An eigen NTS–manifold with a harmonic contact form is called harmonic, and the number $\chi = -\frac{1}{2n}\delta\eta$ is its characteristics.*

We consider a harmonic NTS–manifold. The Lie form of such a manifold is closed [7], which means that the characteristics of such a manifold is constant, i.e., $d\chi = 0$. Moreover, since $\bar{\chi} = \chi$, the function χ is real.

The complete group of structural equations of a harmonic NTS–manifold reads [7]

$$\begin{aligned}
&1)\quad d\theta^a = -\theta^a_b \wedge \theta^b + C^{abc}\theta_b \wedge \theta_c + \chi\delta^a_b\theta^b \wedge \theta;\\
&2)\quad d\theta_a = \theta^b_a \wedge \theta_b + C_{abc}\theta^b \wedge \theta^c + \chi\delta^b_a\theta_b \wedge \theta;\\
&3)\quad d\theta = 0;\\
&4)\quad d\theta^a_b + \theta^a_c \wedge \theta^c_b = (A^{ad}_{bc} - 2C^{adh}C_{hbc})\theta^c \wedge \theta_d,
\end{aligned} \tag{2.1}$$

where $\{A^{ad}_{bc}\}$ is a family of functions on the space of the associated G-structure, which serve as components of the so-called curvature tensor of the associated Q-algebra [1], or the structural tensor of second kind, and,

$$\begin{aligned}
&1)\ A^{ad}_{[bc]} = 0; \quad 2)\ A^{[bd]}_{ac} = 0; \quad 3)\ \overline{A^{ad}_{bc}} = A^{bc}_{ad};\\
&4)\ C^{[abc]} = C^{abc}; \quad 5)\ C_{[abc]} = C_{abc}.
\end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned}
&1)\quad dC^{abc} + C^{dbc}\theta^a_d + C^{adc}\theta^b_d + C^{abd}\theta^c_d = C^{abcd}\theta_d + \chi C^{abc}\theta;\\
&2)\quad dC_{abc} - C_{dbc}\theta^d_a - C_{adc}\theta^d_b - C_{abd}\theta^d_c = C_{abcd}\theta^d + \chi C_{abc}\theta;\\
&3)\quad d\chi = 0,
\end{aligned} \tag{2.3}$$

where C^{abcd}, C_{abcd} are appropriate functions on the space of associated G-structure, and,

$$1)\ C^{a[bcd]} = 0; \quad 2)\ C_{a[bcd]} = 0.$$

Making external differentiation of fourth equation in (2.1), we obtain

$$dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\theta^h + A^{adh}_{bc}\theta_h + 2\chi A^{ad}_{bc}\theta, \tag{2.4}$$

where

$$\begin{aligned} &1)\quad A^{ad}_{b[ch]} = A^{a[dh]}_{bc} = 0;\\ &2)\quad (A^{a[d}_{bc} - 2C^{a[d|h}C_{hbc})C^{c|fg]} = 0;\\ &3)\quad (A^{ad}_{b[c} - 2C^{adf}C_{fb[c})C_{d|hg]} = 0. \end{aligned} \tag{2.5}$$

Making external differentiation of identity 1) in (2.3), we obtain

$$dC^{abcd} + C^{hbcd}\theta^a_h + C^{abcd}\theta^d_h + C^{abhd}\theta^c_d + C^{abch}\theta^d_h = C^{abcdh}\theta_h + 2\chi C^{abcd}\theta,$$

where

$$1)\ C^{abc[dh]} = 0; \qquad 2)\ C^{abcg}C_{gdh} = 0. \tag{2.6}$$

We call identity 2) in (2.6) the first fundamental identity, and identity 3) in (2.5) the second fundamental identity of the harmonic NTS–manifold.

It is convenient to adopt the following notation

$$\begin{aligned} &C^{ad}_{bc} = C^{adh}C_{hbc}, \qquad C^a_b = C^{ac}_{bc}, \qquad A^a_b = A^{ac}_{bc},\\ &A^{ac}_b = A^{hac}_{hb}, \qquad A^a_{bc} = A^{ha}_{hbc}. \end{aligned}$$

Lemma 2.1. *For a harmonic NTS–manifold, $C = C^a_a$ is nonnegative. Moreover, $C = 0$ if and only if the manifold is either cosymplectic or a Kenmotsu manifold, i.e., a manifold obtained from a cosymplectic manifold by a canonical concircular transformation.*

Proof. Since $\overline{C^{abc}} = C_{abc}$, we obtain $C = C^{abc}C_{abc} = \sum_{abc} |C_{abc}|^2 \geqslant 0$, and $C = 0$ if and only if $C^{abc} = C_{abc} = 0$. Then, by Theorem 6 in [7], a harmonic NTS–manifold is either a cosymplectic manifold or a Kenmotsu manifold, i.e., a manifold obtained from a cosymplectic manifold by the canonical concircular transformation [2]. It is easy to see that the converse is also true. The proof is complete. □

According to (2.3) and (2.4) we have

$$\begin{aligned} &1)\quad dC^{ad}_{bc} + C^{gd}_{bc}\theta^a_g + C^{ag}_{bc}\theta^d_g - C^{ad}_{gc}\theta^g_b - C^{ad}_{bg}\theta^g_c = C^{ad}_{bcg}\theta^g + C^{adg}_{bc}\theta_g + 2\chi C^{ad}_{bc}\theta;\\ &2)\quad dC^a_b + C^h_b\theta^a_h - C^a_h\theta^h_b = C^a_{bh}\theta^h + C^{ah}_b\theta_h + 2\chi C^a_b\theta;\\ &3)\quad dA^a_b + A^h_b\theta^a_h - A^a_h\theta^h_b = A^a_{bh}\theta^h + A^{ah}_b\theta_h + 2\chi A^a_b\theta. \end{aligned} \tag{2.7}$$

We contract identity 3) in (2.5), which is the first fundamental identity of the harmonic NTS–manifold, with respect to the indices a and b

$$A^d_c C_{hgd} + A^d_h C_{gcd} + A^d_g C_{chd} - 2C^d_c C_{hgd} - 2C^d_h C_{gcd} - 2C^d_g C_{chd} = 0. \tag{2.8}$$

Now we contract the third identity in (2.5) with respect to the indices a and c and rename b to c. Then, taking into account the symmetry properties of objects A and C (2.2), we obtain

$$A^d_c C_{hgd} + 2C^d_c C_{hgd} - 2C^d_g C_{chd} + 2C^d_h C_{cgd} = 0. \tag{2.9}$$

We deduct term–by–term (2.9) from (2.8), and in view of symmetry properties of C, see (2.2), we obtain the identity

$$A^d_{[h} C_{g]cd} - 2C^d_c C_{hgd} = 0. \tag{2.10}$$

We call identity (2.10) the third fundamental identity of the harmonic NTS–manifold.

The components of the Ricci tensor of the harmonic NTS–manifold on the space of the associated G-structure read [8]

$$\begin{aligned} &1) \quad S_{00} = -2n\chi^2; \\ &2) \quad S_{a\hat{b}} = A^{bc}_{ac} - 3C^{bcd}C_{dca} - 2n\chi^2\delta^b_a; \\ &3) \quad S_{\hat{a}b} = A^{ac}_{bc} - 3C^{acd}C_{dcb} - 2n\chi^2\delta^a_b. \end{aligned} \tag{2.11}$$

All other components are zero.

It follows from identity 1) in formula (2.11) that a Ricci–flat harmonic NTS–manifold is an closely cosymplectic manifold, and therefore it is locally equivalent to the product of a nearly Kähler manifold and the real line. If the manifold is simply connected, then these equivalences can be chosen to be global.

The scalar curvature of a harmonic NTS–manifold is

$$r = 2A^{ab}_{ab} - 6C^{abc}C_{cba} - 2n(n+1)\chi^2. \tag{2.12}$$

It follows from (2.1) that each Kenmotsu manifold is a harmonic NTS–manifold of characteristics $\chi = -1$, and each closely cosymplectic manifold is a harmonic NTS–manifold of characteristics $\chi = 0$. Kenmotsu manifolds and closely cosymplectic manifolds are the most interesting and well–studied examples of harmonic NTS–manifolds. One should add the special Kenmotsu manifolds of the second kind to these examples.

Theorem 2.1. *A harmonic NTS–manifold is an Einstein manifold if and only if the identity*

$$A^a_b = 3C^a_b \tag{2.13}$$

holds on the space of the associated G-structure.

Proof. Let a harmonic NTS–manifold be an Einstein manifold with the cosmological constant ϵ. Then the components of Ricci tensor S and the components of metric tensor g are related by the identity $S_{ij} = \epsilon g_{ij}$, where $\epsilon = \text{const}$. In view of (2.11), these relations on the space of associated G-structure are written as

$$\begin{aligned} &1) \quad -2n\chi^2 = \epsilon; \\ &2) \quad A^b_a - 3C^b_a - 2n\chi^2\delta^b_a = \epsilon\delta^b_a, \end{aligned} \tag{2.14}$$

that is, $A^a_b = 3C^a_b$. The proof is complete. □

Corollary 2.1. *A harmonic Einstein NTS–manifold is a manifold of non–positive scalar curvature.*

Theorem 2.2. *A complete harmonic Einstein NTS–manifold is either a Ricci–flat closely cosymplectic manifold, and hence is holomorphically isometrically covered by the product of a Ricci-flat nearly Kähler manifold with the real line, or is compact and has a finite fundamental group.*

Proof. For $\epsilon = 0$ it follows from identity 1) in (2.14) that $\chi = 0$, i.e., the manifold is Ricci–flat, closely cosymplectic, and therefore locally holomorphically isometric to a manifold of the form $N^{2n} \times \mathbb{R}$, where N^{2n} is a Ricci–flat nearly Kähler manifold [2].

If $\epsilon < 0$, then, according to the classical Myers theorem [3], in the case of completeness the manifold is compact and has a finite fundamental group. The proof is complete. □

Theorem 2.3. *A complete harmonic Einstein NTS–manifold is locally equivalent to the product $N^{2n} \times \mathbb{R}$ or canonically concircular to the manifold $N^{2n} \times \mathbb{R}$ equipped with a cosymplectic structure, where N^{2n} is a Kähler manifold that is an Einstein manifold.*

Proof. Let a harmonic NTS–manifold be an Einstein manifold with cosmological constant ϵ. Then, taking into account (2.13), third fundamental identity (2.10) can be written as

$$3C^d_h C_{gcd} - 3C^d_g C_{hcd} - 4C^d_c C_{hgd} = 0.$$

Symmetrizing this identity in the indices h and c, we obtain

$$C^d_h C_{gcd} + C^d_g C_{hcd} = 0. \tag{2.15}$$

Since (C^d_h) is a Hermitian matrix, at each point of this harmonic NTS–manifold there exists an A-frame [2] in which $C^d_h = C_h \delta^d_h$, where $\{C_c\}$ are the eigenvalues of this matrix. Then identity (2.15) can be written as $C_h \delta^d_h C_{gcd} + C_g \delta^d_g C_{hcd} = 0$. We contract the resulting identity with the object C^{cdf}, then

$$\begin{aligned} &C_h \delta^d_h C_{gcd} C^{cdf} + C_g \delta^d_g C_{hcd} C^{cdf} = 0, \\ &C_h C_g \delta^f_g + C_g C_h \delta^f_h = 0, \\ &2C_g C_h = 0. \end{aligned}$$

Hence, $C_h = 0$ and then $C^d_h = 0$. Contracting this identity in the indices d and h, we obtain

$$\sum_{abc} |C_{abc}|^2 = C^{abc} C_{abc} = C^c_c = 0,$$

and hence, $C_{abc} = 0$, and the considered manifold is either a cosymplectic manifold or a Kenmotsu manifold.

Since a cosymplectic manifold is locally equivalent to the product of a Kähler manifold with the real line [2], and the class of Kenmotsu manifolds coincides with the class of almost contact metric manifolds obtained from cosymplectic manifolds by the canonical concircular transformation of the cosymplectic structure [2], we obtain the required statements. The proof is complete. □

We consider an Einstein harmonic NTS–manifold. Then its Ricci tensor on the space of associated G-structure has the components

$$S_{ij} = a g_{ij} + b \delta^0_i \delta^0_j. \tag{2.16}$$

We are going to obtain expressions for a and b. In view of (2.11), relations (2.16) can be written as

$$\begin{aligned} &1) \quad -2n\chi^2 = a + b; \\ &2) \quad A^{bc}_{ac} - 3C^{bcd} C_{dca} - 2n\chi^2 \delta^b_a = a\delta^b_a. \end{aligned} \tag{2.17}$$

We contract identity 2) in (2.17) in the indices a and b, then in view of (2.12) we obtain

$$a = \frac{r}{2n} - (n-1)\chi^2. \tag{2.18}$$

Substituting (2.18) into the first identity in formula (2.17), we obtain

$$b = -\frac{r}{2n} - (n+1)\chi^2.$$

Definition 2.4. *We call the Ricci tensor of an AC–manifold Φ-invariant if*

$$\Phi Q = Q\Phi. \tag{2.19}$$

Writing identity (2.19) on the space of associated G-structure, we find that for an AC–manifold with Φ-invariant Ricci tensor the following relations hold

$$\begin{aligned} &1) \quad S_{0a} = S_{0\hat{a}} = S_{a0} = S_{\hat{a}0} = 0; \\ &2) \quad S_{ab} = S_{\hat{a}\hat{b}} = 0. \end{aligned} \tag{2.20}$$

And vice versa, if relations (2.20) hold on the space of associated G-structure, then the AC–manifold has a Φ-invariant Ricci tensor. Thus, we have proved the following theorem.

Theorem 2.4. *An AC–manifold has a Φ-invariant Ricci tensor if and only if the following equalities hold on the space of associated G-structure*

$$1)\quad S_{0a} = S_{0\hat{a}} = S_{a0} = S_{\hat{a}0} = 0;$$
$$2)\quad S_{ab} = S_{\hat{a}\hat{b}} = 0.$$

We consider a harmonic NTS–manifold. By (2.11), definition 2.4, and Theorem 2.4 we obtain the following assertion.

Theorem 2.5. *The Ricci tensor of a harmonic NTS–manifold is Φ–invariant.*

3. Identities for Ricci tensor

We recall that the tensor components of Riemannian connection form for a harmonic NTS–manifold on the space of associated G-structure read [8]

$$\begin{array}{llll} 1)\ \theta^{\hat{a}}_{b} = C^{abc}\theta_c; & 2)\ \theta^{a}_{\hat{b}} = C_{abc}\theta^c; & 3)\ \theta^{a}_{0} = -\chi\delta^a_b\theta^b; & 4)\ \theta^{\hat{a}}_{0} = -\chi\delta^b_a\theta_b; \\ 5)\ \theta^{0}_{a} = \chi\delta^b_a\theta_b; & 6)\ \theta^{0}_{\hat{a}} = \chi\delta^a_b\theta^b; & 7)\ \theta^0_0 = 0; & 8)\ \theta^i_j + \theta^{\hat{j}}_{\hat{i}} = 0. \end{array} \tag{3.1}$$

Since the Ricci tensor is a tensor of type (2,0), by the Fundamental Theorem of Tensor Analysis, its components on the space of principal frame bundle over the considered manifold satisfy the relations [2]

$$dS_{ij} - S_{kj}\theta^k_i - S_{ik}\theta^k_j = S_{ij,k}\theta^k, \tag{3.2}$$

where $\{S_{ij,k}\}$ a system of smooth functions that serve as components of the tensor ∇S.

Writing (3.2) on the space on the space of the associated G-structure, in view of (3.1), (2.11), (2.3:3) and (2.7) we obtain

$$\begin{array}{ll} 1)\ S_{0a,\hat{b}} = S_{a0,\hat{b}} = \chi(A^b_a - 3C^b_a); & 2)\ S_{0\hat{a},b} = S_{\hat{a}0,b} = \chi(A^a_b - 3C^a_b); \\ 3)\ S_{ab,c} = -2(A^d_{(a} - 3C^d_{(a})C_{|d|b)c}; & 4)\ S_{a\hat{b},0} = S_{\hat{b}a,0} = 2\chi(A^b_a - 3C^b_a); \\ 5)\ S_{a\hat{b},c} = S_{\hat{b}a,c} = A^b_{ac} - 3C^b_{ac}; & 6)\ S_{a\hat{b},\hat{c}} = S_{\hat{b}a,\hat{c}} = A^{bc}_a - 3C^{bc}_a; \\ 7)\ S_{\hat{a}\hat{b},\hat{c}} = -2(A^{(a}_d - 3C^{(a}_d)C^{|d|b)c}, & \end{array} \tag{3.3}$$

while other components are zero.

Theorem 3.1. *The Ricci tensor of a harmonic NTS–manifold satisfies the identities*

1) $\nabla_X S(\xi, \xi) = 0;$
2) $\nabla_\xi S(\chi, \xi) = 0;$
3) $\nabla_{\Phi X}(S)(\xi, \Phi Y) - \nabla_X(S)(\xi, Y) = 0;$
4) $\nabla_\xi(S)(\Phi X, \Phi Y) - \nabla_\xi(S)(X, Y) = 0;$
5) $\nabla_{\Phi^2 X}(S)(\Phi^2 Y, \Phi^2 Z) - \nabla_{\Phi^2 X}(S)(\Phi Y, \Phi Z) + \nabla_{\Phi X}(S)(\Phi Y, \Phi^2 Z) + \nabla_{\Phi X}(S)(\Phi^2 Y, \Phi Z) = 0.$

Proof. It follows from (3.3) that

$$1)\ \nabla_X S(\xi, \xi) = 0; \qquad 2)\ \nabla_\xi S(\chi, \xi) = 0.$$

Applying the identity recovery procedure [1], [2] to the identity $S_{0a,b} = 0$, we obtain

$$\nabla_{\Phi^2 X}(S)(\xi, \Phi^2 Y) - \nabla_{\Phi X}(S)(\xi, \Phi Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M).$$

This identity can be written as

$$\nabla_{\Phi X}(S)(\xi, \Phi Y) - \nabla_X(S)(\xi, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.4}$$

Applying the identity recovery procedure to the identity $S_{ab,0} = 0$, we get

$$\nabla_\xi(S)(\Phi^2 X, \Phi^2 Y) - \nabla_\xi(S)(\Phi X, \Phi Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M),$$

which can be written as

$$\nabla_\xi(S)(\Phi X, \Phi Y) - \nabla_\xi(S)(X, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.5}$$

Applying the identity recovery procedure to the identity $S_{ab,\hat{c}} = 0$, we get

$$\begin{aligned} \nabla_{\Phi^2 X}(S)(\Phi^2 Y, \Phi^2 Z) &- \nabla_{\Phi^2 X}(S)(\Phi Y, \Phi Z) \\ &+ \nabla_{\Phi X}(S)(\Phi Y, \Phi^2 Z) + \nabla_{\Phi X}(S)(\Phi^2 Y, \Phi Z) = 0, \quad \forall X, Y \in \mathcal{X}(M). \end{aligned} \tag{3.6}$$

The proof is complete. □

Main non–zero components of the tensor ∇S are defined by the following pairs of expressions

$$\begin{array}{ll} 1)\ S_{0a,\hat{b}} = \chi(A_a^b - 3C_a^b); & 2)\ S_{a\hat{b},0} = 2\chi(A_a^b - 3C_a^b); \\ 3)\ S_{ab,c} = -2(A^d_{(a} - 3C^d_{(a})C_{|d|b)c}; & 4)\ S_{a\hat{b},c} = A^b_{ac} - 3C^b_{ac} \end{array} \tag{3.7}$$

and by their adjoint. The most interesting is the study of geometric meaning of vanishing of these components.

Let $S_{0a,\hat{b}} = \chi(A_a^b - 3C_a^b) = 0$. Then either $\chi = 0$, or $A_a^b - 3C_a^b = 0$. In the first case, the manifold is closely cosymplectic, in the second case (by Theorem 2.4) it is an Einstein manifold with cosmological constant $\epsilon = -2n\chi^2$.

Applying the identity recovery procedure to the identity [1], [2] to the identity $S_{0a,\hat{b}} = 0$, we obtain

$$\nabla_{\Phi^2 X}(S)(\xi, \Phi^2 Y) + \nabla_{\Phi X}(S)(\xi, \Phi Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M).$$

This identity can be written as

$$\nabla_{\Phi X}(S)(\xi, \Phi Y) + \nabla_X(S)(\xi, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.8}$$

Since (3.4) holds satisfied for each harmonic NTS–manifold, identity (3.8) is equivalent to

$$\nabla_{\Phi X}(S)(\xi, \Phi Y) = \nabla_X(S)(\xi, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M).$$

Hence, the identity $S_{0a,\hat{b}} = 0$ is equivalent to

$$\nabla_X(S)(\xi, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M) \tag{3.9}$$

The above facts are summarized in the following theorem.

Theorem 3.2. *A harmonic NTS–manifold, the Ricci tensor of which satisfies identity (3.9) is either a precisely cosymplectic manifold or an Einstein manifold with cosmological constant* $\epsilon = -2n\chi^2$.

In view of Theorem 2.3, Theorem 3.2 can be formulated as follows.

Theorem 3.3. *A harmonic NTS–manifold, the Ricci tensor of which satisfies identity (3.9) is locally equivalent to the product of a nearly Kähler manifold with the real line, or canonically concircular to the product of a Kähler manifold with the real line equipped with a cosymplectic structure.*

Let $S_{a\hat{b},0} = 2\chi(A_a^b - 3C_a^b) = 0$, that is $S_{a\hat{b},0} = 0$. This identity is equivalent to

$$\nabla_\xi(S)(\Phi^2 X, \Phi^2 Y) + \nabla_\xi(S)(\Phi X, \Phi Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.10}$$

By (3.5) and (3.10) we obtain

$$\nabla_\xi(S)(\Phi X, \Phi Y) = \nabla_\xi(S)(X, Y) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.11}$$

It follows from (3.7) that $2S_{0a,\hat{b}} = S_{a\hat{b},0}$. Thus, a harmonic NTS–manifold, the Ricci tensor of which satisfies identity (3.9), also satisfies identity (3.11), and vice versa.

Theorem 3.4. *The class of harmonic NTS–manifolds, the Ricci tensor of which satisfies the identity (3.9), coincides with the class of harmonic NTS–manifolds, the Ricci tensor of which satisfies identity (3.11).*

Let $S_{ab,c} = -2(A^d_{(a} - 3C^d_{(a})C_{|d|b)c} = 0$. In view of (3.6), identity $S_{ab,c} = 0$ is equivalent to

$$\nabla_{\Phi^2 X}(S)(\Phi^2 Y, \Phi^2 Z) - \nabla_{\Phi^2 X}(S)(\Phi Y, \Phi Z) = 0, \qquad \forall X, Y \in \mathcal{X}(M),$$

which in view of (3.5) is written as

$$\nabla_X(S)(\Phi Y, \Phi Z) - \nabla_X(S)(Y, Z) = 0, \qquad \forall X, Y \in \mathcal{X}(M). \tag{3.12}$$

Since

$$(A^d_{(a} - 3C^d_{(a})C_{|d|b)c} = 0. \tag{3.13}$$

It follows from (2.10) and (3.13) that

$$2A_a^d C_{bcd} = 3C_a^d C_{bcd} + 3C_b^d C_{acd} + 4C_c^d C_{abd}.$$

We symmetrize the obtained identity in the indices b and c, then by symmetry properties (2.2) of C we obtain

$$C_b^d C_{cad} + C_c^d C_{bad} = 0.$$

In view of this identity, identity (3.13) becomes

$$A_a^d C_{bcd} = 4C_c^d C_{abd}. \tag{3.14}$$

We alternate this identity in the indices a and b, then, in view of the symmetry properties of object C, we obtain the identity

$$A^d_{[a} C_{b]cd} = 4C_c^d C_{abd}. \tag{3.15}$$

By (2.10) and (3.15) we have

$$C_c^d C_{abd} = 0.$$

Since (C_c^d) is the Hermitian matrix, at each point of the manifold, there exists an A–frame [2], in which $C_c^d = C_c \delta_c^d$, where $\{C_c\}$ are the eigenvalues of this matrix. Contracting the identity $C_c C_{abc} = 0$ with the object C^{abd}, we obtain $(C_c)^2 \delta_c^d = 0$, and therefore $C_c = 0$. But then $C_c^d = 0$. Contracting this identity over the indices c and d, we obtain

$$\sum_{abc} |C_{abc}|^2 = C^{abc} C_{abc} = C_c^c = 0,$$

and hence, $C_{abc} = 0$, that is, the manifold is either a cosymplectic manifold or a Kenmotsu manifold [7].

Similarly, by (3.14) one can show that $A_c^c = 0$, and hence a harmonic NTS–manifold, the Ricci tensor of which satisfies identity (3.12), according to (2.12), is a manifold of constant scalar curvature $r = -2n(n+1)\chi^2$ since there are no Kenmotsu manifolds of constant curvature different from (-1), and a Kenmotsu manifold is a space of constant curvature (-1) if and only if it is canonically concircular to a manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with a cosymplectic structure [2]. It is well–known that a cosymplectic manifold is locally equivalent to the product of a Kähler manifold and the real line [2]. The Kähler component of a cosymplectic manifold is locally holomorphically isometric to $\mathbb{C}^n$, and hence the cosymplectic manifold is locally equivalent to the product $\mathbb{C}^n \times \mathbb{R}$.

The above facts are summarized in the next theorem.

Theorem 3.5. *A harmonic NTS–manifold, the Ricci tensor of which satisfies the identity (3.9), is locally equivalent to the product $\mathbb{C}^n \times \mathbb{R}$ or canonically concircular to the manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with a cosymplectic structure.*

Definition 3.1 ([4]). *The Ricci tensor S of an AC–manifold is called parallel if $\nabla S = 0$ and η–parallel if $\nabla_X(S)(\Phi Y, \Phi Z) = 0$ for all $X, Y, Z \in \mathcal{X}(M)$.*

Let a harmonic NTS–manifold have a parallel Ricci tensor, i.e. $\nabla S = 0$. The next statement follows directly from Theorems 3.3 and 3.5.

Theorem 3.6. *A harmonic NTS–manifold with parallel Ricci tensor is locally equivalent to the product $\mathbb{C}^n \times \mathbb{R}$ equipped with a cosymplectic structure.*

Next we consider a harmonic NTS–manifold with η–parallel Ricci tensor. On the space of associated G-structure, the η–parallel condition, i.e., the identity

$$\nabla_X(S)(\Phi Y, \Phi Z) = 0, \qquad \forall X, Y, Z \in \mathcal{X}(M)$$

is written as

$$S_{ij,k}\Phi^i_r\Phi^j_l X^k Y^r Z^l = 0,$$

which is equivalent to the relations

$$\begin{aligned} &1) \quad S_{ab,c} = -2(A^d_{(a} - 3C^d_{(a})C_{|d|b)c};\\ &2) \quad S_{a\hat{b},0} = 2\chi(A^b_a - 3C^b_a);\\ &3) \quad S_{a\hat{b},c} = S_{\hat{b}a,c} = A^b_{ac} - 3C^b_{ac}, \end{aligned} \tag{3.16}$$

i.e., the Ricci tensor of a harmonic NTS–manifold with η–parallel Ricci tensor is parallel. Therefore, for a harmonic NTS–manifold with η–parallel Ricci tensor, Theorem 3.6 holds.

Gray in [5] introduced two classes of Riemannian manifolds defined by the covariant derivative of the Ricci tensor. Class A consists of all Riemannian manifolds, the Ricci tensor of which S is a Killing tensor, i.e.

$$\nabla_X(S)(Y, Z) + \nabla_Y(S)(X, Z) + \nabla_Z(S)(X, Y) = 0; \qquad \forall X, Y, Z \in \mathcal{X}(M).$$

The second class B consists of all Riemannian manifolds, the Ricci tensor of which is the Codazzi tensor, i.e.

$$\nabla_X(S)(Y, Z) = \nabla_Y(S)(X, Z); \qquad \forall X, Y, Z \in \mathcal{X}(M).$$

Definition 3.2 ([10], [6]). *A symmetric 2-tensor field T is called a Codazzi tensor if $dT = 0$, i.e., if T satisfies the Codazzi equation*

$$\nabla_X(T)(Y, Z) = \nabla_Y(T)(X, Z), \qquad \forall X, Y, Z \in \mathcal{X}(M).$$

Let M^{2n+1} be a harmonic NTS–manifold, the Ricci tensor is the Codazzi tensor. Then the identity holds

$$\nabla_X(S)(Y, Z) = \nabla_Y(S)(X, Z); \qquad \forall X, Y, Z \in \mathcal{X}(M). \tag{3.17}$$

Identity (3.17) on the space of associated G-structure is written as

$$S_{ij,k} = S_{kj,i}. \tag{3.18}$$

In particular, by (3.7) and (3.18) we have $S_{ab,c} = S_{cb,a}$, that is,

$$(A^d_{(a} - 3C^d_{(a)})C_{|d|b)c} = (A^d_{(c} - 3C^d_{(c)})C_{|d|b)a},$$
$$A^d_a C_{dbc} + A^d_b C_{dac} - 3C^d_a C_{dbc} - 3C^d_b C_{dac} = A^d_c C_{dba} + A^d_b C_{dca} - 3C^d_c C_{dba} - 3C^d_b C_{dca},$$
$$A^d_c C_{abd} - A^d_a C_{cbd} + 2A^d_b C_{acd} = 3C^d_a C_{bcd} + 3C^d_c C_{abd} + 6C^d_b C_{acd}.$$

In view of the third fundamental identity, the resulting identity can be written as

$$4C^d_b C_{cad} + 2A^d_b C_{acd} = 3C^d_a C_{bcd} + 3C^d_c C_{abd} + 6C^d_b C_a cd,$$
$$2A^d_b C_{acd} = 3C^d_a C_{bcd} + 3C^d_c C_{abd} + 10C^d_b C_{acd}.$$

Alternating the last identity by indices a and b, taking into account the third fundamental identity and the properties of the object C^{abc}, we obtain

$$C^d_c C_{abd} = 7C^d_{[b} C_{a]cd}. \tag{3.19}$$

Proceeding as in the proof of Theorems 3.3 and 3.5, by identity (3.19) we obtain that $C_{abc} = 0$, i.e. the manifold is either a cosymplectic manifold or a Kenmotsu manifold.

By (3.7) and (3.18) we also have $S_{\hat{b}a,0} = S_{0a,\hat{b}}$, that is,

$$2\chi(A^b_a - 3C^b_a) = \chi(A^b_a - 3C^b_a),$$
$$\chi(A^b_a - 3C^b_a) = 0.$$

Arguing as in the proof of Theorem 3.5, we obtain the next statement.

Theorem 3.7. *A harmonic NTS–manifold, the Ricci tensor of which is the Codazzi tensor, is locally equivalent to the product $\mathbb{C}^n \times \mathbb{R}$ or canonically concircular to the manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with a cosymplectic structure.*

Definition 3.3 ([10], [6]). *A symmetric 2-tensor field T is called the Killing tensor if*

$$\nabla_X(T)(Y,Z) + \nabla_Y(T)(X,Z) + \nabla_Z(T)(X,Y) = 0; \qquad \forall X,Y,Z \in \mathcal{X}(M).$$

Now let M^{2n+1} be a harmonic NTS–manifold, the Ricci tensor of which is the Killing tensor. Then the identity holds

$$\nabla_X(S)(Y,Z) + \nabla_Y(S)(X,Z) + \nabla_Z(S)(X,Y) = 0; \qquad \forall X,Y,Z \in \mathcal{X}(M).$$

On the space of associated G-structure this identity is written as

$$S_{ij,k} + S_{jk,i} + S_{ki,j} = 0. \tag{3.20}$$

In particular, it follows from (3.7) and (3.20) that

$$S_{0a,\hat{b}} + S_{a\hat{b},0} + S_{\hat{b}0,a} = 0,$$
$$\chi(A^b_a - 3C^b_a) + 2\chi(A^b_a - 3C^b_a) + \chi(A^b_a - 3C^b_a) = 0,$$
$$\chi(A^b_a - 3C^b_a) = 0.$$

Arguing as in the proof of Theorem 3.5 and using the third fundamental identity, we obtain the following theorem.

Theorem 3.8. *A harmonic NTS–manifold, the Ricci tensor of which is a Killing tensor, is locally equivalent to the product $\mathbb{C}^n \times \mathbb{R}$ or canonically concircular to the manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with a cosymplectic structure.*

Remark 3.1. *Theorems 3.6, 3.7, 3.8 are invertible.*

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